

KY FAN'S INEQUALITY VIA CONVEXITY

JAMAL ROOIN

Department of Mathematics
Institute for Advanced Studies in Basic Sciences
Zanjan, Iran
EMail: rooin@iasbs.ac.ir

Received: 20 October, 2007

Accepted: 11 December, 2007

Communicated by: **P.S. Bullen**

2000 AMS Sub. Class.: 26D15.

Key words: Convexity, Ky Fan's Inequality.

Abstract: In this note, using the strict convexity and concavity of the function $f(x) = \frac{1}{1+e^x}$ on $[0, \infty)$ and $(-\infty, 0]$ respectively, we prove Ky Fan's inequality by separating the left and right hands of it by $\frac{1}{G_n + G'_n}$.



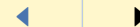
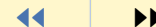
**Ky Fan's Inequality via
Convexity**

Jamal Rooin

vol. 9, iss. 1, art. 23, 2008

[Title Page](#)

[Contents](#)



Page 1 of 4

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 2 of 4

Go Back

Full Screen

Close

Let x_1, \dots, x_n in $(0, 1/2]$ and $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$. We denote by A_n and G_n , the arithmetic and geometric means of x_1, \dots, x_n respectively, i.e.

$$(1) \quad A_n = \sum_{i=1}^n \lambda_i x_i, \quad G_n = \prod_{i=1}^n x_i^{\lambda_i},$$

and also by A'_n and G'_n , the arithmetic and geometric means of $1 - x_1, \dots, 1 - x_n$ respectively, i.e.

$$(2) \quad A'_n = \sum_{i=1}^n \lambda_i (1 - x_i), \quad G'_n = \prod_{i=1}^n (1 - x_i)^{\lambda_i}.$$

In 1961 the following remarkable inequality, due to Ky Fan, was published for the first time in the well-known book *Inequalities* by Beckenbach and Bellman [2, p. 5]: If $x_i \in (0, 1/2]$, then

$$(3) \quad \frac{A'_n}{G'_n} \leq \frac{A_n}{G_n},$$

with equality holding if and only if $x_1 = \dots = x_n$.

Inequality (1) has evoked the interest of several mathematicians and in numerous articles new proofs, extensions, refinements and various related results have been published; see the survey paper [1]. Also, for some recent results, see [6] – [10].

In this note, using the strict convexity and concavity of the function $f(x) = \frac{1}{1+e^x}$ on $[0, \infty)$ and $(-\infty, 0]$ respectively, we prove Ky Fan's inequality (3) by separating the left and right hand sides of (3) by $\frac{1}{G_n + G'_n}$:

$$(4) \quad \frac{A'_n}{G'_n} \leq \frac{1}{G_n + G'_n} \leq \frac{A_n}{G_n}.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 4

Go Back

Full Screen

Close

Moreover, we show equality holds in each inequality in (4), if and only $x_1 = \dots = x_n$.

It is noted that, since for $a, b, c, d > 0$ the inequality $\frac{a}{b} \leq \frac{c}{d}$ implies $\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$, considering $A_n + A'_n = 1$, the inequalities (3) and (4) are equivalent.

Indeed, since $f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3}$, the function f has the foregoing convexity properties. Now, using Jensen's inequality

$$f\left(\sum_{i=1}^n \lambda_i y_i\right) \leq \sum_{i=1}^n \lambda_i f(y_i),$$

for $y_i = \ln \frac{1-x_i}{x_i} \geq 0$ ($1 \leq i \leq n$), we get the right hand of (4) with equality holding if and only if $\ln \frac{1-x_1}{x_1} = \dots = \ln \frac{1-x_n}{x_n}$, or equivalently $x_1 = \dots = x_n$. The left hand of (4) is handled by using Jensen's inequality for the convex function $-f$ on $(-\infty, 0]$ with $y_i = \ln \frac{x_i}{1-x_i} \leq 0$ ($1 \leq i \leq n$).

It might be noted that it suffices to prove either of the two inequalities in (4) as $\frac{a}{b} \leq \frac{c}{d}$ is equivalent to both $\frac{a}{b} \leq \frac{a+c}{b+d}$ and $\frac{a+c}{b+d} \leq \frac{c}{d}$.

It was pointed out by a referee that the use of the function f , or rather its inverse $g(x) = \ln((1-x)/x)$, to prove Ky Fan's inequality can be found in the literature; see [4], [3, pp. 31, 154], [5].

References

- [1] H. ALZER, The inequality of Ky Fan and related results, *Acta Appl. Math.*, **38** (1995), 305–354.
- [2] E.F. BECKENBACH AND R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin, 1961.
- [3] P. BILER AND A. WITKOWSKI, *Problems in Mathematical Analysis*, Marcel Dekker, Inc., 1990.
- [4] K.K. CHONG, On Ky Fan's inequality and some related inequalities between means, *Southeast Asian Bull. Math.*, **29** (1998), 363–372.
- [5] A.McD. MERCER, A short proof of Ky Fan's arithmetic-geometric inequality, *J. Math. Anal. Appl.*, **204** (1996), 940–942.
- [6] J. ROOIN, An approach to Ky Fan type inequalities from binomial expansion, (accepted).
- [7] J. ROOIN, Ky Fan's inequality with binomial expansion, *Elemente Der Mathematik*, **60** (2005), 171–173.
- [8] J. ROOIN, On Ky Fan's inequality and its additive analogues, *Math. Inequal. & Applics.*, **6** (2003), 595–604.
- [9] J. ROOIN, Some new proofs of Ky Fan's inequality, *International Journal of Applied Mathematics* **20** (2007), 285–291.
- [10] J. ROOIN AND A.R. MEDGHALCHI, New proofs for Ky Fan's inequality and two of its variants, *International Journal of Applied Mathematics*, **10** (2002), 51–57.



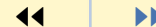
Ky Fan's Inequality via
Convexity

Jamal Rooin

vol. 9, iss. 1, art. 23, 2008

Title Page

Contents



Page 4 of 4

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756