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JORDAN-TYPE INEQUALITIES FOR GENERALIZED BESSEL FUNCTIONS

ÁRPÁD BARICZ

BABEŞ-BOLYAI UNIVERSITY, FACULTY OF ECONOMICS
RO-400591 CLUJ-NAPOCA, ROMANIA
bariczocsi@yahoo.com

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Dedicated to my son Koppány.

ABSTRACT. In this note our aim is to present some Jordan-type inequalities for generalized Bessel functions in order to extend some recent results concerning generalized and sharp versions of the well-known Jordan's inequality.

Key words and phrases: Bessel functions, modified Bessel functions, Jordan's inequality.

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1. INTRODUCTION AND PRELIMINARIES

The following inequality is known in the literature as Jordan's inequality [8, p. 33]

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad 0 < x \leq \frac{\pi}{2}.$$

This inequality plays an important role in many areas of mathematics and it has been studied by several mathematicians. Recently many authors including, for example A. McD. Mercer, U. Abel and D. Caccia [7], F. Yuefeng [17], F. Qi and Q.D. Hao [10], L. Debnath and C.J. Zhao [5], S.H. Wu [15], J. Sándor [12], X. Zhang, G. Wang and Y. Chu [18], L. Zhu [19, 20], A.Y. Özban [9], S. Wu and L. Debnath [16], W.D. Jiang and H. Yun [6] (see also the references therein) have improved Jordan's inequality. For the history of this the interested reader is referred to the survey articles of J. Sándor [13] and F. Qi [11].

In a recent work [2] we pointed out that the improvements of Jordan's inequality can be confined as particular cases of some inequalities concerning Bessel and modified Bessel functions.

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Our aim in this paper is to continue our investigation related to extensions of Jordan's inequality. The main motivation to write this note is the publication of S. Wu and L. Debnath [16], which we wish to complement. For this let us recall some basic facts about generalized Bessel functions.

The generalized Bessel function of the first kind v_p is defined [4] as a particular solution of the generalized Bessel differential equation

$$x^2y''(x) + bxy'(x) + \left[cx^2 - p^2 + (1-b)p\right] y(x) = 0,$$

where $b, p, c \in \mathbb{R}$, and v_p has the infinite series representation

$$v_p(x) = \sum_{n\geq 0} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \cdot \left(\frac{x}{2}\right)^{2n+p} \quad \text{for all } x \in \mathbb{R}.$$

This function permits us to study the classical Bessel function J_p [14, p. 40] and the modified Bessel function I_p [14, p. 77] together. For $c = 1$ ($c = -1$ respectively) and $b = 1$ the function v_p reduces to the function J_p (I_p respectively). Now the generalized and normalized (with conditions $u_p(0) = 1$ and $u_p'(0) = -c/(4\kappa)$) Bessel function of the first kind is defined [4] as follows

$$u_p(x) = 2^p\Gamma(\kappa) \cdot x^{-p/2}v_p(x^{1/2}) = \sum_{n\geq 0} \frac{(-c/4)^n}{(\kappa)_n}\frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R},$$

where $\kappa := p+(b+1)/2 \neq 0, -1, -2, \ldots$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$, $a \neq 0, -1, -2, \ldots$ is the well-known Pochhammer (or Appell) symbol defined in terms of Euler's gamma function. This function is related in fact to an obvious transform of the well-known hypergeometric function ${}_0F_1$, i.e. $u_p(x) = {}_0F_1(\kappa, -cx/4)$, and satisfies the following differential equation

$$xy''(x) + \kappa y'(x) + (c/4)y(x) = 0.$$

Now let us consider the function $\lambda_p : \mathbb{R} \to \mathbb{R}$, defined by

$$\lambda_p(x) := u_p(x^2) = \sum_{n\geq 0} \frac{(-c/4)^n}{(\kappa)_n}\frac{x^{2n}}{n!}.$$

It is worth mentioning that if $c = b = 1$, then λ_p reduces to the function $\mathcal{J}_p : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x).$$

Moreover, if $c = -1$ and $b = 1$, then λ_p becomes $\mathcal{I}_p : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x).$$

For later use we note that in particular (for $p = 1/2$, $p = 3/2$ respectively) the functions $\mathcal{J}_p$ and $\mathcal{I}_p$ reduce to some elementary functions, like [14, p. 54]

$$\begin{aligned} \mathcal{J}_{\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2x}} \cdot J_{\frac{1}{2}}(x) = \frac{\sin x}{x}, \\ \mathcal{J}_{\frac{3}{2}}(x) &= \frac{3}{x}\sqrt{\frac{\pi}{2x}} \cdot J_{\frac{3}{2}}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \end{aligned} \tag{1.1}$$

$$\begin{aligned} \mathcal{I}_{\frac{1}{2}}(x) &= \sqrt{\frac{\pi}{2x}} \cdot I_{\frac{1}{2}}(x) = \frac{\sinh x}{x}, \\ \mathcal{I}_{\frac{3}{2}}(x) &= \frac{3}{x}\sqrt{\frac{\pi}{2x}} \cdot I_{\frac{3}{2}}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right). \end{aligned} \tag{1.2}$$

2. EXTENSIONS OF JORDAN'S INEQUALITY TO BESSEL FUNCTIONS

The following theorems are extensions of Theorem 1 due to S. Wu and L. Debnath [16] to generalized Bessel functions of the first kind.

Theorem 2.1. *If $\kappa > 0$ and $c \in [0, 1]$, then for all $0 < x \le r \le \pi/2$ we have*

$$\begin{aligned} \lambda_p(r) + \frac{c}{2\kappa}x(r-x)\lambda_{p+1}(r) + \left[\frac{1-\lambda_p(r)}{r^2}\right](r-x)^2 \\ \le \lambda_p(x) \le \lambda_p(r) + \frac{c}{4\kappa}(r^2-x^2)\lambda_{p+1}(r) - \frac{c}{4\kappa}r(r-x)^2\lambda'_{p+1}(r). \end{aligned}$$

Moreover, if $\kappa > 0$ and $c \le 0$, then the above inequalities hold for all $0 < x \le r$, and equality holds if and only if $x = r$.

Proof. When $x = r$, clearly we have equality. Assume that $x \ne r$ and fix r. Let us consider the functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : (0, r) \to \mathbb{R}$, defined by

$$\varphi_1(x) := \lambda_p(x) - \lambda_p(r) - \frac{c}{4\kappa}(r^2 - x^2)\lambda_{p+1}(r), \quad \varphi_2(x) := \left(1 - \frac{x}{r}\right)^2,$$

$$\varphi_3(x) := \lambda_{p+1}(r) - \lambda_{p+1}(x) \quad \text{and} \quad \varphi_4(x) := 1 - \frac{r}{x}.$$

Then we have

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \frac{cr^2}{4\kappa} \cdot \frac{\varphi_3(x) - \varphi_3(r)}{\varphi_4(x) - \varphi_4(r)} \quad \text{and} \quad \frac{\varphi_3'(x)}{\varphi_4'(x)} = \frac{x^2\varphi_3'(x)}{r}.$$

Here we applied the derivative formula

$$\lambda_p'(x) = -\frac{cx}{2\kappa}\lambda_{p+1}(x), \tag{2.1}$$

which follows immediately from the series representation of λ_p. Suppose that $c \in [0, 1]$. It is known [3] that the function $\mathcal{J}_p$ is decreasing and concave on $[0, \pi/2]$ when $p \ge -1/2$. On the other hand, $\lambda_p(x) = \mathcal{J}_{\kappa-1}(x\sqrt{c})$ and thus λ_p is decreasing and concave on $[0, \pi/2]$ when $\kappa \ge 1/2$. From this we obtain that φ_3 is increasing and convex when $\kappa > 0$. Thus φ_3'/φ_4' is increasing too as a product of two positive and increasing functions. Using the monotone form of the l'Hospital rule due to G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [1, Lemma 2.2] we obtain that φ_1'/φ_2' is also increasing on $(0, r)$.

Now assume that $c \leq 0$. Then clearly φ_3 is decreasing and concave, since all coefficients of the corresponding power series are negative. Consequently φ_3'/φ_4' is decreasing and hence φ_1'/φ_2' is increasing on $(0, r)$. Using again the monotone form of the l'Hospital rule [1, Lemma 2.2] this implies that the function $\phi_1 : (0, r) \to \mathbb{R}$, defined by

$$\phi_1(x) := \frac{\varphi_1(x) - \varphi_1(r)}{\varphi_2(x) - \varphi_2(r)},$$

is increasing. Moreover from the l'Hospital rule we obtain that

$$\phi_1(0^+) = 1 - \lambda_p(r) - \frac{cr^2}{4\kappa}\lambda_{p+1}(r) \quad \text{and} \quad \phi_1(r^-) = -\frac{cr^3}{4\kappa}\lambda'_{p+1}(r).$$

Hence for all $\kappa > 0$, $c \in [0, 1]$ and $0 < x \leq r \leq \pi/2$ we have the following inequality: $\phi_1(0^+) \leq \phi_1(x) \leq \phi_1(r^-)$. Moreover, when $\kappa > 0$, $c \leq 0$ and $0 < x \leq r$, the above inequality also holds, hence the required result follows. □

Theorem 2.2. *If $\kappa > 0$ and $c \in [0, 1]$, then for all $0 < x \leq r \leq \pi/2$ we have*

$$\lambda_p(r) + \frac{c}{4\kappa}(r^2 - x^2)\lambda_{p+1}(r) - \frac{c}{16\kappa r}(r^2 - x^2)^2\lambda'_{p+1}(r) \\ \leq \lambda_p(x) \leq \lambda_p(r) + \frac{c}{4\kappa}\frac{x^2}{r^2}(r^2 - x^2)\lambda_{p+1}(r) + \left[\frac{1 - \lambda_p(r)}{r^4}\right](r^2 - x^2)^2.$$

Moreover, if $\kappa > 0$ and $c \leq 0$, then the above inequalities are reversed for all $0 < x \leq r$, and equality holds if and only if $x = r$.

Proof. The proof of this theorem is similar to the proof of Theorem 2.1, so we sketch the proof. Let us consider the functions $\varphi_5, \varphi_6 : (0, r) \to \mathbb{R}$, defined by

$$\varphi_5(x) := \left(1 - \frac{x^2}{r^2}\right)^2 \quad \text{and} \quad \varphi_6(x) := x^2 - r^2.$$

In view of (2.1), easy computations show that

$$\frac{\varphi_1'(x)}{\varphi_5'(x)} = \frac{cr^4}{8\kappa} \cdot \frac{\varphi_3(x) - \varphi_3(r)}{\varphi_6(x) - \varphi_6(r)} \quad \text{and}$$
$$\frac{\varphi_3'(x)}{\varphi_6'(x)} = -\frac{\lambda'_{p+1}(x)}{2x} = \frac{c}{4(\kappa + 1)}\lambda_{p+2}(x).$$

Suppose that $c \in [0, 1]$. Since λ_p is decreasing on $[0, \pi/2]$ when $\kappa \geq 1/2$, we get that φ_3'/φ_6' is decreasing on $(0, r)$, when $\kappa > 0$. Thus from the monotone form of the l'Hospital rule [1, Lemma 2.2], φ_1'/φ_5' is also decreasing on $(0, r)$.

Now assume that $c \leq 0$. Then clearly φ_3'/φ_6' is decreasing and consequently φ_1'/φ_5' is increasing on $(0, r)$. Now consider the function $\phi_2 : (0, r) \to \mathbb{R}$, defined by

$$\phi_2(x) := \frac{\varphi_1(x) - \varphi_1(r)}{\varphi_5(x) - \varphi_5(r)}.$$

Then from the monotone form of the l'Hospital rule [1, Lemma 2.2], we conclude that ϕ_2 is decreasing when $\kappa > 0$ and $c \in [0,1]$, and is increasing when $\kappa > 0$ and $c \leq 0$. In addition, from the usual l'Hospital rule we have that $\phi_2(0^+) = \phi_1(0^+)$ and $4\phi_2(r^-) = \phi_1(r^-)$. Now for all $\kappa > 0$, $c \in [0,1]$ and $0 < x \leq r \leq \pi/2$, using the inequality $\phi_2(0^+) \geq \phi_2(x) \geq \phi_2(r^-)$, the asserted result follows. Finally, when $\kappa > 0$, $c \leq 0$ and $0 < x \leq r$ the above inequalities are reversed. Thus the proof is complete. $\square$

Remark 1. First note that taking $c = b = 1$ and $p = 1/2$ in Theorems 2.1 and 2.2 and using (1.1), we reobtain the results of S. Wu and L. Debnath [16, Theorem 1], but just for $0 < x \leq r \leq \pi/2$. Moreover, the inequalities in Theorem 2.2 are improvements of inequalities established in [2, Theorem 5.14]. More precisely in [2, Theorem 5.14] we proved that if $\kappa > 0$, $c \in [0,1]$ and $0 < x \leq r \leq \pi/2$, then

$$\lambda_p(r) + \left[\left(\frac{c}{4\kappa}\right)\lambda_{p+1}(r)\right](r^2 - x^2) \leq \lambda_p(x) \leq \lambda_p(r) + \left[\frac{1-\lambda_p(r)}{r^2}\right](r^2 - x^2). \tag{2.2}$$

Easy computations show that

$$-\frac{c}{16\kappa r}(r^2 - x^2)^2\lambda'_{p+1}(r) \geq 0,$$

$$\frac{c}{4\kappa}\frac{x^2}{r^2}(r^2 - x^2)\lambda'_{p+1}(r) + \left[\frac{1-\lambda_p(r)}{r^4}\right](r^2 - x^2)^2 \leq \left[\frac{1-\lambda_p(r)}{r^2}\right](r^2 - x^2),$$

where $\kappa > 0$, $c \in [0,1]$ and $0 < x \leq r \leq \pi/2$. Thus Theorem 2.2 provides an improvement of (2.2).

Finally taking $c = -1$, $b = 1$ and $p = 1/2$ in Theorems 2.1 and 2.2 and using (1.2), we obtain the hyperbolic counterpart of Theorem 1 due to S. Wu and L. Debnath [16].

Corollary 2.3. *If $0 < x \leq r$, then*

$$\frac{\sinh r}{r} + \frac{1}{2}\left(\frac{\sinh r}{r} - \cosh r\right)\left(1 - \frac{x^2}{r^2}\right) - \frac{3}{2}\left(-\frac{1}{3}r\sinh r + \cosh r - \frac{\sinh r}{r}\right)\left(1 - \frac{x}{r}\right)^2$$
$$\geq \frac{\sinh x}{x} \geq \frac{\sinh r}{r} + \frac{1}{2}\left(\frac{\sinh r}{r} - \cosh r\right)\left(1 - \frac{x^2}{r^2}\right) + \frac{3}{2}\left(\frac{2}{3} + \frac{\cosh r}{3} - \frac{\sinh r}{r}\right)\left(1 - \frac{x}{r}\right)^2,$$

where the equality holds if and only if $x = r$ and the values

$$\frac{3}{2}\left(\frac{2}{3} + \frac{\cosh r}{3} - \frac{\sinh r}{r}\right) \quad \textit{and} \quad \frac{3}{2}\left(-\frac{1}{3}r\sinh r + \cosh r - \frac{\sinh r}{r}\right)$$

are the best constants. Moreover,

$$\frac{\sinh r}{r} + \frac{1}{2}\left(\frac{\sinh r}{r} - \cosh r\right)\left(1 - \frac{x^2}{r^2}\right) - \frac{3}{8}\left(-\frac{1}{3}r\sinh r + \cosh r - \frac{\sinh r}{r}\right)\left(1 - \frac{x^2}{r^2}\right)^2$$
$$\geq \frac{\sinh x}{x} \geq \frac{\sinh r}{r} + \frac{1}{2}\left(\frac{\sinh r}{r} - \cosh r\right)\left(1 - \frac{x^2}{r^2}\right) + \frac{3}{2}\left(\frac{2}{3} + \frac{\cosh r}{3} - \frac{\sinh r}{r}\right)\left(1 - \frac{x^2}{r^2}\right)^2,$$

where equality holds if and only if $x = r$ and the values

$$-\frac{3}{8}\left(-\frac{1}{3}r\sinh r + \cosh r - \frac{\sinh r}{r}\right) \quad \textit{and} \quad \frac{3}{2}\left(\frac{2}{3} + \frac{\cosh r}{3} - \frac{\sinh r}{r}\right)$$

are the best constants.

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