ON NEW INEQUALITIES OF HADAMARD-TYPE FOR LIPSCHITZIAN MAPPINGS AND THEIR APPLICATIONS

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Abstract: In this paper, we study some new inequalities of Hadamard's Type for Lipschitzian mappings. some applications are also included.

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1. Introduction

Let $f : [a,b] \to \mathbb{R}$ ($a < b$) be a continuous function.

If f is convex on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities in (1.1) are known as the Hermite-Hadamard inequality [1].

For some recent results which generalize, improve, and extend this classic inequality, see references of [2] – [7]. In order to refine inequalities of (1.1), the author of this paper in [2] defined the following some notations, symbols and mappings. we list these notations and symbols by

$Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, $T_n = t_1 + t_2 + \cdots + t_n$; $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{1} = (1, 1, \ldots, 1)$, $\frac{\mathbf{1}}{\mathbf{n}} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ and $(1_i, 0) = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is ith component, $i = 1, 2, \ldots, n$) are special points in $\mathbb{R}^n$; $G = \left[0, \frac{1}{n}\right] \times \left[0, \frac{1}{n}\right] \times \cdots \times \left[0, \frac{1}{n}\right]$, $I = [0,1] \times [0,1] \times \cdots \times [0,1]$, $V = [a,b] \times [a,b] \times \cdots \times [a,b]$, $D = [a, x_1] \times [x_1, x_2] \times \cdots \times [x_{n-1}, b]$ ($x_i = a + \frac{(b-a)i}{n}$, $i = 0, 1, \ldots, n$; $x_0 = a$, $x_n = b$), $H = \{\mathbf{t} \in I | T_n \le 1\}$ and $L = \{\mathbf{t} \in I | T_n = 1\}$ are subsets in $\mathbb{R}^n$.

We list these mappings by

$$R_n : I \mapsto \mathbb{R}, \quad R_n(\mathbf{t}) \triangleq \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^n \left(t_i y_i + (1-t_i)\frac{x_{i-1}+x_i}{2}\right)\right) dY,$$

$$S_n : H \mapsto \mathbb{R}, \quad S_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n}\int_V f\left(\sum_{i=1}^n t_i y_i + (1-T_n)\frac{a+b}{2}\right) dY$$

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and

$$P_n : L \mapsto \mathbb{R}, \quad P_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n} \int_V f\left(\sum_{i=1}^{n} t_i y_i\right) dY.$$

We write P_{n+1} in the following equivalent form

$$P_{n+1} : H \mapsto \mathbb{R}, \quad P_{n+1}(\mathbf{t}) \triangleq \frac{1}{(b-a)^{n+1}} \int_V \left[\int_a^b f\left(\sum_{i=1}^{n} t_i y_i + (1-T_n)x\right) dx\right] dY.$$

Let $g : A \subseteq \mathbb{R}^n \to \mathbb{R}$. For all $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \dots, t_n^{(j)}\right) \in A (j = 1, 2)$ with $t_i^{(1)} \le t_i^{(2)}$ $(i = 1, 2, \dots, n)$, if $g(\mathbf{t}^{(1)}) \le g(\mathbf{t}^{(2)})$, then we call g increasing on A.

For these mappings and if f is convex on $[a, b]$, L.-C. Wang in [2] gave the following properties and inequalities:

P_n is convex on L; R_n and S_n are convex, increasing on I and G, respectively;

$$\text{(1.2)} \qquad f\left(\frac{a+b}{2}\right) = R_n(\mathbf{0}) \le R_n(\mathbf{t}) \le R_n(\mathbf{1}) = \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY \le \frac{1}{b-a}\int_a^b f(x)dx$$

for any $\mathbf{t} \in I$,

$$\text{(1.3)} \quad f\left(\frac{a+b}{2}\right) = S_n(\mathbf{0}) \le S_n(\mathbf{t}) \le S_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) = \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY$$

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for all $\mathbf{t} \in G$,

$$S_n(\mathbf{t}) \le P_{n+1}(\mathbf{t}) \tag{1.4}$$

for all $\mathbf{t} \in H$, and

$$S_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) = P_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \le P_n(\mathbf{t}) \le P_n(1_i, 0) = \frac{1}{b-a}\int_a^b f(x)dx \tag{1.5}$$

for all $\mathbf{t} \in L$.

(1.2) – (1.5) are refinements of (1.1).

Recently, Dragomir *et al.* [3], Yang and Tseng [5], Matic and Pečarić [6] and L.-C. Wang [7] proved some results for Lipschitzian mappings related to (1.1). In this paper, we will prove some new inequalities for Lipschitzian mappings related to the mappings R_n (or (1.2)), S_n (or (1.3)) and P_n (or (1.5) and (1.4)). Finally, some applications are given.

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2. Main Results

A function $f : [a,b] \to \mathbb{R}$ is called an $M-$Lipschitzian mapping, if for every two elements $x, y \in [a,b]$ and $M > 0$ we have

$$|f(x) - f(y)| \leq M|x - y|.$$

For the mapping $R_n(\mathbf{t})$, we have the following theorem:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be an $M-$Lipschitzian mapping, then we have*

$$\left|R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)})\right| \leq \frac{M}{4n^2}(b-a)\sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.1}$$

for any $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \dots, t_n^{(j)}\right) \in I (j = 1, 2)$,

$$\left|f\left(\frac{a+b}{2}\right) - R_n(\mathbf{t})\right| \leq \frac{M}{4n^2}(b-a)T_n \tag{2.2}$$

and

$$\left|R_n(\mathbf{t}) - \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY\right| \leq \frac{M}{4n^2}(b-a)\left(n - T_n\right) \tag{2.3}$$

for all $t \in I$, and

$$\left|\left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M(n^2-1)}{3n^2}(b-a). \tag{2.4}$$

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Proof. (1) For $x_i = \frac{(b-a)i}{n}$ $(i = 0, 1, \ldots, n;\, x_0 = a, x_n = b)$, from integral properties, we have

$$\begin{aligned}
&\left|R_n\left(\mathbf{t}^{(2)}\right) - R_n\left(\mathbf{t}^{(1)}\right)\right| \\
&\le \left(\frac{n}{b-a}\right)^n \int_D \left| f\left(\frac{1}{n}\sum_{i=1}^{n}\left(t_i^{(2)} y_i + (1-t_i^{(2)})\frac{x_{i-1}+x_i}{2}\right)\right)\right. \\
&\qquad\qquad \left. - f\left(\frac{1}{n}\sum_{i=1}^{n}\left(t_i^{(1)} y_i + (1-t_i^{(1)})\frac{x_{i-1}+x_i}{2}\right)\right)\right| dY \\
&\le \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \int_D \left|\sum_{i=1}^{n}\left(t_i^{(2)} - t_i^{(1)}\right)\left(y_i - \frac{x_{i-1}+x_i}{2}\right)\right| dY \\
&\le \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \int_D \left|y_i - \frac{x_{i-1}+x_i}{2}\right| dY \\
&= \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \left(\frac{b-a}{n}\right)^{n-1} \int_{x_{i-1}}^{x_i} \left|y_i - \frac{x_{i-1}+x_i}{2}\right| dy_i \\
&= \frac{M}{b-a} \sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right| \left[\int_{x_{i-1}}^{\frac{x_{i-1}+x_i}{2}} \left(\frac{x_{i-1}+x_i}{2} - y_i\right) dy_i \right. \\
&\qquad\qquad \left. + \int_{\frac{x_{i-1}+x_i}{2}}^{x_i} \left(y_i - \frac{x_{i-1}+x_i}{2}\right) dy_i \right] \\
&= \frac{M}{4n^2}(b-a) \sum_{i=1}^{n}\left|t_i^{(2)} - t_i^{(1)}\right|.
\end{aligned}$$

This completes the proof of (2.1).

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(2) The inequalities (2.2) and (2.3) follow from (2.1) by choosing $\mathbf{t}^{(1)} = \mathbf{0}$, $\mathbf{t}^{(2)} = \mathbf{t}$ and $\mathbf{t}^{(1)} = \mathbf{1}$, $\mathbf{t}^{(2)} = \mathbf{t}$, respectively. This completes the proof of (2.2) and (2.3).

(3) From integral properties, we have

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(y_i)dy_i = \left(\frac{n}{b-a}\right)^{n-1} \sum_{i=1}^{n} \int_D f(y_i)dY. \tag{2.5}$$

Using (2.5) and integral properties, we obtain

$$\begin{aligned}
&\left| \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq \left(\frac{n}{b-a}\right)^n \int_D \left| \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=1}^{n} y_j\right) - \frac{1}{n}\sum_{i=1}^{n} f(y_i) \right| dY \\
&\leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \int_D \sum_{i=1}^{n} \left| \frac{1}{n}\sum_{j=1}^{n} y_j - y_i \right| dY \\
&\leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n^2} \sum_{i=1}^{n} \int_D \sum_{j=1}^{n} |y_j - y_i| \, dY \\
&= \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n^2} \sum_{i=1}^{n} \int_D \left[\sum_{j=1}^{i-1} (y_i - y_j) + \sum_{j=i+1}^{n} (y_j - y_i) \right] dY \\
&= \frac{n}{b-a} \cdot \frac{M}{n^2} \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} \left(\int_{x_{i-1}}^{x_i} y_i dy_i - \int_{x_{j-1}}^{x_j} y_j dy_j \right) \right.
\end{aligned}$$

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$$+\sum_{j=i+1}^{n}\left(\int_{x_{j-1}}^{x_j} y_j dy_j - \int_{x_{i-1}}^{x_i} y_i dy_i\right)\Bigg]$$

$$=\frac{n}{b-a}\cdot\frac{M}{n^2}\sum_{i=1}^{n}\left[\left(\frac{b-a}{n}\right)^2\left(\sum_{j=1}^{i-1}(i-j)+\sum_{j=i+1}^{n}(j-i)\right)\right]$$

$$=\frac{M(n^2-1)}{3n^2}(b-a).$$

This completes the proof of (2.4).

This completes the proof of Theorem 2.1. ∎

Corollary 2.2. *Let f be convex on $[a,b]$, with $f'_+(a)$ and $f'_-(b)$ existing. Then we obtain*

$$0 \le R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)}) \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)\sum_{i=1}^{n}\left(t_i^{(2)}-t_i^{(1)}\right) \tag{2.6}$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)},\dots,t_n^{(j)}) \in I\,(j=1,2)$ *with* $t_i^{(2)} \ge t_i^{(1)}\,(i=1,2,\dots,n)$,

$$0 \le R_n(\mathbf{t}) - f\left(\frac{a+b}{2}\right) \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)T_n \tag{2.7}$$

and

$$0 \le \left(\frac{n}{b-a}\right)^n\int_D f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)dY - R_n(\mathbf{t}) \le \frac{\max\{|f'_+(a)|,|f'_-(b)|\}}{4n^2}(b-a)(n-T_n) \tag{2.8}$$

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for all $\mathbf{t} \in I$, *and*

$$(2.9) \qquad 0 \le \frac{1}{b-a}\int_a^b f(x)dx - \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n}\sum_{i=1}^n y_i\right) dY \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}(n^2-1)}{3n^2}(b-a).$$

Proof. For any $x, y \in [a, b]$, from properties of convex functions, we have the following $\max\{|f'_+(a)|, |f'_-(b)|\}-$Lipschitzian inequality (see [8]):

$$(2.10) \qquad |f(x) - f(y)| \le \max\{|f'_+(a)|, |f'_-(b)|\}|x-y|.$$

Since R_n is increasing on I, using (1.2), (2.10) and Theorem 2.1, we obtain (2.6)-(2.9).

This completes the proof of Corollary (2.2). ∎

For the mapping $S_n(\mathbf{t})$, we have the following theorem:

Theorem 2.3. *Let* f *be defined as in Theorem 2.1, then we obtain*

$$(2.11) \qquad \left|S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)})\right| \le \frac{M}{4}(b-a)\sum_{i=1}^n \left|t_i^{(2)} - t_i^{(1)}\right|$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in H (j = 1, 2)$,

$$(2.12) \qquad \left|f\left(\frac{a+b}{2}\right) - S_n(\mathbf{t})\right| \le \frac{M}{4}(b-a)T_n$$

and

$$(2.13) \qquad \left|S_n(\mathbf{t}) - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right) dY\right| \le \frac{M}{4n}(b-a)\sum_{i=1}^n |nt_i - 1|$$

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for all $\mathbf{t} \in H$, *and*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^n} \int_V f\left(\frac{1}{n}\sum_{i=1}^n y_i\right) dY \right| \le \frac{M}{4}(b-a). \tag{2.14}$$

Proof. (1) From integral properties, we obtain

$$\begin{aligned}
&\left| S_n\left(\mathbf{t}^{(2)}\right) - S_n\left(\mathbf{t}^{(1)}\right) \right| \\
&\le \frac{1}{(b-a)^n} \int_V \left| f\left(\sum_{i=1}^n t_i^{(2)} y_i + \left(1 - \sum_{i=1}^n t_i^{(2)}\right) \frac{a+b}{2} \right) \right. \\
&\qquad \left. - f\left(\sum_{i=1}^n t_i^{(1)} y_i + \left(1 - \sum_{i=1}^n t_i^{(1)}\right) \frac{a+b}{2} \right) \right| dY \\
&\le \frac{M}{(b-a)^n} \int_V \left| \sum_{i=1}^n \left(t_i^{(2)} - t_i^{(1)}\right) \left(y_i - \frac{a+b}{2}\right) \right| dY \\
&\le \frac{M}{(b-a)^n} \sum_{i=1}^n \left| t_i^{(2)} - t_i^{(1)} \right| \int_V \left| y_i - \frac{a+b}{2} \right| dY \\
&= \frac{M}{b-a} \sum_{i=1}^n \left| t_i^{(2)} - t_i^{(1)} \right| \int_a^b \left| y_i - \frac{a+b}{2} \right| dy_i \\
&= \frac{M}{b-a} \sum_{i=1}^n \left| t_i^{(2)} - t_i^{(1)} \right| \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - y_i \right) dy_i + \int_{\frac{a+b}{2}}^b \left(y_i - \frac{a+b}{2} \right) dy_i \right] \\
&= \frac{M}{4}(b-a) \sum_{i=1}^n \left| t_i^{(2)} - t_i^{(1)} \right| .
\end{aligned}$$

This completes the proof of (2.11).

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(2) The inequalities (2.12) and (2.13) follow from (2.11) by choosing $\mathbf{t}^{(1)} = \mathbf{0}$, $\mathbf{t}^{(2)} = \mathbf{t}$ and $\mathbf{t}^{(1)} = \frac{\mathbf{1}}{\mathbf{n}}$, $\mathbf{t}^{(2)} = \mathbf{t}$, respectively. The inequalities (2.14) follow from (2.12) by choosing $\mathbf{t}^{(1)} = \frac{\mathbf{1}}{\mathbf{n}}$. This completes the proof of (2.12)-(2.14).

This completes the proof of Theorem 2.3. ∎

Corollary 2.4. *Let f be defined as in Corollary 2.2, then we have*

$$0 \le S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)}) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4}(b-a)\sum_{i=1}^{n}\left(t_i^{(2)} - t_i^{(1)}\right) \tag{2.15}$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in G(j = 1, 2)$ *with* $t_i^{(2)} \ge t_i^{(1)}$ $(i = 1, 2, \dots, n)$,

$$0 \le S_n(\mathbf{t}) - f\left(\frac{a+b}{2}\right) \le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4}(b-a)T_n \tag{2.16}$$

and

$$\begin{aligned} 0 &\le \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - S_n(\mathbf{t}) \\ &\le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4}(b-a)(1-T_n) \end{aligned} \tag{2.17}$$

for all $\mathbf{t} \in G$, *and*

$$\begin{aligned} 0 &\le \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY - f\left(\frac{a+b}{2}\right) \\ &\le \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4}(b-a). \end{aligned} \tag{2.18}$$

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Proof. Since S_n is increasing on G, using (1.3), (2.10) and Theorem 2.3, we obtain (2.15) – (2.18).

This completes the proof of Corollary (2.4). ∎

For the mapping $P_n(\mathbf{t})$, we have the following theorem:

Theorem 2.5. *Let f be defined as in Theorem 2.1. For $n \geq 2$, then we obtain*

$$\left|p_n(\mathbf{t}^{(2)}) - p_n(\mathbf{t}^{(1)})\right| \leq \frac{M}{3}(b-a)\sum_{i=1}^{n-1}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.19}$$

for any $\mathbf{t}^{(j)} = \left(t_1^{(j)}, \dots, t_n^{(j)}\right) \in L (j = 1, 2)$,

$$\left|\frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)dY - p_n(\mathbf{t})\right| \leq \frac{M}{3n}(b-a)\sum_{i=1}^{n-1}|nt_i - 1| \tag{2.20}$$

and

$$\left|p_n(\mathbf{t}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M}{3}(b-a)\sum_{i=1}^{n-1} t_i \tag{2.21}$$

for all $\mathbf{t} \in L$, *and*

$$\left|\frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)dY - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M(n-1)}{3n}(b-a). \tag{2.22}$$

For $n \geq 1$ and all $\mathbf{t} \in H$, then we have

$$|S_n(\mathbf{t}) - P_{n+1}(\mathbf{t})| \leq \frac{M}{4}(b-a)(1 - T_n). \tag{2.23}$$

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Proof. (1) Since $n \geq 2$ and $T_n = t_1 + \cdots + t_{n-1} + t_n = 1$, we can write $P_n(\mathbf{t})$ in the following equivalent form

$$P_n(\mathbf{t}) = \frac{1}{(b-a)^n} \int_V f\left(\sum_{i=1}^{n-1} t_i y_i + \left(1 - \sum_{i=1}^{n-1} t_i\right) y_n\right) dY. \tag{2.24}$$

Using (2.24) and integral properties, we obtain

$$\begin{aligned}
&\left|P_n\left(\mathbf{t}^{(2)}\right) - P_n\left(\mathbf{t}^{(1)}\right)\right| \\
&\leq \frac{1}{(b-a)^n} \int_V \left| f\left(\sum_{i=1}^{n-1} t_i^{(2)} y_i + \left(1 - \sum_{i=1}^{n-1} t_i^{(2)}\right) y_n\right) \right. \\
&\qquad\qquad \left. - f\left(\sum_{i=1}^{n-1} t_i^{(1)} y_i + \left(1 - \sum_{i=1}^{n-1} t_i^{(1)}\right) y_n\right) \right| dY \\
&\leq \frac{M}{(b-a)^n} \int_V \left| \sum_{i=1}^{n-1} \left(t_i^{(2)} - t_i^{(1)}\right)(y_i - y_n) \right| dY \\
&\leq \frac{M}{(b-a)^n} \sum_{i=1}^{n-1} \left|t_i^{(2)} - t_i^{(1)}\right| (b-a)^{n-2} \int_a^b \int_a^b |y_i - y_n|\, dy_i dy_n \\
&= \frac{M}{(b-a)^2} \sum_{i=1}^{n-1} \left|t_i^{(2)} - t_i^{(1)}\right| \int_a^b \left[\int_a^x (x-y)dy + \int_x^b (y-x)dy\right] dx \\
&= \frac{M}{3}(b-a) \sum_{i=1}^{n-1} \left|t_i^{(2)} - t_i^{(1)}\right|.
\end{aligned}$$

This completes the proof of (2.19).

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(2) The inequalities (2.20) and (2.21) follow from (2.19) by choosing $\mathbf{t}^{(1)} = \frac{\mathbf{1}}{\mathbf{n}}$, $\mathbf{t}^{(2)} = \mathbf{t}$ and $\mathbf{t}^{(1)} = (1_n, 0) = (0, \dots, 0, 1)$, $\mathbf{t}^{(2)} = \mathbf{t}$, respectively. The inequalities (2.22) follow from (2.21) by choosing $\mathbf{t} = \frac{\mathbf{1}}{\mathbf{n}}$. This completes the proof of (2.20) – (2.22).

(3) Using integral properties, we write $S_n(\mathbf{t})$ in the following equivalent form

$$S_n(\mathbf{t}) = \frac{1}{(b-a)^{n+1}} \int_V \left[\int_a^b f\left(\sum_{i=1}^n t_i y_i + (1-T_n)\frac{a+b}{2} \right) dx \right] dY. \tag{2.25}$$

Using (2.25) and integral properties, we obtain

$$\begin{aligned}
&|S_n(\mathbf{t}) - P_{n+1}(\mathbf{t})| \\
&\le \frac{1}{(b-a)^{n+1}} \int_V \left[\int_a^b \left| f\left(\sum_{i=1}^n t_i y_i + (1-T_n)\frac{a+b}{2} \right) \right. \right. \\
&\qquad\qquad \left. \left. - f\left(\sum_{i=1}^n t_i y_i + (1-T_n)x \right) \right| dx \right] dY \\
&\le \frac{M}{(b-a)^{n+1}}(1-T_n) \int_V \left[\int_a^b \left| \frac{a+b}{2} - x \right| dx \right] dY \\
&= \frac{M}{b-a}(1-T_n) \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right) dx + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2} \right) dx \right] \\
&= \frac{M}{4}(b-a)(1-T_n).
\end{aligned}$$

This completes the proof of (2.23).

This completes the proof of Theorem 2.5. ∎

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Corollary 2.6. *Let f be defined as in Corollary 2.2. For $n \geq 2$, then we have*

$$\left|P_n(\mathbf{t}^{(2)}) - P_n(\mathbf{t}^{(1)})\right| \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3}(b-a)\sum_{i=1}^{n-1}\left|t_i^{(2)} - t_i^{(1)}\right| \tag{2.26}$$

for any $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in L(j = 1, 2)$ *,*

$$\begin{aligned} 0 &\leq P_n(\mathbf{t}) - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY \\ &\leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3n}(b-a)\sum_{i=1}^{n-1}|nt_i - 1| \end{aligned} \tag{2.27}$$

and

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - P_n(\mathbf{t}) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3}(b-a)\sum_{i=1}^{n-1} t_i \tag{2.28}$$

for all $\mathbf{t} \in L$*, and*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^n}\int_V f\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) dY \\ &\leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}(n-1)}{3n}(b-a). \end{aligned} \tag{2.29}$$

For $n \geq 1$ and all $\mathbf{t} \in H$, we have

$$0 \leq P_{n+1}(\mathbf{t}) - S_n(\mathbf{t}) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4}(b-a)(1 - T_n). \tag{2.30}$$

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Proof. Using (1.5), (1.4), (2.10) and Theorem 2.5, we obtain (2.26) – (2.30).

This completes the proof of Corollary (2.6). ∎

Remark 1. The condition in Corollary 2.2 (or Corollary 2.4 or 2.6) is better than the condition in Corollary 2.2 (or Corollary 4.2 or Theorem 3.3) of [3]. This is due to the fact that f is a differentiable convex function on $[a, b]$ with $M = \sup_{t\in[a,b]} |f'(t)| < \infty$.

Remark 2. When $n = 1$, (2.1) and (2.11), (2.2) and (2.12), (2.3) and (2.13) and (2.23) reduce to (3.4), (3.2), (3.1) and (4.3) of [3], respectively. When $n = 2$, (2.19), (2.20), and (2.21) reduce to (4.6), (4.1) and (4.2) of [3], respectively.

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3. Applications

In this section, we agree that when $t_i = 0$,

$$\frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] = \frac{\ln b - \ln a}{n^2} \quad \text{and} \quad \frac{b^{t_i} - a^{t_i}}{t_i} = \ln b - \ln a.$$

For $b > a > 0$, $1 \geq t_i^{(2)} \geq t_i^{(1)} \geq 0$ and $1 \geq t_i \geq 0$ $(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned} 0 &\leq \prod_{i=1}^{n} \frac{1}{t_i^{(2)}}\left[\left(\frac{b}{a}\right)^{\frac{t_i^{(2)}}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i^{(2)}}{2n^2}}\right] - \prod_{i=1}^{n} \frac{1}{t_i^{(1)}}\left[\left(\frac{b}{a}\right)^{\frac{t_i^{(1)}}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i^{(1)}}{2n^2}}\right] \\ &\leq \frac{1}{4}\left(\frac{\ln b - \ln a}{n^2}\right)^{n+1}\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{n}\left(t_i^{(2)} - t_i^{(1)}\right), \end{aligned} \tag{3.1}$$

$$\begin{aligned} 0 &\leq \left(\frac{n^2}{\ln b - \ln a}\right)^{n} \prod_{i=1}^{n} \frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] - 1 \\ &\leq \frac{\ln b - \ln a}{4n^2}\left(\frac{b}{a}\right)^{\frac{1}{2}} T_n \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} 0 &\leq \prod_{i=1}^{n}\left[\left(\frac{b}{a}\right)^{\frac{1}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{1}{2n^2}}\right] - \prod_{i=1}^{n} \frac{1}{t_i}\left[\left(\frac{b}{a}\right)^{\frac{t_i}{2n^2}} - \left(\frac{a}{b}\right)^{\frac{t_i}{2n^2}}\right] \\ &\leq \frac{1}{4}\left(\frac{\ln b - \ln a}{n^2}\right)^{n+1}\left(\frac{b}{a}\right)^{\frac{1}{2}} (n - T_n). \end{aligned} \tag{3.3}$$

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For $b > a > 0$, $\frac{1}{n} \geq t_i^{(2)} \geq t_i^{(1)} \geq 0$ and $\frac{1}{n} \geq t_i \geq 0$ $(i = 1, 2, \dots, n)$, we have

$$(3.4)\qquad 0 \leq (ab)^{\frac{1-\sum_{i=1}^n t_i^{(2)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left(b^{t_i^{(2)}} - a^{t_i^{(2)}} \right) - (ab)^{\frac{1-\sum_{i=1}^n t_i^{(1)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left(b^{t_i^{(1)}} - a^{t_i^{(1)}} \right) \leq \frac{b\left(\ln b - \ln a\right)^{n+1}}{4} \sum_{i=1}^n \left(t_i^{(2)} - t_i^{(1)} \right),$$

$$(3.5)\qquad 0 \leq \frac{1}{\left(\ln b - \ln a\right)^n} (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i} \right) - (ab)^{\frac{1}{2}} \leq \frac{b\left(\ln b - \ln a\right)}{4} T_n$$

and

$$(3.6)\qquad 0 \leq \left(n b^{\frac{1}{n}} - n a^{\frac{1}{n}} \right)^n - (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left(b^{t_i} - a^{t_i} \right) \leq \frac{b\left(\ln b - \ln a\right)^{n+1}}{4} \left(1 - T_n\right).$$

For $b > a > 0$, $1 \geq t_i^{(j)} \geq 0$ $(i = 1, 2, \dots, n; n \geq 2)$ and $t_1^{(j)} + t_2^{(j)} + \cdots + t_n^{(j)} = 1$

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$(j=1,2)$, we have

$$\left|\prod_{i=1}^{n}\frac{1}{t_i^{(2)}}\left(b^{t_i^{(2)}}-a^{t_i^{(2)}}\right)-\prod_{i=1}^{n}\frac{1}{t_i^{(1)}}\left(b^{t_i^{(1)}}-a^{t_i^{(1)}}\right)\right| \le \frac{b\left(\ln b-\ln a\right)^{n+1}}{3}\sum_{i=1}^{n-1}\left|t_i^{(2)}-t_i^{(1)}\right|. \tag{3.7}$$

For $b>a>0$, $1\ge t_i\ge 0$ $(i=1,2,\ldots,n;\ n\ge 2)$ and $T_n=1$, we have

$$0\le\prod_{i=1}^{n}\frac{1}{t_i}\left(b^{t_i}-a^{t_i}\right)-\left(nb^{\frac{1}{n}}-na^{\frac{1}{n}}\right)^{n}\le\frac{b\left(\ln b-\ln a\right)^{n+1}}{3n}\sum_{i=1}^{n-1}|nt_i-1| \tag{3.8}$$

and

$$0\le\frac{b-a}{\ln b-\ln a}-\frac{1}{(\ln b-\ln a)^{n}}\prod_{i=1}^{n}\frac{1}{t_i}\left(b^{t_i}-a^{t_i}\right)\le\frac{b\left(\ln b-\ln a\right)}{3}\sum_{i=1}^{n-1}t_i. \tag{3.9}$$

For $b>a>0$, we have

$$\begin{aligned}0\le\frac{b-a}{\ln b-\ln a}&-\left(\frac{n^2}{\ln b-\ln a}\right)^{n}(ab)^{\frac12}\prod_{i=1}^{n}\left[\left(\frac{b}{a}\right)^{\frac{1}{2n^2}}-\left(\frac{a}{b}\right)^{\frac{1}{2n^2}}\right]\\&\le\frac{b(n^2-1)}{3n^2}\left(\ln b-\ln a\right),\end{aligned} \tag{3.10}$$

$$0\le\left(\frac{nb^{\frac1n}-na^{\frac1n}}{\ln b-\ln a}\right)^{n}-(ab)^{\frac12}\le\frac{b\left(\ln b-\ln a\right)}{4} \tag{3.11}$$

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and

$$0 \leq \frac{b-a}{\ln b - \ln a} - \left(\frac{n\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right)}{\ln b - \ln a} \right)^n \leq \frac{b(n-1)}{3n} (\ln b - \ln a) . \tag{3.12}$$

Indeed, (3.1) – (3.12) follow from (2.6) – (2.8), (2.15) – (2.17), (2.26) – (2.28), (2.9), (2.18) and (2.29) applied to the convex function $f : [\ln a, \ln b] \mapsto [a, b]$, $f(x) = e^x$, with some simple manipulations, respectively.

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