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GENERALIZED MULTIVARIATE JENSEN-TYPE INEQUALITY

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Abstract

A multivariate Jensen-type inequality is generalized.

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Contents

1. Introduction

The following theorem was proved in [1] with $S = (0,\infty)^n$, $g_1, \dots, g_n$ real-valued functions on S, $f(x) = \sum_{i=1}^{n} x_i g_i(x)$ for any column vector $x = (x_1, \dots, x_n)^T \in S$, and e_i the i^{th} unit column vector in $\mathbb{R}^n$.

Theorem 1.1. *Let $g_1, \dots, g_n$ be convex on S, and let $X = (X_1, \dots, X_n)^T$ be a random column vector in S with $E(X) = \mu = (\mu_1, \dots, \mu_n)^T$ and $E\left(XX^T\right) = \Sigma + \mu\mu^T$ for covariance matrix Σ. Then,*

$$E(f(X)) \geq \sum_{i=1}^{n} \mu_i \, g_i\left(\frac{\Sigma e_i}{\mu_i} + \mu\right)$$

and the bound is sharp.

2. Generalized Result

Theorem 2.1. *Let $g_1, \ldots, g_n$ be convex on S, F convex on $\mathbb{R}^n$ and nondecreasing in each argument, and $f(x) = F(x_1 g_1(x), \ldots, x_n g_n(x))$. Let $X = (X_1, \ldots, X_n)^T$ be a random column vector in S with $E(X) = \mu = (\mu_1, \ldots, \mu_n)^T$ and $E(XX^T) = \Sigma + \mu\mu^T$ for covariance matrix Σ. Then,*

$$E(f(X)) \geq F(\mu_1 g_1(\xi_1), \ldots, \mu_n g_n(\xi_n)) \tag{2.1}$$

where $\xi_i = E\left(\frac{XX_i}{\mu_i}\right) = \frac{\Sigma e_i}{\mu_i} + \mu$ and the bound is sharp.

Proof. By Jensen's inequality, we have

$$E(f(X)) \geq F(E(X_1 g_1(X)), \ldots, E(X_n g_n(X)))$$

and it is proved in [1] that $E(X_i g_i(X)) \geq \mu_i g_i(\xi_i)$ is the best possible lower bound for each i. Since F is nondecreasing in each argument, (2.1) follows and the bound is obviously attained when X is concentrated at μ. □

Theorem 1.1 is a special case of Theorem 2.1 with $F(u_1, \ldots, u_n) = \sum_{i=1}^n u_i$. A simple generalization puts

$$F(u_1, \ldots, u_n) = \sum_{i=1}^n k_i(u_i)$$

where each k_i is convex nondecreasing on R. Alternatively, we can put

$$F(u_1, \ldots, u_n) = \max_i k_i(u_i)$$

since convexity is preserved under maxima.

Drawing on an example in [1], let

$$g_i(x) = \rho_i \prod_{j=1}^{n} x_j^{-\gamma_{ij}}$$

with $\rho_i > 0$ and $\gamma_{ij} > 0$ where the g_i represent Cournot-type price functions (inverse demand functions) for quasi-substitutable products. x_i is the supply of product i and $g_i(x_1, \ldots, x_n)$ is the equilibrium price of product i, given its supply and the supplies of its alternates. Then, $x_i\, g_i(x)$ represents the revenue from product i and $f(x) = \max_i\, x_i\, g_i(x)$ represents maximum revenue across the ensemble of products. Then, with probabilistic supplies, we have

$$E(f(X)) \geq \max_i\, \mu_i\, g_i\left(\frac{\Sigma e_i}{\mu_i} + \mu\right) = \max_i\, \mu_i\, \rho_i \prod_{j=1}^{n} \left(\frac{\sigma_{ij}}{\mu_i} + \mu_j\right)^{-\gamma_{ij}},$$

where σ_{ij} is the ij^{th} element of Σ.

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