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# A NEW OBSTRUCTION TO MINIMAL ISOMETRIC IMMERSIONS INTO A REAL SPACE FORM

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ABSTRACT. In the theory of minimal submanifolds, the following problem is fundamental: *when does a given Riemannian manifold admit (or does not admit) a minimal isometric immersion into an Euclidean space of arbitrary dimension?* S.S. Chern, in his monograph [6] *Minimal submanifolds in a Riemannian manifold*, remarked that the result of Takahashi (*the Ricci tensor of a minimal submanifold into a Euclidean space is negative semidefinite)* was the only known Riemannian obstruction to minimal isometric immersions in Euclidean spaces. A second obstruction was obtained by B.Y. Chen as an immediate application of his fundamental inequality [1]: *the scalar curvature and the sectional curvature of a minimal submanifold into a Euclidean space satisfies the inequality* $\tau \leq k$. We find a new relation between the Chen invariant, the dimension of the submanifold, the length of the mean curvature vector field and a deviation parameter. This result implies a new obstruction: *the sectional curvature of a minimal submanifold into a Euclidean space also satisfies the inequality* $k \leq -\tau$.

*Key words and phrases:* Constrained maximum, Chen's inequality, Minimal submanifolds.

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## 1. OPTIMIZATIONS ON RIEMANNIAN MANIFOLDS

Let $(N, \widetilde{g})$ be a Riemannian manifold, $(M, g)$ a Riemannian submanifold, and $f \in \mathcal{F}(N)$. To these ingredients we attach the optimum problem

$$\min_{x \in M} f(x). \tag{1.1}$$

The fundamental properties of such programs are given in the papers [7] – [9]. For the interest of this paper we recall below a result obtained in [7].

**Theorem 1.1.** *If* $x_0 \in M$ *is a solution of the problem* $(1.1)$, *then*

i) $(\operatorname{grad} f)(x_0) \in T^{\perp}_{x_0} M,$

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ii) *the bilinear form*

$$\alpha : T_{x_0}M \times T_{x_0}M \to R,$$
$$\alpha(X,Y) = \mathrm{Hess}\,_f(X,Y) + \widetilde{g}(h(X,Y), (\mathrm{grad}\, f)(x_0))$$

*is positive semidefinite, where $h$ is the second fundamental form of the submanifold $M$ in $N$.*

**Remark 1.2.** The bilinear form $\alpha$ is nothing else but $\mathrm{Hess}\,_{f|M}(x_0)$.

## 2. CHEN'S INEQUALITY

Let $(M, g)$ be a Riemannian manifold of dimension $n$, and $x$ a point in $M$. We consider the orthonormal frame $\{e_1, e_2, \dots, e_n\}$ in $T_xM$.

The *scalar curvature* at $x$ is defined by

$$\tau = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j).$$

We denote

$$\delta_M = \tau - \min(k),$$

where $k$ is the *sectional curvature* at the point $x$. The invariant $\delta_M$ is called the *Chen's invariant* of Riemannian manifold $(M, g)$.

The Chen's invariant was estimated as the following: "$(M, g)$ is a Riemannian submanifold in a real space form $\widetilde{M}(c)$, varying with $c$ and the length of the mean curvature vector field of $M$ in $\widetilde{M}(c)$."

**Theorem 2.1.** *Consider $(\widetilde{M}(c), \widetilde{g})$ a real space form of dimension $m$, $M \subset \widetilde{M}(c)$ a Riemannian submanifold of dimension $n \ge 3$. The Chen's invariant of $M$ satisfies*

$$\delta_M \le \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1)c \right\},$$

*where $H$ is the mean curvature vector field of submanifold $M$ in $\widetilde{M}(c)$. Equality is attained at a point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_xM$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^{\perp}M$ in which the Weingarten operators take the following form*

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & h_{33}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn}^{n+1} \end{pmatrix},$$

*with $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$ and*

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad r \in \overline{n+2, m}.$$

**Corollary 2.2.** *If the Riemannian manifold $(M, g)$, of dimension $n \geq 3$, admits a minimal isometric immersion into a real space form $\widetilde{M}(c)$, then*

$$k \geq \tau - \frac{(n-2)(n+1)c}{2}.$$

The aim of this paper is threefold:

- to formulate a new theorem regarding the relation between $\delta_M$, the dimension $n$, the length of the mean curvature vector field, and a deviation parameter $a$;
- to prove this new theorem using the technique of Riemannian programming;
- to obtain a new obstruction, $k \leq -\tau + \frac{(n^2-n+2)c}{2}$, for minimal isometric immersions in real space forms.

## 3. A NEW OBSTRUCTION TO MINIMAL ISOMETRIC IMMERSIONS INTO A REAL SPACE FORM

Let $(M, g)$ be a Riemannian manifold of dimension $n$, and $a$ a real number. We define the following invariants

$$\delta_M^a = \begin{cases} \tau - a \min k, & \text{for } a \geq 0, \\ \tau - a \max k, & \text{for } a < 0, \end{cases}$$

where $\tau$ is the scalar curvature, and $k$ is the sectional curvature.

With these ingredients we obtain

**Theorem 3.1.** *For any real number $a \in [-1, 1]$, the invariant $\delta_M^a$ of a Riemannian submanifold $(M, g)$, of dimension $n \geq 3$, into a real space form $\widetilde{M}(c)$, of dimension $m$, verifies the inequality*

$$\delta_M^a \leq \frac{(n^2-n-2a)c}{2} + \frac{n(a+1)-3a-1}{n(a+1)-2a}\frac{n^2\left\|H\right\|^2}{2},$$

*where $H$ is the mean curvature vector field of submanifold $M$ in $\widetilde{M}(c)$.*

*If $a \in (-1, 1)$, equality is attained at the point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_xM$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ in which the Weingarten operators take the form*

$$A_r = \begin{pmatrix} h_{11}^r & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^r & 0 & \cdots & 0 \\ 0 & 0 & h_{33}^r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn}^r \end{pmatrix},$$

*with $(a+1)h_{11}^r = (a+1)h_{22}^r = h_{33}^r = \cdots = h_{nn}^r, \forall\, r \in \overline{n+1, m}$.*

*Proof.* Consider $x \in M$, $\{e_1, e_2, \dots, e_n\}$ an orthonormal frame in $T_xM$, $\{e_{n+1}, e_{n+2}, \dots, e_m\}$ an orthonormal frame in $T_x^\perp M$ and $a \in (-1, 1)$.

From Gauss' equation it follows

$$\tau - ak(e_1 \wedge e_2) = \frac{(n^2-n-2a)c}{2} + \sum_{r=n+1}^{m} \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) - a \sum_{r=n+1}^{m} (h_{11}^r h_{22}^r - (h_{12}^r)^2).$$

Using the fact that $a \in (-1,1)$, we obtain

$$\tau - ak(e_1 \wedge e_2) \le \frac{(n^2-n-2a)c}{2} + \sum_{r=n+1}^{m} \sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r - a \sum_{r=n+1}^{m} h_{11}^r h_{22}^r. \tag{3.1}$$

For $r \in \overline{n+1,m}$, let us consider the quadratic form

$$f_r : R^n \to R,$$

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = \sum_{1\le i<j\le n} (h_{ii}^r h_{jj}^r) - a h_{11}^r h_{22}^r$$

and the constrained extremum problem

$$\max f_r,$$

$$\text{subject to } P : h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r,$$

where $k^r$ is a real constant.

The first three partial derivatives of the function $f_r$ are

$$\frac{\partial f_r}{\partial h_{11}^r} = \sum_{2\le j\le n} h_{jj}^r - a h_{22}^r, \tag{3.2}$$

$$\frac{\partial f_r}{\partial h_{22}^r} = \sum_{j\in \overline{1,n}\setminus\{2\}} h_{jj}^r - a h_{11}^r, \tag{3.3}$$

$$\frac{\partial f_r}{\partial h_{33}^r} = \sum_{j\in \overline{1,n}\setminus\{3\}} h_{jj}^r \,. \tag{3.4}$$

As for a solution $(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r)$ of the problem in question, the vector $(\mathrm{grad})(f_1)$ being normal at $P$, from (3.2) and (3.3) we obtain

$$\sum_{j=1}^{n} h_{jj}^r - h_{11}^r - a h_{22}^r = \sum_{j=1}^{n} h_{jj}^r - h_{22}^r - a h_{11}^r,$$

therefore

$$h_{11}^r = h_{22}^r = b^r. \tag{3.5}$$

From (3.2) and (3.4), it follows

$$\sum_{j=1}^{n} h_{jj}^r - h_{11}^r - a h_{22}^r = \sum_{j=1}^{n} h_{jj}^r - h_{33}^r.$$

By using (3.5) we obtain $h_{33}^r = b^r(a+1)$. Similarly one gets

$$h_{jj}^r = b^r(a+1), \qquad \forall j \in \overline{3,n}. \tag{3.6}$$

As $h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, from (3.5) and (3.6) we obtain

$$b^r = \frac{k^r}{n(a+1)-2a}\,. \tag{3.7}$$

We fix an arbitrary point $p \in P$.

The 2-form $\alpha : T_pP \times T_pP \to R$ has the expression

$$\alpha(X,Y) = \mathrm{Hess}_{f_r}(X,Y) + \langle h'(X,Y), (\mathrm{grad}\ f_r)(p)\rangle\,,$$

where $h'$ is the second fundamental form of $P$ in $R^n$ and $\langle\cdot,\cdot\rangle$ is the standard inner-product on $R^n$.

In the standard frame of $R^n$, the Hessian of $f_r$ has the matrix

$$\mathrm{Hess}_{f_r} = \begin{pmatrix} 0 & 1-a & 1 & \cdots & 1 \\ 1-a & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

As $P$ is totally geodesic in $R^n$, considering a vector $X$ tangent to $P$ at the arbitrary point $p$, that is, verifying the relation $\sum_{i=1}^n X^i = 0$, we have

$$\begin{aligned} \alpha(X,X) &= 2 \sum_{1 \le i < j \le n} X^i X^j - 2aX^1X^2 \\ &= \left(\sum_{i=1}^n X^i\right)^2 - \sum_{i=1}^n (X^i)^2 - 2aX^1X^2 \\ &= -\sum_{i=1}^n (X^i)^2 - a(X^1+X^2)^2 + a(X^1)^2 + a(X^2)^2 \\ &= -\sum_{i=3}^n (X^i)^2 - a(X^1+X^2)^2 - (1-a)(X^1)^2 - (1-a)(X^2)^2 \le 0. \end{aligned}$$

So $\mathrm{Hess}_{f|M}$ is everywhere negative semidefinite, therefore the point $(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r)$, which satisfies (3.5), (3.6), (3.7) is a global maximum point.

From (3.5) and (3.6), it follows

$$\begin{aligned} f_r &\le (b^r)^2 + 2b^r(n-2)b^r(a+1) + C_{n-2}^2 (b^r)^2(a+1)^2 - a(b^r)^2 \\ &= \frac{(b^r)^2}{2}[n^2(a+1)^2 - n(a+1)(5a+1) + 6a^2 + 2a] \\ &= \frac{(b^r)^2}{2}[n(a+1) - 3a - 1][n(a+1) - 2a]. \end{aligned} \tag{3.8}$$

By using (3.7) and (3.8), we obtain

$$\begin{aligned} f_r &\le \frac{(k^r)^2}{2[n(a+1)-2a]}[n(a+1) - 3a - 1] \\ &= \frac{n^2(H^r)^2}{2} \cdot \frac{n(a+1)-3a-1}{n(a+1)-2a}. \end{aligned} \tag{3.9}$$

The relations (3.1) and (3.9) imply

$$\tau - ak(e_1 \wedge e_2) \le \frac{(n^2-n-2a)c}{2} + \frac{n(a+1)-3a-1}{n(a+1)-2a} \cdot \frac{n^2 \|H\|^2}{2}. \tag{3.10}$$

In (3.10) we have equality if and only if the same thing occurs in the inequality (3.1) and, in addition, (3.5) and (3.6) occur. Therefore

$$h_{ij}^r = 0, \qquad \forall r \in \overline{n+1,m},\ \forall i,j \in \overline{1,n}, \quad \text{with} \quad i \ne j \tag{3.11}$$

and

$$(a+1)h_{11}^r = (a+1)h_{22}^r = h_{33}^r = \cdots = h_{nn}^r, \forall r \in \overline{n+1,m}. \tag{3.12}$$

The relations (3.10), (3.11) and (3.12) imply the conclusion of the theorem. □

**Remark 3.2.**

i) Making $a$ to converge at $1$ in the previous inequality, we obtain **Chen's Inequality.** The conditions for which we have equality are obtained in [1] and [7].

ii) For $a = 0$ we obtain the well-known inequality

$$\tau \leq \frac{n(n-1)}{2}(\|H\|^2 + c).$$

The equality is attained at the point $x \in M$ if and only if $x$ is a totally umbilical point.

iii) Making $a$ converge at $-1$ in the previous inequality, we obtain

$$\delta_M^{-1} \leq \frac{(n^2 - n + 2)c}{2} + \frac{n^2 \|H\|^2}{2}.$$

The equality is attained at the point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in $T_x M$ and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^{\perp} M$ in which the Weingarten operators take the following form

$$A_r = \begin{pmatrix} h_{11}^r & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

with $h_{11}^r = h_{22}^r, \forall\, r \in \overline{n+1, m}$.

**Corollary 3.3.** *If the Riemannian manifold $(M, g)$, of dimension $n \geq 3$, admits a minimal isometric immersion into a real space form $\widetilde{M}(c)$, then*

$$\tau - \frac{(n-2)(n+1)c}{2} \leq k \leq -\tau + \frac{(n^2 - n + 2)c}{2}.$$

**Corollary 3.4.** *If the Riemannian manifold $(M, g)$, of dimension $n \geq 3$, admits a minimal isometric immersion into a Euclidean space, then*

$$\tau \leq k \leq -\tau.$$

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