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# ON NEW INEQUALITIES OF HADAMARD-TYPE FOR LIPSCHITZIAN MAPPINGS AND THEIR APPLICATIONS

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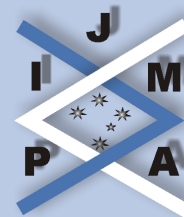
*Key words:* Lipschitzian mappings, Hadamard inequality, Convex function.

*Abstract:* In this paper, we study some new inequalities of Hadamard's Type for Lipschitzian mappings. some applications are also included.

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**Inequalities of Hadamard-type  
for Lipschitzian Mappings**

Liang-Cheng Wang

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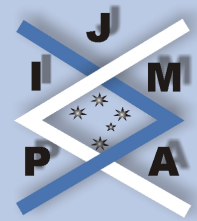
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# 1. Introduction

Let  $f : [a, b] \rightarrow \mathbb{R}$  ( $a < b$ ) be a continuous function.

If  $f$  is convex on  $[a, b]$ , then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

The inequalities in (1.1) are known as the Hermite-Hadamard inequality [1].

For some recent results which generalize, improve, and extend this classic inequality, see references of [2] – [7]. In order to refine inequalities of (1.1), the author of this paper in [2] defined the following some notations, symbols and mappings. We list these notations and symbols by

$Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ ,  $\mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$ ,  $T_n = t_1 + t_2 + \dots + t_n$ ;  $\mathbf{0} = (0, 0, \dots, 0)$ ,  $\mathbf{1} = (1, 1, \dots, 1)$ ,  $\frac{\mathbf{1}}{n} = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$  and  $(1_i, 0) = (0, \dots, 0, 1, 0, \dots, 0)$  ( $1$  is  $i$ th component,  $i = 1, 2, \dots, n$ ) are special points in  $\mathbb{R}^n$ ;  $G = [0, \frac{1}{n}] \times [0, \frac{1}{n}] \times \dots \times [0, \frac{1}{n}]$ ,  $I = [0, 1] \times [0, 1] \times \dots \times [0, 1]$ ,  $V = [a, b] \times [a, b] \times \dots \times [a, b]$ ,  $D = [a, x_1] \times [x_1, x_2] \times \dots \times [x_{n-1}, b]$  ( $x_i = a + \frac{(b-a)i}{n}$ ,  $i = 0, 1, \dots, n$ ;  $x_0 = a, x_n = b$ ),  $H = \{\mathbf{t} \in I | T_n \leq 1\}$  and  $L = \{\mathbf{t} \in I | T_n = 1\}$  are subsets in  $\mathbb{R}^n$ .

We list these mappings by

$$R_n : I \mapsto \mathbb{R}, \quad R_n(\mathbf{t}) \triangleq \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n} \sum_{i=1}^n \left(t_i y_i + (1-t_i) \frac{x_{i-1} + x_i}{2}\right)\right) dY,$$

$$S_n : H \mapsto \mathbb{R}, \quad S_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n} \int_V f\left(\sum_{i=1}^n t_i y_i + (1-T_n) \frac{a+b}{2}\right) dY$$



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and

$$P_n : L \mapsto \mathbb{R}, \quad P_n(\mathbf{t}) \triangleq \frac{1}{(b-a)^n} \int_V f \left( \sum_{i=1}^n t_i y_i \right) dY.$$

We write  $P_{n+1}$  in the following equivalent form

$$P_{n+1} : H \mapsto \mathbb{R}, \quad P_{n+1}(\mathbf{t}) \triangleq \frac{1}{(b-a)^{n+1}} \int_V \left[ \int_a^b f \left( \sum_{i=1}^n t_i y_i + (1 - T_n)x \right) dx \right] dY.$$

Let  $g : A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ . For all  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in A (j = 1, 2)$  with  $t_i^{(1)} \leq t_i^{(2)} (i = 1, 2, \dots, n)$ , if  $g(\mathbf{t}^{(1)}) \leq g(\mathbf{t}^{(2)})$ , then we call  $g$  increasing on  $A$ .

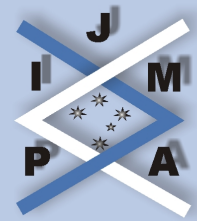
For these mappings and if  $f$  is convex on  $[a, b]$ , L.-C. Wang in [2] gave the following properties and inequalities:

$P_n$  is convex on  $L$ ;  $R_n$  and  $S_n$  are convex, increasing on  $I$  and  $G$ , respectively;

$$\begin{aligned} (1.2) \quad f \left( \frac{a+b}{2} \right) &= R_n(\mathbf{0}) \leq R_n(\mathbf{t}) \leq R_n(\mathbf{1}) \\ &= \left( \frac{n}{b-a} \right)^n \int_D f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \end{aligned}$$

for any  $\mathbf{t} \in I$ ,

$$(1.3) \quad f \left( \frac{a+b}{2} \right) = S_n(\mathbf{0}) \leq S_n(\mathbf{t}) \leq S_n \left( \frac{\mathbf{1}}{\mathbf{n}} \right) = \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY$$



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for all  $\mathbf{t} \in G$ ,

$$(1.4) \quad S_n(\mathbf{t}) \leq P_{n+1}(\mathbf{t})$$

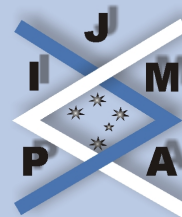
for all  $\mathbf{t} \in H$ , and

$$(1.5) \quad S_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) = P_n\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \leq P_n(\mathbf{t}) \leq P_n(1_i, 0) = \frac{1}{b-a} \int_a^b f(x) dx$$

for all  $\mathbf{t} \in L$ .

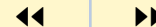
(1.2) – (1.5) are refinements of (1.1).

Recently, Dragomir *et al.* [3], Yang and Tseng [5], Matic and Pečarić [6] and L.-C. Wang [7] proved some results for Lipschitzian mappings related to (1.1). In this paper, we will prove some new inequalities for Lipschitzian mappings related to the mappings  $R_n$  (or (1.2)),  $S_n$  (or (1.3)) and  $P_n$  (or (1.5) and (1.4)). Finally, some applications are given.



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## 2. Main Results

A function  $f : [a, b] \rightarrow \mathbb{R}$  is called an  $M$ -Lipschitzian mapping, if for every two elements  $x, y \in [a, b]$  and  $M > 0$  we have

$$|f(x) - f(y)| \leq M|x - y|.$$

For the mapping  $R_n(\mathbf{t})$ , we have the following theorem:

**Theorem 2.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian mapping, then we have*

$$(2.1) \quad |R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)})| \leq \frac{M}{4n^2}(b - a) \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}|$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in I (j = 1, 2)$ ,

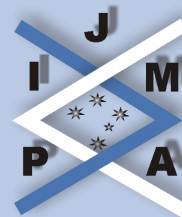
$$(2.2) \quad \left| f\left(\frac{a+b}{2}\right) - R_n(\mathbf{t}) \right| \leq \frac{M}{4n^2}(b - a)T_n$$

and

$$(2.3) \quad \left| R_n(\mathbf{t}) - \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY \right| \leq \frac{M}{4n^2}(b - a)(n - T_n)$$

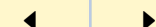
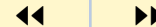
for all  $t \in I$ , and

$$(2.4) \quad \left| \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M(n^2 - 1)}{3n^2}(b - a).$$



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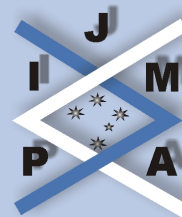
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*Proof.* (1) For  $x_i = \frac{(b-a)i}{n}$  ( $i = 0, 1, \dots, n; x_0 = a, x_n = b$ ), from integral properties, we have

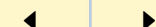
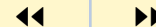
$$\begin{aligned}
 & |R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)})| \\
 & \leq \left(\frac{n}{b-a}\right)^n \int_D \left| f\left(\frac{1}{n} \sum_{i=1}^n \left(t_i^{(2)} y_i + (1-t_i^{(2)}) \frac{x_{i-1} + x_i}{2}\right)\right) \right. \\
 & \quad \left. - f\left(\frac{1}{n} \sum_{i=1}^n \left(t_i^{(1)} y_i + (1-t_i^{(1)}) \frac{x_{i-1} + x_i}{2}\right)\right) \right| dY \\
 & \leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \int_D \left| \sum_{i=1}^n (t_i^{(2)} - t_i^{(1)}) \left(y_i - \frac{x_{i-1} + x_i}{2}\right) \right| dY \\
 & \leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \int_D \left| y_i - \frac{x_{i-1} + x_i}{2} \right| dY \\
 & = \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \left(\frac{b-a}{n}\right)^{n-1} \int_{x_{i-1}}^{x_i} \left| y_i - \frac{x_{i-1} + x_i}{2} \right| dy_i \\
 & = \frac{M}{b-a} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \left[ \int_{x_{i-1}}^{\frac{x_{i-1} + x_i}{2}} \left(\frac{x_{i-1} + x_i}{2} - y_i\right) dy_i \right. \\
 & \quad \left. + \int_{\frac{x_{i-1} + x_i}{2}}^{x_i} \left(y_i - \frac{x_{i-1} + x_i}{2}\right) dy_i \right] \\
 & = \frac{M}{4n^2} (b-a) \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}|.
 \end{aligned}$$

This completes the proof of (2.1).



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(2) The inequalities (2.2) and (2.3) follow from (2.1) by choosing  $\mathbf{t}^{(1)} = \mathbf{0}$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$  and  $\mathbf{t}^{(1)} = \mathbf{1}$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$ , respectively. This completes the proof of (2.2) and (2.3).

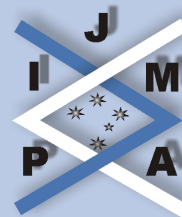
(3) From integral properties, we have

$$(2.5) \quad \int_a^b f(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(y_i)dy_i = \left(\frac{n}{b-a}\right)^{n-1} \sum_{i=1}^n \int_D f(y_i)dY.$$

Using (2.5) and integral properties, we obtain

$$\begin{aligned} & \left| \left(\frac{n}{b-a}\right)^n \int_D f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \left(\frac{n}{b-a}\right)^n \int_D \left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{1}{n} \sum_{j=1}^n y_j\right) - \frac{1}{n} \sum_{i=1}^n f(y_i) \right| dY \\ & \leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n} \int_D \sum_{i=1}^n \left| \frac{1}{n} \sum_{j=1}^n y_j - y_i \right| dY \\ & \leq \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n^2} \sum_{i=1}^n \int_D \sum_{j=1}^n |y_j - y_i| dY \\ & = \left(\frac{n}{b-a}\right)^n \cdot \frac{M}{n^2} \sum_{i=1}^n \int_D \left[ \sum_{j=1}^{i-1} (y_i - y_j) + \sum_{j=i+1}^n (y_j - y_i) \right] dY \\ & = \frac{n}{b-a} \cdot \frac{M}{n^2} \sum_{i=1}^n \left[ \sum_{j=1}^{i-1} \left( \int_{x_{i-1}}^{x_i} y_i dy_i - \int_{x_{j-1}}^{x_j} y_j dy_j \right) \right] \end{aligned}$$





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$$\begin{aligned}
 & + \sum_{j=i+1}^n \left( \int_{x_{j-1}}^{x_j} y_j dy_j - \int_{x_{i-1}}^{x_i} y_i dy_i \right) \\
 = & \frac{n}{b-a} \cdot \frac{M}{n^2} \sum_{i=1}^n \left[ \left( \frac{b-a}{n} \right)^2 \left( \sum_{j=1}^{i-1} (i-j) + \sum_{j=i+1}^n (j-i) \right) \right] \\
 = & \frac{M(n^2-1)}{3n^2} (b-a).
 \end{aligned}$$

This completes the proof of (2.4).

This completes the proof of Theorem 2.1. ■

**Corollary 2.2.** *Let  $f$  be convex on  $[a, b]$ , with  $f'_+(a)$  and  $f'_-(b)$  existing. Then we obtain*

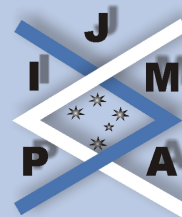
$$\begin{aligned}
 (2.6) \quad 0 & \leq R_n(\mathbf{t}^{(2)}) - R_n(\mathbf{t}^{(1)}) \\
 & \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4n^2} (b-a) \sum_{i=1}^n (t_i^{(2)} - t_i^{(1)})
 \end{aligned}$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in I$  ( $j = 1, 2$ ) with  $t_i^{(2)} \geq t_i^{(1)}$  ( $i = 1, 2, \dots, n$ ),

$$(2.7) \quad 0 \leq R_n(\mathbf{t}) - f\left(\frac{a+b}{2}\right) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4n^2} (b-a) T_n$$

and

$$\begin{aligned}
 (2.8) \quad 0 & \leq \left( \frac{n}{b-a} \right)^n \int_D f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY - R_n(\mathbf{t}) \\
 & \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4n^2} (b-a) (n - T_n)
 \end{aligned}$$



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for all  $\mathbf{t} \in I$ , and

$$(2.9) \quad 0 \leq \frac{1}{b-a} \int_a^b f(x) dx - \left( \frac{n}{b-a} \right)^n \int_D f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY \\ \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\} (n^2 - 1)}{3n^2} (b-a).$$

*Proof.* For any  $x, y \in [a, b]$ , from properties of convex functions, we have the following  $\max\{|f'_+(a)|, |f'_-(b)|\}$ -Lipschitzian inequality (see [8]):

$$(2.10) \quad |f(x) - f(y)| \leq \max\{|f'_+(a)|, |f'_-(b)|\} |x - y|.$$

Since  $R_n$  is increasing on  $I$ , using (1.2), (2.10) and Theorem 2.1, we obtain (2.6)-(2.9).

This completes the proof of Corollary (2.2). ■

For the mapping  $S_n(\mathbf{t})$ , we have the following theorem:

**Theorem 2.3.** Let  $f$  be defined as in Theorem 2.1, then we obtain

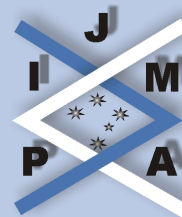
$$(2.11) \quad |S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)})| \leq \frac{M}{4} (b-a) \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}|$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in H$  ( $j = 1, 2$ ),

$$(2.12) \quad \left| f \left( \frac{a+b}{2} \right) - S_n(\mathbf{t}) \right| \leq \frac{M}{4} (b-a) T_n$$

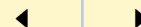
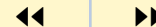
and

$$(2.13) \quad \left| S_n(\mathbf{t}) - \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY \right| \leq \frac{M}{4n} (b-a) \sum_{i=1}^n |nt_i - 1|$$



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for all  $\mathbf{t} \in H$ , and

$$(2.14) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)^n} \int_V f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY \right| \leq \frac{M}{4}(b-a).$$

*Proof.* (1) From integral properties, we obtain

$$\begin{aligned} & |S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)})| \\ & \leq \frac{1}{(b-a)^n} \int_V \left| f\left(\sum_{i=1}^n t_i^{(2)} y_i + \left(1 - \sum_{i=1}^n t_i^{(2)}\right) \frac{a+b}{2}\right) \right. \\ & \quad \left. - f\left(\sum_{i=1}^n t_i^{(1)} y_i + \left(1 - \sum_{i=1}^n t_i^{(1)}\right) \frac{a+b}{2}\right) \right| dY \\ & \leq \frac{M}{(b-a)^n} \int_V \left| \sum_{i=1}^n (t_i^{(2)} - t_i^{(1)}) \left(y_i - \frac{a+b}{2}\right) \right| dY \\ & \leq \frac{M}{(b-a)^n} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \int_V \left| y_i - \frac{a+b}{2} \right| dY \\ & = \frac{M}{b-a} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \int_a^b \left| y_i - \frac{a+b}{2} \right| dy_i \\ & = \frac{M}{b-a} \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}| \left[ \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - y_i\right) dy_i + \int_{\frac{a+b}{2}}^b \left(y_i - \frac{a+b}{2}\right) dy_i \right] \\ & = \frac{M}{4}(b-a) \sum_{i=1}^n |t_i^{(2)} - t_i^{(1)}|. \end{aligned}$$

This completes the proof of (2.11).



(2) The inequalities (2.12) and (2.13) follow from (2.11) by choosing  $\mathbf{t}^{(1)} = \mathbf{0}$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$  and  $\mathbf{t}^{(1)} = \frac{1}{n}$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$ , respectively. The inequalities (2.14) follow from (2.12) by choosing  $\mathbf{t}^{(1)} = \frac{1}{n}$ . This completes the proof of (2.12)-(2.14).

This completes the proof of Theorem 2.3. ■

**Corollary 2.4.** *Let  $f$  be defined as in Corollary 2.2, then we have*

$$(2.15) \quad 0 \leq S_n(\mathbf{t}^{(2)}) - S_n(\mathbf{t}^{(1)}) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4} (b-a) \sum_{i=1}^n (t_i^{(2)} - t_i^{(1)})$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in G$  ( $j = 1, 2$ ) with  $t_i^{(2)} \geq t_i^{(1)}$  ( $i = 1, 2, \dots, n$ ),

$$(2.16) \quad 0 \leq S_n(\mathbf{t}) - f\left(\frac{a+b}{2}\right) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4} (b-a) T_n$$

and

$$(2.17) \quad 0 \leq \frac{1}{(b-a)^n} \int_V f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY - S_n(\mathbf{t}) \\ \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4} (b-a)(1 - T_n)$$

for all  $\mathbf{t} \in G$ , and

$$(2.18) \quad 0 \leq \frac{1}{(b-a)^n} \int_V f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) dY - f\left(\frac{a+b}{2}\right) \\ \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4} (b-a).$$

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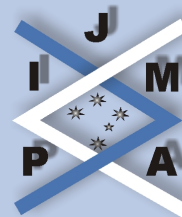
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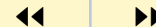
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*Proof.* Since  $S_n$  is increasing on  $G$ , using (1.3), (2.10) and Theorem 2.3, we obtain (2.15) – (2.18).

This completes the proof of Corollary (2.4). ■

For the mapping  $P_n(\mathbf{t})$ , we have the following theorem:

**Theorem 2.5.** *Let  $f$  be defined as in Theorem 2.1. For  $n \geq 2$ , then we obtain*

$$(2.19) \quad |p_n(\mathbf{t}^{(2)}) - p_n(\mathbf{t}^{(1)})| \leq \frac{M}{3}(b-a) \sum_{i=1}^{n-1} |t_i^{(2)} - t_i^{(1)}|$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in L (j = 1, 2)$ ,

$$(2.20) \quad \left| \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY - p_n(\mathbf{t}) \right| \leq \frac{M}{3n}(b-a) \sum_{i=1}^{n-1} |nt_i - 1|$$

and

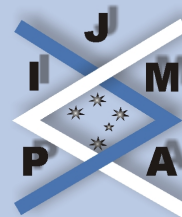
$$(2.21) \quad \left| p_n(\mathbf{t}) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M}{3}(b-a) \sum_{i=1}^{n-1} t_i$$

for all  $\mathbf{t} \in L$ , and

$$(2.22) \quad \left| \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M(n-1)}{3n}(b-a).$$

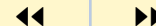
For  $n \geq 1$  and all  $\mathbf{t} \in H$ , then we have

$$(2.23) \quad |S_n(\mathbf{t}) - P_{n+1}(\mathbf{t})| \leq \frac{M}{4}(b-a)(1 - T_n).$$



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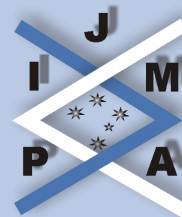
*Proof.* (1) Since  $n \geq 2$  and  $T_n = t_1 + \cdots + t_{n-1} + t_n = 1$ , we can write  $P_n(\mathbf{t})$  in the following equivalent form

$$(2.24) \quad P_n(\mathbf{t}) = \frac{1}{(b-a)^n} \int_V f \left( \sum_{i=1}^{n-1} t_i y_i + \left( 1 - \sum_{i=1}^{n-1} t_i \right) y_n \right) dY.$$

Using (2.24) and integral properties, we obtain

$$\begin{aligned} & |P_n(\mathbf{t}^{(2)}) - P_n(\mathbf{t}^{(1)})| \\ & \leq \frac{1}{(b-a)^n} \int_V \left| f \left( \sum_{i=1}^{n-1} t_i^{(2)} y_i + \left( 1 - \sum_{i=1}^{n-1} t_i^{(2)} \right) y_n \right) \right. \\ & \quad \left. - f \left( \sum_{i=1}^{n-1} t_i^{(1)} y_i + \left( 1 - \sum_{i=1}^{n-1} t_i^{(1)} \right) y_n \right) \right| dY \\ & \leq \frac{M}{(b-a)^n} \int_V \left| \sum_{i=1}^{n-1} (t_i^{(2)} - t_i^{(1)}) (y_i - y_n) \right| dY \\ & \leq \frac{M}{(b-a)^n} \sum_{i=1}^{n-1} |t_i^{(2)} - t_i^{(1)}| (b-a)^{n-2} \int_a^b \int_a^b |y_i - y_n| dy_i dy_n \\ & = \frac{M}{(b-a)^2} \sum_{i=1}^{n-1} |t_i^{(2)} - t_i^{(1)}| \left[ \int_a^b \left[ \int_a^x (x-y) dy + \int_x^b (y-x) dy \right] dx \right. \\ & \left. = \frac{M}{3} (b-a) \sum_{i=1}^{n-1} |t_i^{(2)} - t_i^{(1)}|. \right. \end{aligned}$$

This completes the proof of (2.19).



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(2) The inequalities (2.20) and (2.21) follow from (2.19) by choosing  $\mathbf{t}^{(1)} = \frac{1}{n}$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$  and  $\mathbf{t}^{(1)} = (1_n, 0) = (0, \dots, 0, 1)$ ,  $\mathbf{t}^{(2)} = \mathbf{t}$ , respectively. The inequalities (2.22) follow from (2.21) by choosing  $\mathbf{t} = \frac{1}{n}$ . This completes the proof of (2.20) – (2.22).

(3) Using integral properties, we write  $S_n(\mathbf{t})$  in the following equivalent form

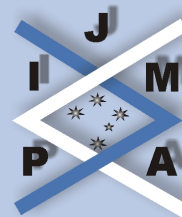
$$(2.25) \quad S_n(\mathbf{t}) = \frac{1}{(b-a)^{n+1}} \int_V \left[ \int_a^b f \left( \sum_{i=1}^n t_i y_i + (1 - T_n) \frac{a+b}{2} \right) dx \right] dY.$$

Using (2.25) and integral properties, we obtain

$$\begin{aligned} & |S_n(\mathbf{t}) - P_{n+1}(\mathbf{t})| \\ & \leq \frac{1}{(b-a)^{n+1}} \int_V \left[ \int_a^b \left| f \left( \sum_{i=1}^n t_i y_i + (1 - T_n) \frac{a+b}{2} \right) \right. \right. \\ & \quad \left. \left. - f \left( \sum_{i=1}^n t_i y_i + (1 - T_n) x \right) \right| dx \right] dY \\ & \leq \frac{M}{(b-a)^{n+1}} (1 - T_n) \int_V \left[ \int_a^b \left| \frac{a+b}{2} - x \right| dx \right] dY \\ & = \frac{M}{b-a} (1 - T_n) \left[ \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - x \right) dx + \int_{\frac{a+b}{2}}^b \left( x - \frac{a+b}{2} \right) dx \right] \\ & = \frac{M}{4} (b-a) (1 - T_n). \end{aligned}$$

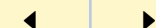
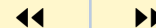
This completes the proof of (2.23).

This completes the proof of Theorem 2.5. ■



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**Corollary 2.6.** Let  $f$  be defined as in Corollary 2.2. For  $n \geq 2$ , then we have

$$(2.26) \quad |P_n(\mathbf{t}^{(2)}) - P_n(\mathbf{t}^{(1)})| \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3} (b-a) \sum_{i=1}^{n-1} |t_i^{(2)} - t_i^{(1)}|$$

for any  $\mathbf{t}^{(j)} = (t_1^{(j)}, \dots, t_n^{(j)}) \in L (j = 1, 2)$ ,

$$(2.27) \quad 0 \leq P_n(\mathbf{t}) - \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY \\ \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3n} (b-a) \sum_{i=1}^{n-1} |nt_i - 1|$$

and

$$(2.28) \quad 0 \leq \frac{1}{b-a} \int_a^b f(x) dx - P_n(\mathbf{t}) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{3} (b-a) \sum_{i=1}^{n-1} t_i$$

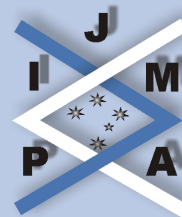
for all  $\mathbf{t} \in L$ , and

$$(2.29) \quad 0 \leq \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{(b-a)^n} \int_V f \left( \frac{1}{n} \sum_{i=1}^n y_i \right) dY \\ \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\} (n-1)}{3n} (b-a).$$

For  $n \geq 1$  and all  $\mathbf{t} \in H$ , we have

$$(2.30) \quad 0 \leq P_{n+1}(\mathbf{t}) - S_n(\mathbf{t}) \leq \frac{\max\{|f'_+(a)|, |f'_-(b)|\}}{4} (b-a) (1 - T_n).$$





*Proof.* Using (1.5), (1.4), (2.10) and Theorem 2.5, we obtain (2.26) – (2.30).

This completes the proof of Corollary (2.6). ■

*Remark 1.* The condition in Corollary 2.2 (or Corollary 2.4 or 2.6) is better than the condition in Corollary 2.2 (or Corollary 4.2 or Theorem 3.3) of [3]. This is due to the fact that  $f$  is a differentiable convex function on  $[a, b]$  with  $M = \sup_{t \in [a, b]} |f'(t)| < \infty$ .

*Remark 2.* When  $n = 1$ , (2.1) and (2.11), (2.2) and (2.12), (2.3) and (2.13) and (2.23) reduce to (3.4), (3.2), (3.1) and (4.3) of [3], respectively. When  $n = 2$ , (2.19), (2.20), and (2.21) reduce to (4.6), (4.1) and (4.2) of [3], respectively.

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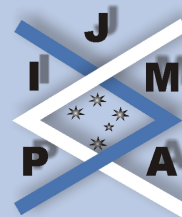
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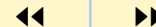
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### 3. Applications

In this section, we agree that when  $t_i = 0$ ,

$$\frac{1}{t_i} \left[ \left( \frac{b}{a} \right)^{\frac{t_i}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{t_i}{2n^2}} \right] = \frac{\ln b - \ln a}{n^2} \quad \text{and} \quad \frac{b^{t_i} - a^{t_i}}{t_i} = \ln b - \ln a.$$

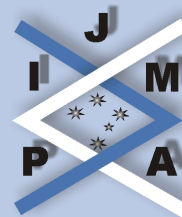
For  $b > a > 0$ ,  $1 \geq t_i^{(2)} \geq t_i^{(1)} \geq 0$  and  $1 \geq t_i \geq 0$  ( $i = 1, 2, \dots, n$ ), we have

$$(3.1) \quad 0 \leq \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left[ \left( \frac{b}{a} \right)^{\frac{t_i^{(2)}}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{t_i^{(2)}}{2n^2}} \right] - \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left[ \left( \frac{b}{a} \right)^{\frac{t_i^{(1)}}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{t_i^{(1)}}{2n^2}} \right] \\ \leq \frac{1}{4} \left( \frac{\ln b - \ln a}{n^2} \right)^{n+1} \left( \frac{b}{a} \right)^{\frac{1}{2}} \sum_{i=1}^n (t_i^{(2)} - t_i^{(1)}),$$

$$(3.2) \quad 0 \leq \left( \frac{n^2}{\ln b - \ln a} \right)^n \prod_{i=1}^n \frac{1}{t_i} \left[ \left( \frac{b}{a} \right)^{\frac{t_i}{2n^2}} \left( \frac{a}{b} \right)^{\frac{t_i}{2n^2}} \right] - 1 \\ \leq \frac{\ln b - \ln a}{4n^2} \left( \frac{b}{a} \right)^{\frac{1}{2}} T_n$$

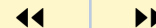
and

$$(3.3) \quad 0 \leq \prod_{i=1}^n \left[ \left( \frac{b}{a} \right)^{\frac{1}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{1}{2n^2}} \right] - \prod_{i=1}^n \frac{1}{t_i} \left[ \left( \frac{b}{a} \right)^{\frac{t_i}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{t_i}{2n^2}} \right] \\ \leq \frac{1}{4} \left( \frac{\ln b - \ln a}{n^2} \right)^{n+1} \left( \frac{b}{a} \right)^{\frac{1}{2}} (n - T_n).$$



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For  $b > a > 0$ ,  $\frac{1}{n} \geq t_i^{(2)} \geq t_i^{(1)} \geq 0$  and  $\frac{1}{n} \geq t_i \geq 0$  ( $i = 1, 2, \dots, n$ ), we have

$$(3.4) \quad \begin{aligned} 0 &\leq (ab)^{\frac{1-\sum_{i=1}^n t_i^{(2)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left( b^{t_i^{(2)}} - a^{t_i^{(2)}} \right) \\ &\quad - (ab)^{\frac{1-\sum_{i=1}^n t_i^{(1)}}{2}} \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left( b^{t_i^{(1)}} - a^{t_i^{(1)}} \right) \\ &\leq \frac{b(\ln b - \ln a)^{n+1}}{4} \sum_{i=1}^n \left( t_i^{(2)} - t_i^{(1)} \right), \end{aligned}$$

$$(3.5) \quad \begin{aligned} 0 &\leq \frac{1}{(\ln b - \ln a)^n} (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left( b^{t_i} - a^{t_i} \right) - (ab)^{\frac{1}{2}} \\ &\leq \frac{b(\ln b - \ln a)}{4} T_n \end{aligned}$$

and

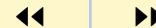
$$(3.6) \quad \begin{aligned} 0 &\leq \left( nb^{\frac{1}{n}} - na^{\frac{1}{n}} \right)^n - (ab)^{\frac{1-\sum_{i=1}^n t_i}{2}} \prod_{i=1}^n \frac{1}{t_i} \left( b^{t_i} - a^{t_i} \right) \\ &\leq \frac{b(\ln b - \ln a)^{n+1}}{4} (1 - T_n). \end{aligned}$$

For  $b > a > 0$ ,  $1 \geq t_i^{(j)} \geq 0$  ( $i = 1, 2, \dots, n; n \geq 2$ ) and  $t_1^{(j)} + t_2^{(j)} + \dots + t_n^{(j)} = 1$



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( $j = 1, 2$ ), we have

$$(3.7) \quad \left| \prod_{i=1}^n \frac{1}{t_i^{(2)}} \left( b^{t_i^{(2)}} - a^{t_i^{(2)}} \right) - \prod_{i=1}^n \frac{1}{t_i^{(1)}} \left( b^{t_i^{(1)}} - a^{t_i^{(1)}} \right) \right| \leq \frac{b(\ln b - \ln a)^{n+1}}{3} \sum_{i=1}^{n-1} \left| t_i^{(2)} - t_i^{(1)} \right|.$$

For  $b > a > 0$ ,  $1 \geq t_i \geq 0$  ( $i = 1, 2, \dots, n$ ;  $n \geq 2$ ) and  $T_n = 1$ , we have

$$(3.8) \quad 0 \leq \prod_{i=1}^n \frac{1}{t_i} (b^{t_i} - a^{t_i}) - \left( nb^{\frac{1}{n}} - na^{\frac{1}{n}} \right)^n \leq \frac{b(\ln b - \ln a)^{n+1}}{3n} \sum_{i=1}^{n-1} |nt_i - 1|$$

and

$$(3.9) \quad 0 \leq \frac{b-a}{\ln b - \ln a} - \frac{1}{(\ln b - \ln a)^n} \prod_{i=1}^n \frac{1}{t_i} (b^{t_i} - a^{t_i}) \leq \frac{b(\ln b - \ln a)^{n-1}}{3} \sum_{i=1}^{n-1} t_i.$$

For  $b > a > 0$ , we have

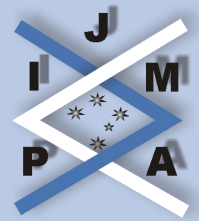
$$(3.10) \quad 0 \leq \frac{b-a}{\ln b - \ln a} - \left( \frac{n^2}{\ln b - \ln a} \right)^n (ab)^{\frac{1}{2}} \prod_{i=1}^n \left[ \left( \frac{b}{a} \right)^{\frac{1}{2n^2}} - \left( \frac{a}{b} \right)^{\frac{1}{2n^2}} \right] \leq \frac{b(n^2 - 1)}{3n^2} (\ln b - \ln a),$$

$$(3.11) \quad 0 \leq \left( \frac{nb^{\frac{1}{n}} - na^{\frac{1}{n}}}{\ln b - \ln a} \right)^n - (ab)^{\frac{1}{2}} \leq \frac{b(\ln b - \ln a)}{4}$$

and

$$(3.12) \quad 0 \leq \frac{b-a}{\ln b - \ln a} - \left( \frac{n \left( b^{\frac{1}{n}} - a^{\frac{1}{n}} \right)}{\ln b - \ln a} \right)^n \leq \frac{b(n-1)}{3n} (\ln b - \ln a).$$

Indeed, (3.1) – (3.12) follow from (2.6) – (2.8), (2.15) – (2.17), (2.26) – (2.28), (2.9), (2.18) and (2.29) applied to the convex function  $f : [\ln a, \ln b] \mapsto [a, b]$ ,  $f(x) = e^x$ , with some simple manipulations, respectively.



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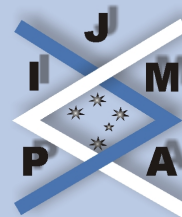
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## References

- [1] J. HADAMARD, Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.*, **58** (1893), 171–215.
- [2] L.-C. WANG, Three mapping related of Hermite-Hadamard inequalities, *J. Sichuan Univ.*, **39** (2002), 652–656. (Chinese).
- [3] S.S. DRAGOMIR, Y.J. CHO AND S.S. KIM, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, *J. Math. Anal. Appl.*, **245** (2000), 489–501.
- [4] L.-C. WANG, On extensions and refinements of Hermite-Hadamard inequalities for convex functions, *Math. Inequal. & Applics.*, **6**(4) (2003), 659–666.
- [5] G.-S. YANG AND K.-L. TSENG, Inequalities of Hadamard's type for Lipschitzian mappings, *J. Math. Anal. Appl.*, **260** (2001), 230–238.
- [6] M. MATIĆ AND J. PEČARIĆ, Note on inequalities of Hadamard's type for Lipschitzian mappings, *Tamkang J. Math.*, **32**(2) (2001), 127–130.
- [7] L.-C. WANG, New inequalities of Hadamard's type for Lipschitzian mappings, *J. Inequal. Pure Appl. Math.*, **6**(2) (2005), Art. 37. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=506>].
- [8] L.-C. WANG, *Convex Functions and Their Inequalities*, Sichuan University Press, Chengdu, China, 2001. (Chinese).

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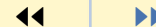
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