

R.A. AGNEW AND J.E. PEČARIĆ

Deerfield, IL 60015-3007, USA

*E-Mail:* [raagnew@aol.com](mailto:raagnew@aol.com)

*URL:* <http://members.aol.com/raagnew>

Faculty of Textile Technology

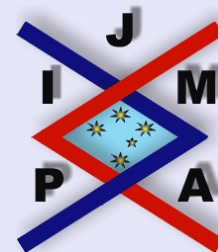
University of Zagreb

Pierottijeva 6, 10000 Zagreb

Croatia

*E-Mail:* [pecaric@mahazu.hazu.hr](mailto:pecaric@mahazu.hazu.hr)

*URL:* <http://mahazu.hazu.hr/DepMPCS/indexJP.html>



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Abstract

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## Abstract

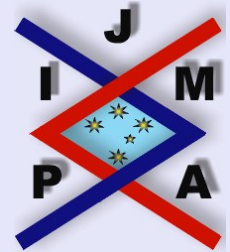
A multivariate Jensen-type inequality is generalized.

*2000 Mathematics Subject Classification:* Primary 26D15.

*Key words:* Convex functions, Tchebycheff methods, Jensen's inequality.

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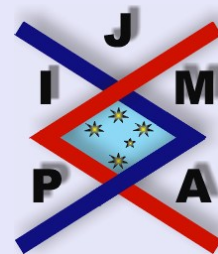
# 1. Introduction

The following theorem was proved in [1] with  $S = (0, \infty)^n$ ,  $g_1, \dots, g_n$  real-valued functions on  $S$ ,  $f(x) = \sum_{i=1}^n x_i g_i(x)$  for any column vector  $x = (x_1, \dots, x_n)^T \in S$ , and  $e_i$  the  $i^{\text{th}}$  unit column vector in  $\mathbb{R}^n$ .

**Theorem 1.1.** *Let  $g_1, \dots, g_n$  be convex on  $S$ , and let  $X = (X_1, \dots, X_n)^T$  be a random column vector in  $S$  with  $E(X) = \mu = (\mu_1, \dots, \mu_n)^T$  and  $E(XX^T) = \Sigma + \mu\mu^T$  for covariance matrix  $\Sigma$ . Then,*

$$E(f(X)) \geq \sum_{i=1}^n \mu_i g_i \left( \frac{\sum e_i}{\mu_i} + \mu \right)$$

*and the bound is sharp.*



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## 2. Generalized Result

**Theorem 2.1.** Let  $g_1, \dots, g_n$  be convex on  $S$ ,  $F$  convex on  $\mathbb{R}^n$  and nondecreasing in each argument, and  $f(x) = F(x_1 g_1(x), \dots, x_n g_n(x))$ . Let  $X = (X_1, \dots, X_n)^T$  be a random column vector in  $S$  with  $E(X) = \mu = (\mu_1, \dots, \mu_n)^T$  and  $E(XX^T) = \Sigma + \mu\mu^T$  for covariance matrix  $\Sigma$ . Then,

$$(2.1) \quad E(f(X)) \geq F(\mu_1 g_1(\xi_1), \dots, \mu_n g_n(\xi_n))$$

where  $\xi_i = E\left(\frac{XX_i}{\mu_i}\right) = \frac{\sum e_i}{\mu_i} + \mu$  and the bound is sharp.

*Proof.* By Jensen's inequality, we have

$$E(f(X)) \geq F(E(X_1 g_1(X)), \dots, E(X_n g_n(X)))$$

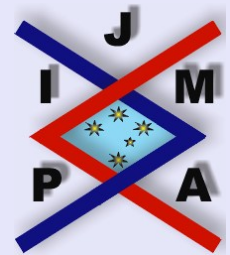
and it is proved in [1] that  $E(X_i g_i(X)) \geq \mu_i g_i(\xi_i)$  is the best possible lower bound for each  $i$ . Since  $F$  is nondecreasing in each argument, (2.1) follows and the bound is obviously attained when  $X$  is concentrated at  $\mu$ .  $\square$

Theorem 1.1 is a special case of Theorem 2.1 with  $F(u_1, \dots, u_n) = \sum_{i=1}^n u_i$ . A simple generalization puts

$$F(u_1, \dots, u_n) = \sum_{i=1}^n k_i(u_i)$$

where each  $k_i$  is convex nondecreasing on  $R$ . Alternatively, we can put

$$F(u_1, \dots, u_n) = \max_i k_i(u_i)$$



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since convexity is preserved under maxima.

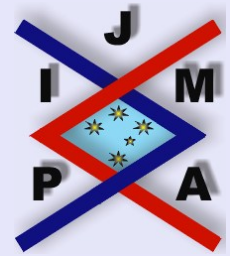
Drawing on an example in [1], let

$$g_i(x) = \rho_i \prod_{j=1}^n x_j^{-\gamma_{ij}}$$

with  $\rho_i > 0$  and  $\gamma_{ij} > 0$  where the  $g_i$  represent Cournot-type price functions (inverse demand functions) for quasi-substitutable products.  $x_i$  is the supply of product  $i$  and  $g_i(x_1, \dots, x_n)$  is the equilibrium price of product  $i$ , given its supply and the supplies of its alternates. Then,  $x_i g_i(x)$  represents the revenue from product  $i$  and  $f(x) = \max_i x_i g_i(x)$  represents maximum revenue across the ensemble of products. Then, with probabilistic supplies, we have

$$E(f(X)) \geq \max_i \mu_i g_i \left( \frac{\sum e_i}{\mu_i} + \mu \right) = \max_i \mu_i \rho_i \prod_{j=1}^n \left( \frac{\sigma_{ij}}{\mu_i} + \mu_j \right)^{-\gamma_{ij}},$$

where  $\sigma_{ij}$  is the  $ij^{th}$  element of  $\Sigma$ .




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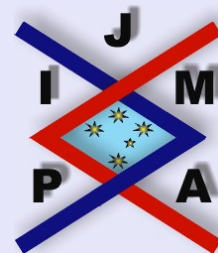
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## References

- [1] R.A. AGNEW, Multivariate version of a Jensen-type inequality, *J. Inequal. in Pure and Appl. Math.*, **6**(4) (2005), Art. 120. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=594>].
- [2] R.A. AGNEW, Inequalities with application in economic risk analysis, *J. Appl. Prob.*, **9** (1972), 441–444.
- [3] D. BROOK, Bounds for moment generating functions and for extinction probabilities, *J. Appl. Prob.*, **3** (1966), 171–178.
- [4] B. GULJAS, C.E.M. PEARCE AND J. PEČARIĆ, Jensen's inequality for distributions possessing higher moments, with applications to sharp bounds for Laplace-Stieltjes transforms, *J. Austral. Math. Soc. Ser. B*, **40** (1998), 80–85.
- [5] S. KARLIN AND W.J. STUDDEN, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Wiley Interscience, 1966.
- [6] J.F.C. KINGMAN, On inequalities of the Tchebychev type, *Proc. Camb. Phil. Soc.*, **59** (1963), 135–146.
- [7] C.E.M. PEARCE AND J.E. PEČARIĆ, An integral inequality for convex functions, with application to teletraffic congestion problems, *Math. Oper. Res.*, **20** (1995), 526–528.
- [8] A.O. PITTENGER, Sharp mean-variance bounds for Jensen-type inequalities, *Stat. & Prob. Letters*, **10** (1990), 91–94.



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