MONOTONICITY OF RATIOS INVOLVING INCOMPLETE GAMMA FUNCTIONS WITH ACTUARIAL APPLICATIONS

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Abstract: Ratios involving incomplete gamma functions and their monotonicity properties play important roles in financial risk analysis. We derive desired monotonicity properties either using Pinelis' Calculus Rules or applying probabilistic techniques. As a consequence, we obtain several inequalities involving conditional expectations that have been of interest in actuarial science.

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Contents

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1. Introduction

The gamma function $\Gamma(u) = \int_0^\infty x^{u-1}e^{-x}dx$ and its numerous variations (e.g., upper and lower incomplete, regularized, inverted, etc., gamma functions) have played major roles in research and applications. The ratio of two gamma functions has also been a prominent research topic for a long time. For a collection of results, references, notes, and insightful comments in the area, we refer to [10].

When working on insurance related problems (see Section 3) we discovered that solutions of these problems hinge on monotonicity properties of the functions

$$\mathcal{R}_c(u,v) = \frac{\Gamma\big(u+c,v\big)}{\Gamma\big(u,v\big)} \quad \text{and} \quad \mathcal{Q}_c(u,v) = \frac{\mathcal{R}_c(u,v)}{u},$$

where $c > 0$ is a constant and $\Gamma(u,v) = \int_v^\infty x^{u-1}e^{-x}dx$ is the upper incomplete gamma function. Note that when $v = 0$, then the functions $\mathcal{R}_c(u,v)$ and $\mathcal{Q}_c(u,v)$ reduce, respectively, to the ratios $\Gamma(u+c)/\Gamma(u)$ and $\Gamma(u+c)/\Gamma(u+1)$. We refer to [9] and [10] for monotonicity properties, inequalities, and references concerning the latter two ratios and their variations. Monotonicity results and inequalities for upper and lower incomplete gamma functions have been studied in [9]; see also the references therein.

Note that the monotonicity of $\mathcal{R}_c(u,v)$ and $\mathcal{Q}_c(u,v)$ with respect to v follows immediately from Pinelis' Calculus Rules, which have been reported in a series of papers in the *Journal of Inequalities in Pure and Applied Mathematics* during the period 2001–2007. Indeed, both the numerator and the denominator of the ratio $\mathcal{R}_c(u,v)$ converge to 0 when $v \to \infty$, and the ratio $\Gamma'_v\big(u+c,v\big)/\Gamma'_v\big(u,v\big)$, which is equal to v^c, is increasing, where $\Gamma'_v(u,v)$ is the derivative of $\Gamma(u,v)$ with respect to v. Hence, according to Proposition 1.1 in [8], we have that

$$\mathcal{R}_c(u,v+\epsilon) > \mathcal{R}_c(u,v) \quad \text{for every} \quad \epsilon > 0. \tag{1.1}$$

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The same argument applies to the function $v \mapsto \mathcal{Q}_c(u,v)$, and thus the same monotonicity property holds for this function as well.

Pinelis' Calculus Rules, however, do not seem to be easily applicable for deriving monotonicity properties of the functions $u \mapsto \mathcal{R}_c(u,v)$ and $u \mapsto \mathcal{Q}_c(u,v)$. Therefore, in the current paper we use 'probabilistic' arguments to arrive at the desired results. The arguments are based on so-called weighted distributions, which are of interest on their own. We also present a description of insurance related problems that have led us to the research in the present paper.

vol. 9, iss. 3, art. 61, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Monotonicity of $u \mapsto \mathcal{R}_c(u, v)$ and $u \mapsto \mathcal{Q}_c(u, v)$

The following general bound has been proved in [5] (see also [3] for uses in insurance)

$$\mathbf{E}[\alpha(X)\beta(X)] \geq \mathbf{E}[\alpha(X)]\mathbf{E}[\beta(X)] \tag{2.1}$$

for non-decreasing functions $\alpha(x)$ and $\beta(x)$. We shall see in the proof below that the bound (2.1) is helpful in the context of the present paper.

Proposition 2.1. *For any positive c, u and v we have that*

$$\mathcal{R}_c(u + \epsilon, v) > \mathcal{R}_c(u, v) \quad \textit{for every} \quad \epsilon > 0. \tag{2.2}$$

Proof. Statement (2.2) means that the function $\rho(u) = \mathcal{R}_c(u, v)$ is increasing. To verify the monotonicity property, we check that $\rho'(u) > 0$, which is equivalent to the inequality

$$\int_v^\infty \log(x) x^c q(x) dx \int_v^\infty q(x) dx > \int_v^\infty \log(x) q(x) dx \int_v^\infty x^c q(x) dx \tag{2.3}$$

with $q(x) = x^{u-1}e^{-x}$. (It is interesting to point out, as has been noted by a referee of this paper, that inequality (2.2) is equivalent to (2.3) with $\log(x)$ replaced by x^ϵ; the proof that follows is valid with this change as well.) Let X_q be a random variable whose density function is $x \mapsto q(x)/\int_v^\infty q(y)dy$ on the interval $[v, \infty)$. We rewrite bound (2.3) as

$$\mathbf{E}[\log(X_q) X_q^c] > \mathbf{E}[\log(X_q)]\mathbf{E}[X_q^c]. \tag{2.4}$$

With the functions $\alpha(x) = \log(x)$ and $\beta(x) = x^c$, we have from (2.1) that bound (2.4) holds with '$\geq$' instead of '$>$', which is a weaker result than desired. Therefore,

Ratios of Gammas
Edward Furman and Ričardas Zitikis
vol. 9, iss. 3, art. 61, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

we next show that the equality $\mathbf{E}[\log(X_q)X_q^c] = \mathbf{E}[\log(X_q)]\mathbf{E}[X_q^c]$ is impossible. To this end we proceed with the equation (due to W. Hoeffding; see [5])

$$\mathbf{E}[\log(X_q)X_q^c] - \mathbf{E}[\log(X_q)]\mathbf{E}[X_q^c] \\ = \int\int \Big(\mathbf{P}\big[\log(X_q) \le x, X_q^c \le y\big] - \mathbf{P}\big[\log(X_q) \le x\big]\mathbf{P}\big[X_q^c \le y\big]\Big)dxdy.$$

We have that $\mathbf{P}[\log(X_q) \le x, X_q^c \le y] \ge \mathbf{P}[\log(X_q) \le x]\mathbf{P}[X_q^c \le y]$, which is the so-called 'positive dependence' between the random variables $\log(X_q)$ and X_q^c: when one of them increases, the other one also increases. Hence, in order to have the equality $\mathbf{E}[\log(X_q)X_q^c] = \mathbf{E}[\log(X_q)]\mathbf{E}[X_q^c]$, we need to have $\mathbf{P}[\log(X_q) \le x, X_q^c \le y] = \mathbf{P}[\log(X_q) \le x]\mathbf{P}[X_q^c \le y]$ for all x and y. But this means independence of $\log(X_q)$ and X_q^c, which is possible only if X_q is a constant almost surely. The latter, however, is impossible since, by construction, the random variable X_q has a density. This completes the proof of Proposition 2.1. □

Proposition 2.2. *When $c \le 1$, for any positive u and v we have that*

$$\mathcal{Q}_c(u+\epsilon, v) < \mathcal{Q}_c(u, v) \quad \textit{for every} \quad \epsilon > 0. \tag{2.5}$$

Proof. Since $\Gamma(u,v)u = \Gamma(u+1,v) - v^u e^{-v}$, we have that

$$\mathcal{Q}_c(u,v) = \frac{\Gamma(u+c,v)}{\Gamma(u+1,v) - v^u e^{-v}} = \frac{1}{a(u) - (v^{-c}e^{-v})/b(u)},$$

where

$$a(u) = \frac{\Gamma(u+1,v)}{\Gamma(u+c,v)} \quad \text{and} \quad b(u) = \frac{\Gamma(u+c,v)}{v^{u+c}}.$$

Note that $a(u) = \mathcal{R}_{1-c}(u+c,v)$, which is constant if $c = 1$ and an increasing function of u if $c < 1$ by Proposition 2.1. Note also the equality $b(u) = \int_1^\infty x^u e^{-vx}dx$.

journal of inequalities in pure and applied mathematics
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The latter integral is increasing with respect to u. Hence, $u \mapsto \mathcal{Q}_c(u, v)$ is a decreasing function. This finishes the proof of Proposition 2.2. $\square$

It is natural to ask whether the condition $c \leq 1$ in Proposition 2.2 is necessary. Computer aided graphics indicate that when $c > 1$, the function $u \mapsto \mathcal{Q}_c(u, v)$ is initially decreasing and then increasing either concavely or convexly, depending on the magnitude of $c > 1$. The problem of finding the minimum point of the function $u \mapsto \mathcal{Q}_c(u, v)$ and deriving its monotonicity patterns for $c > 1$ are interesting problems, whose resolutions would aid in risk measurement and management. Indeed, values $c > 1$ do show up when considering tail moments of higher orders than those considered in the next section: the applications we consider there require $c = 1$ only.

journal of inequalities in pure and applied mathematics
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3. Applications

Assume that an insurance portfolio consists of K risks, which are non-negative random variables $X_1, \dots, X_K$. Let the random variables be independent but possibly not identically distributed. In fact, assume that each X_k has the gamma distribution $Ga(\gamma_k, \alpha)$ with parameters $\gamma_k > 0$ and $\alpha > 0$, that is,

$$F_{X_k}(t) = 1 - \frac{\Gamma(\gamma_k, \alpha t)}{\Gamma(\gamma_k)}, \tag{3.1}$$

where $\Gamma(\gamma_k) = \Gamma(\gamma_k, 0)$ is the complete gamma function. We note in passing that the gamma distribution is natural and thus frequently utilized in actuarial science. Indeed, many total insurance claim distributions have roughly the same shape as the gamma distribution: they are non-negatively supported, unimodal, and skewed to the right. For applications of the gamma distribution, we refer, e.g., to [2] and [4], as well as to the references therein.

Consider the situation when an insurer is concerned with the overall portfolio risk

$$S = \sum_{j=1}^{K} X_j$$

that exceeds a certain threshold. Such situations arise when dealing with policies involving deductibles and reinsurance contracts. That is, given a pre-specified threshold t, we are concerned with those risks for which $S > t$ holds. We are then interested in the total risk and also in the average contribution of each risk X_k, or the unions of several X_k's, to the total risk of the portfolio. Mathematically, these problems can be formulated as the conditional expectations $\mathbf{E}[S|\, S > t]$ and $\mathbf{E}[X_k|\, S > t]$, or the sum of $\mathbf{E}[X_k|\, S > t]$ over all $k \in \Delta$ for some $\Delta \subseteq \{1, \dots, K\}$. In particular, we are interested in comparing the expectations $\mathbf{E}[S|\, S > t]$ and

vol. 9, iss. 3, art. 61, 2008

journal of inequalities in pure and applied mathematics
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$\mathbf{E}[S_\Delta |\, S_\Delta > t]$, and also $\mathbf{E}[S_\Delta |\, S > t]$ and $\mathbf{E}[S_\Delta |\, S_\Delta > t]$, where

$$S_\Delta = \sum_{j \in \Delta} X_j.$$

A motivation for such comparisons arises when testing theoretical properties of risk capital allocation procedures. For related discussions, we refer to [1].

The following proposition, which generalizes Proposition 1 in [2] to arbitrary random variables, is particularly useful in quantifying the above noted conditional expectations. The presented proof of the proposition below is also much simpler than that in [2].

Proposition 3.1. *Let $\xi_1, \dots, \xi_K$ be independent (but not necessarily identically distributed) non-negative random variables with positive and finite means. Then, for every $1 \le k \le K$,*

$$\mathbf{E}\left[\xi_k \Big|\, \sum_{j=1}^{K} \xi_j > t\right] = \mathbf{E}[\xi_k] \frac{1 - F_{\sum_{j \neq k}^{K} \xi_j + \xi_k^*}(t)}{1 - F_{\sum_{j=1}^{K} \xi_j}(t)}, \tag{3.2}$$

where $\xi_k^* \ge 0$ is an independent of $\xi_1, \dots, \xi_K$ random variable whose distribution function is

$$F_k^*(x) = \frac{\mathbf{E}[\xi_k \mathbf{1}\{\xi_k \le x\}]}{\mathbf{E}[\xi_k]}.$$

Proof. The equations

$$\mathbf{E}\left[\xi_k \Big|\, \sum_{j=1}^{K} \xi_j > t\right] = \frac{\mathbf{E}\left[\xi_k \mathbf{1}\left\{\sum_{j \neq k}^{K} \xi_j + \xi_k > t\right\}\right]}{\mathbf{E}\left[\mathbf{1}\left\{\sum_{j=1}^{K} \xi_j > t\right\}\right]} = \mathbf{E}[\xi_k] \frac{\mathbf{P}\left[\sum_{j \neq k}^{K} \xi_j + \xi_k^* > t\right]}{1 - F_{\sum_{j=1}^{K} \xi_j}(t)}$$

prove the proposition. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

A notable property of the gamma distribution is that of 'closure under convolutions', meaning that the distribution of the sum $\sum_{k\in\Delta} X_k$ has again a gamma distribution, which is $Ga\left(\sum_{j\in\Delta}\gamma_j,\alpha\right)$. Another useful property is the 'closure under the size-biased transform', which we explain next.

To start with, note that the distribution $F_k^*(x)$ of X_k^* is a special case of the more general weighted distribution (see [7] and [11], as well as the references therein)

$$F_w^*(x) = \frac{\mathbf{E}[w(X_k)\mathbf{1}\{X_k \le x\}]}{\mathbf{E}[w(X_k)]},$$

where $w(x)$ is a non-negative function such that the expectation $\mathbf{E}[w(X_k)]$ is positive and finite. When $w(x) = x^c$ for a constant $c > 0$, the distribution F_w^* is called 'size-biased'. We check (see [6]) that in this case the distribution F_w^* is $Ga(\gamma_k + c, \alpha)$, provided of course that F_{X_k} is $Ga(\gamma_k, \alpha)$, as assumed in (3.1). In particular, when $c = 1$, then $X_k^* \backsim Ga(\gamma_k + 1, \alpha)$ and so, in turn, $\sum_{j\neq k}^K X_j + X_k^* \sim Ga\left(\sum_{j=1}^K \gamma_j + 1, \alpha\right)$. Combining these notes with equations (3.1) and (3.2), and also utilizing the fact that $\mathbf{E}[X_k] = \gamma_k/\alpha$, we have that

$$\mathbf{E}[X_k|\, S > t] = \frac{\gamma_k}{\alpha\sum_{j=1}^K \gamma_j} \frac{\Gamma\left(\sum_{j=1}^K \gamma_j + 1, \alpha t\right)}{\Gamma\left(\sum_{j=1}^K \gamma_j, \alpha t\right)} = \frac{\gamma_k}{\alpha\sum_{j=1}^K \gamma_j}\mathcal{R}_1\left(\sum_{j=1}^K \gamma_j, \alpha t\right), \tag{3.3}$$

where $\mathcal{R}_1$ is $\mathcal{R}_c$ with $c = 1$. Hence,

$$\mathbf{E}[S|\, S > t] = \frac{1}{\alpha}\mathcal{R}_1\left(\sum_{j=1}^K \gamma_j, \alpha t\right). \tag{3.4}$$

vol. 9, iss. 3, art. 61, 2008

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Likewise, we derive the equation

$$\mathbf{E}[S_\Delta |\, S_\Delta > t] = \frac{1}{\alpha}\mathcal{R}_1\left(\sum_{j\in\Delta}\gamma_j, \alpha t\right). \tag{3.5}$$

To compare the right-hand sides of equations (3.4) and (3.5), we apply Proposition 2.1 and arrive at the following corollary.

Corollary 3.2. *We have that*

$$\mathbf{E}[S |\, S > t] \geq \mathbf{E}[S_\Delta |\, S_\Delta > t] \tag{3.6}$$

with the strong inequality '$>$' holding if $\sum_{j\in\complement\Delta}\gamma_j > 0$, *where* $\complement\Delta$ *is the complement of* Δ *in* $\{1,\dots,K\}$.

Inequality (3.6) is intuitive from the actuarial point of view since it implies that more risks mean higher expected losses.

It is also important to compare the expectations $\mathbf{E}[S_\Delta |\, S > t]$ and $\mathbf{E}[S_\Delta |\, S_\Delta > t]$. Loosely speaking, the former expectation refers to the risk contribution of the risk-set Δ to the total risk when the risk-set Δ is a part of a portfolio. The expectation $\mathbf{E}[S_\Delta |\, S_\Delta > t]$ refers to the risk contribution when the risk-set Δ is a stand-alone risk. To derive an expression for $\mathbf{E}[S_\Delta |\, S > t]$, we use equation (3.3) and obtain

$$\begin{aligned} \mathbf{E}[S_\Delta |\, S > t] &= \frac{\sum_{j\in\Delta}\gamma_j}{\alpha\sum_{j=1}^K\gamma_j}\mathcal{R}_1\left(\sum_{j=1}^K\gamma_j, \alpha t\right) \\ &= \frac{1}{\alpha}\left(\sum_{j\in\Delta}\gamma_j\right)\mathcal{Q}_1\left(\sum_{j=1}^K\gamma_j, \alpha t\right), \end{aligned} \tag{3.7}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

where $\mathcal{Q}_1$ is $\mathcal{Q}_c$ with $c = 1$. Next we rewrite equation (3.5) in terms of the function $\mathcal{Q}_1$ and have that

$$\mathbf{E}[S_\Delta |\, S_\Delta > t] = \frac{1}{\alpha}\left(\sum_{j\in\Delta}\gamma_j\right)\, \mathcal{Q}_1\left(\sum_{j\in\Delta}\gamma_j, \alpha t\right). \tag{3.8}$$

Using Proposition 2.2, we compare the right-hand sides of equations (3.7) and (3.8), and obtain the following corollary.

Corollary 3.3. *We have*

$$\mathbf{E}[S_\Delta |\, S > t] \leq \mathbf{E}[S_\Delta |\, S_\Delta > t] \tag{3.9}$$

with the strong inequality '$<$' holding if $\sum_{j\in \complement\Delta}\gamma_j > 0$.

Inequality (3.9) means that risks, or their unions, are more 'dangerous' when they stand alone than when being a part of a portfolio.

journal of inequalities in pure and applied mathematics
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