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HEISENBERG UNCERTAINTY PRINCIPLES FOR SOME q^2-ANALOGUE FOURIER TRANSFORMS

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ABSTRACT. The aim of this paper is to state q-analogues of the Heisenberg uncertainty principles for some q^2-analogue Fourier transforms introduced and studied in [7, 8].

Key words and phrases: Heisenberg inequality, q-Fourier transforms.

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1. INTRODUCTION

One of the most famous uncertainty principles is the so-called Heisenberg uncertainty principle. With the use of an inequality involving a function and its Fourier transform, it states that in classical Fourier analysis it is impossible to find a function f that is arbitrarily well localized together with its Fourier transform $\widehat{f}$.

In this paper, we will prove that similar to the classical theory, a non-zero function and its q^2-analogue Fourier transform (see [7, 8]) cannot both be sharply localized. For this purpose we will prove a q-analogue of the Heisenberg uncertainly principle. This paper is organized as follows: in Section 2, some notations, results and definitions from the theory of the q^2-analogue Fourier transform are presented. All of these results can be found in [7] and [8]. In Section 3, q-analogues of the Heisenberg uncertainly principle are stated.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we will follow the notations of [7, 8]. We fix $q \in]0,1[$ such that $\frac{Log(1-q)}{Log(q)} \in 2\mathbb{Z}$. For the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions, refer to the book by G. Gasper and M. Rahman [3].

Define

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}.$$

We also denote

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \tag{2.1}$$

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and

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \tag{2.2}$$

The q^2-analogue differential operator (see [8]) is

$$\partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}. \tag{2.3}$$

We remark that if f is differentiable at z, then $\lim_{q\to 1} \partial_q(f)(z) = f'(z)$.

∂_q is closely related to the classical q-derivative operators studied in [3, 5].

The q-trigonometric functions q-cosine and q-sine are defined by (see [7, 8]):

$$\cos(x;q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!} \tag{2.4}$$

and

$$\sin(x;q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}. \tag{2.5}$$

These functions induce a ∂_q-adapted q^2-analogue exponential function by

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2). \tag{2.6}$$

$e(z;q^2)$ is absolutely convergent for all z in the plane since both of its component functions are absolutely convergent. $\lim_{q\to 1^-} e(z;q^2) = e^z$ (exponential function) pointwise and uniformly on compacta.

The q-Jackson integrals are defined by (see [4])

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} \{f(q^n) + f(-q^n)\} q^n \tag{2.7}$$

and

$$\int_{0}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n), \tag{2.8}$$

provided that the sums converge absolutely. Using these q-integrals, we define for $p > 0$,

$$L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}, \tag{2.9}$$

$$L_q^p(\mathbb{R}_{q,+}) = \left\{ f : \left(\int_{0}^{\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\} \tag{2.10}$$

and

$$L_q^{\infty}(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)| < \infty \right\}. \tag{2.11}$$

The following result can be verified by direct computation.

Lemma 2.1. *If $\int_{-\infty}^{\infty} f(t)d_qt$ exists, then*

(1) *for all integers n, $\int_{-\infty}^{\infty} f(q^n t)d_qt = q^{-n}\int_{-\infty}^{\infty} f(t)d_qt$;*
(2) *f odd implies that $\int_{-\infty}^{\infty} f(t)d_qt = 0$;*
(3) *f even implies that $\int_{-\infty}^{\infty} f(t)d_qt = 2\int_0^{\infty} f(t)d_qt$.*

The following lemma lists some useful computational properties of ∂_q, and reflects the sensitivity of this operator to the parity of its argument. The proof is straightforward.

Lemma 2.2.

(1) *If f is odd $\partial_q f(z) = \frac{f(z)-f(qz)}{(1-q)z}$ and if f is even $\partial_q f(z) = \frac{f(q^{-1}z)-f(z)}{(1-q)z}$.*
(2) *We have $\partial_q \sin(x;q^2) = \cos(x;q^2)$, $\partial_q \cos(x;q^2) = -\sin(x;q^2)$ and $\partial_q e(x;q^2) = e(x;q^2)$.*
(3) *If f and g are both odd, then*
$$\partial_q(fg)(z) = q^{-1}(\partial_q f)\left(\frac{z}{q}\right)g(z) + q^{-1}f\left(\frac{z}{q}\right)(\partial_q g)\left(\frac{z}{q}\right).$$
(4) *If f is odd and g is even, then*
$$\partial_q(fg)(z) = (\partial_q f)(z)\,g(z) + qf(qz)\,(\partial_q g)(qz).$$
(5) *If f and g are both even, then*
$$\partial_q(fg)(z) = (\partial_q f)(z)g\left(\frac{z}{q}\right) + f(z)\,(\partial_q g)(z).$$

The following simple result, giving a q-analogue of the integration by parts theorem, can be verified by direct calculation.

Lemma 2.3. *If $\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx$ exists, then*

$$\int_{-\infty}^{\infty}(\partial_q f)(x)g(x)d_qx = -\int_{-\infty}^{\infty} f(x)(\partial_q g)(x)d_qx. \tag{2.12}$$

With the use of the q-Gamma function

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

R.L. Rubin defined in [8] the q^2-analogue Fourier transform as

$$\widehat{f}(x;q^2) = K\int_{-\infty}^{\infty} f(t)e(-itx;q^2)d_qt, \tag{2.13}$$

where $K = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}$.

We define the q^2-analogue Fourier-cosine and Fourier-sine transform as (see [2] and [6])

$$\mathcal{F}_q(f)(x) = 2K\int_0^{\infty} f(t)\cos(xt;q^2)d_qt \tag{2.14}$$

and

$${}_q\mathcal{F}(f)(x) = 2K\int_0^{\infty} f(t)\sin(xt;q^2)d_qt. \tag{2.15}$$

Observe that if f is even then $\widehat{f}(\cdot;q^2) = \mathcal{F}_q$ and if f is odd then $\widehat{f}(\cdot;q^2) =_q \mathcal{F}$.

It was shown in [8] that we have the following theorem.

Theorem 2.4.

(1) *If* $f(u),\ uf(u) \in L^1_q(\mathbb{R}_q)$, *then* $\partial_q\left(\widehat{f}\right)(x;q^2) = (-iuf(u))\widehat{\ }(x;q^2)$.

(2) *If* $f,\ \partial_q f \in L^1_q(\mathbb{R}_q)$, *then* $(\partial_q f)\widehat{\ }(x;q^2) = ix\widehat{f}\,(x;q^2)$

(3) *For* $f \in L^2_q(\mathbb{R}_q)$, $\|\widehat{f}\,(.;q^2)\|_{2,q} = \|f\|_{2,q}$.

3. q-ANALOGUE OF THE HEISENBERG UNCERTAINLY PRINCIPLE

For a function f defined on $\mathbb{R}_q$, we denote by f_0 and f_e its odd and even parts respectively. Let us begin with the following theorem.

Theorem 3.1. *If* f, xf *and* $x\widehat{f}(x;q^2)$ *are in* $L^2_q(\mathbb{R}_q)$, *then*

$$\|f\|^2_{2,q} \le \|x\widehat{f}(x;q^2)\|_{2,q}\left[q\left(1+q^{-\frac{3}{2}}\right)\|xf_o\|_{2,q} + \left(1+q^{\frac{3}{2}}\right)\|xf_e\|_{2,q}\right]. \tag{3.1}$$

Proof. Using the properties of the q^2-analogue differential operator ∂_q, the properties of the q-integrals, the Hölder inequality and Theorem 2.4, we can see that

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} x\partial_q(f\overline{f})(x)d_qx\right| &= \left|\int_{-\infty}^{\infty} x\left(q\overline{f}_0(x) + \overline{f}_e(q^{-1}x)\right)(\partial_q f)(x)d_qx\right.\\
&\qquad \left.+ \int_{-\infty}^{\infty} x\left(qf_0(qx) + f_e(x)\right)(\partial_q\overline{f})(x)d_qx\right|\\
&\le q\int_{-\infty}^{\infty}|xf_0(x)||\partial_q f(x)|d_qx + \int_{-\infty}^{\infty}|xf_e(q^{-1}x)||\partial_q f(x)|d_qx\\
&\qquad + \int_{-\infty}^{\infty}|xf_e(x)||\partial_q f(x)|d_qx + q\int_{-\infty}^{\infty}|xf_0(x)||\partial_q f(x)|d_qx\\
&\le \|\partial_q f\|_{2,q}\left[q\left(\int_{-\infty}^{\infty}|xf_o(x)|^2d_qx\right)^{\frac{1}{2}} + \left(\int_{-\infty}^{\infty}|xf_e(q^{-1}x)|^2d_qx\right)^{\frac{1}{2}}\right.\\
&\qquad \left.+\left(\int_{-\infty}^{\infty}|xf_e(x)|^2d_qx\right)^{\frac{1}{2}} + q\left(\int_{-\infty}^{\infty}|xf_o(qx)|^2d_qx\right)^{\frac{1}{2}}\right]\\
&= \|x\widehat{f}\|_{2,q}\left[q\left(1+q^{-\frac{3}{2}}\right)\|xf_o\|_{2,q} + \left(1+q^{\frac{3}{2}}\right)\|xf_e\|_{2,q}\right].
\end{aligned}$$

On the other hand, using the q-integration by parts theorem, we obtain

$$\int_{-\infty}^{\infty} x\partial_q(f\overline{f})(x)d_qx = -\int_{-\infty}^{\infty}|f(x)|^2d_qx = -\|f\|^2_{2,q},$$

which completes the proof. □

Corollary 3.2. *If* f, xf *and* $x\widehat{f}$ *are in* $L^2_q(\mathbb{R}_q)$, *then*

$$\|xf\|_{2,q}\|x\widehat{f}(x;q^2)\|_{2,q} \ge \frac{1}{q^{-\frac{1}{2}}+1+q+q^{\frac{3}{2}}}\|f\|^2_{2,q}. \tag{3.2}$$

Proof. The properties of the q-integral imply

$$\begin{aligned}
\|xf\|^2_{2,q} &= \int_{-\infty}^{\infty} x^2(f_o(x)+f_e(x))\left(\overline{f}_o(x)+\overline{f}_e(x)\right)d_qx\\
&= \int_{-\infty}^{\infty} x^2 f_o(x)\overline{f}_o(x)d_qx + \int_{-\infty}^{\infty} x^2 f_e(x)\overline{f}_e(x)d_qx\\
&= \|xf_o\|^2_{2,q} + \|xf_e\|^2_{2,q}.
\end{aligned}$$

So, $\|xf_o\|_{2,q} \leq \|xf\|_{2,q}$ and $\|xf_e\|_{2,q} \leq \|xf\|_{2,q}$.

These inequalities together with the previous theorem give the desired result. □

Corollary 3.3.

(1) *If f, xf and $x\mathcal{F}_q$ are in $L^2_q(\mathbb{R}_{q,+})$, then*

$$\left(\int_0^\infty x^2|f(x)|^2 d_qx\right)^{\frac{1}{2}} \left(\int_0^\infty x^2 \left|\mathcal{F}_q(x)\right|^2 d_qx\right)^{\frac{1}{2}} \geq \frac{1}{1+q^{\frac{3}{2}}} \int_0^\infty |f(x)|^2 d_qx. \tag{3.3}$$

(2) *If f, xf and $x\ {}_q\mathcal{F}$ are in $L^2_q(\mathbb{R}_{q,+})$, then*

$$\left(\int_0^\infty x^2|f(x)|^2 d_qx\right)^{\frac{1}{2}} \left(\int_0^\infty x^2 \left|\ {}_q\mathcal{F}(x)\right|^2 d_qx\right)^{\frac{1}{2}} \geq \frac{1}{q\left(1+q^{-\frac{3}{2}}\right)} \int_0^\infty |f(x)|^2 d_qx. \tag{3.4}$$

Proof. The proof is a simple application of the previous theorem on taking $g(x) = f(x)$ if x is positive and $g(x) = f(-x)$ (resp. $g(x) = -f(-x)$) if not in the first case (resp. second case). □

Remark 1. Corollary 3.2 gives a q-analogue of the Heisenberg uncertainty principle for the q^2-analogue Fourier transform $\widehat{f}(\cdot; q^2)$.

Remark 2. Corollary 3.3 gives a q-analogue of the Heisenberg uncertainty principles for the q^2-analogue Fourier-cosine and Fourier-sine transforms. These inequalities are slightly different from those given in [1]. This is due to the related q-analogue of special functions used.

Remark 3. Note that when q tends to 1, these inequalities tend at least formally to the corresponding classical ones.

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