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A CONJECTURE ON GENERAL MEANS

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Abstract

We conjecture that the general mean of two positive numbers, as a function of its order, has one and only one inflection point. No analytic proof seems available due to the extreme complexity of the second derivative of the function. We show the importance of this conjecture in today's economies.

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Let $x_1, \ldots, x_n$ be n positive numbers and $M(p) = \left(\sum_{i=1}^n f_i x_i^p\right)^{\frac{1}{p}}$ the mean of order p of the x_i's; $0 < f_i < 1$ and $\sum_{i=1}^n f_i = 1$. One of the most important theorems about a general mean is that it is an increasing function of its order. A proof can be found in [2, Theorem 16, pp. 26-27]. The proof rests in part on Hölder's inequality and on successive contributions to the theory of inequalities that go back to Halley and Newton. Another, more analytic proof is based upon the convexity of $x \log x$ – see [3, pp. 76-77].

In this note, we make a conjecture about the exact shape of the curve $M(p)$ in (M, p) space. If it is well known that $M(p)$ is increasing with p, it seems that the exact properties of the curve $M(p)$ have not yet been uncovered. We offer here a conjecture and explain its importance.

In (M, p) space the curve $M(p)$ has one and only one inflection point if $n = 2$, irrespective of the size of the x_i's and the f_i's. Between its limiting values, $M(p)$ is in a first phase convex and then turns concave. Due to the extreme complexity of the second derivative of $M(p)$, we could not offer an analytical proof of this property, and we had to rely on numerical calculations only.

We would have liked to extend this conjecture to $n > 2$, but Professor Anthony Pakes of Western Australia University mentioned to us that his colleague, Grant Keady, found a counter-example with $x_i = 1/8, 2/9, 1$ and $f_i = 1/27, 25/27, 1/27$ where the second derivative, although showing very little variation over $[-10, +10]$, has three zeros. We keep the conjecture for $n = 2$, whose analytical proof remains, in our opinion, a formidable challenge.

The importance of this property stems from the following reason. Both theory and empirical observations have led economists to introduce and make the widest use of a general mean of order p in the following form. Let $x_1 \equiv K_t$ denote the stock of capital of a nation at time date t; let $x_2 \equiv L_t$ be the quantity of labour; associated to both variables K_t and L_t is a function which gives output Y_t as the general mean

$$Y_t = [\delta K_t^p + (1-\delta) L_t^p]^{\frac{1}{p}} \tag{1}$$

where δ and $(1-\delta)$ are the weights of K_t and L_t respectively.

Furthermore, the order p of this mean is related to a parameter of fundamental importance, the so-called “elasticity of substitution”, defined as follows. Omitting the time indexes in our notation, let $\overline{Y}$ denote a given level of production. The equation of the level curve, in space (K, L), corresponding to a given value $\overline{Y}$ is given, in implicit form, by:

$$\overline{Y} = [\delta K^p + (1-\delta) L^p]^{\frac{1}{p}} \tag{2}$$

This level curve is called an "isoquant".

Let the capital-labour ratio $K/L \equiv r$, and minus the slope of the level curve be denoted τ:

$$(3) \qquad -\left.\frac{dK}{dL}\right|_{Y=\bar{Y}} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \tau$$

The elasticity of substitution, denoted σ, is the elasticity of r with respect to τ, defined by

$$(4) \qquad \sigma = \frac{d\log r}{d\log \tau} = \frac{dr/r}{d\tau/\tau}.$$

From a geometric point of view, the elasticity of substitution measures, in linear approximation, the relative change along an isoquant of the ratio $r = K/L$ induced by a relative change in the slope of the isoquant[1].

It can be verified from (2), (3), (4) that the order of the mean p is related to σ by $p = 1 - 1/\sigma$. Observations show that σ is close to one, i.e. that p is close to 0. In turn this implies that the mean Y is close to its limiting form when $p \to 0$, the geometric mean $Y = K^{\delta}L^{1-\delta}$. It turns out also that the abscissa of the inflection point, for the usual values of δ (its order of magnitude is 0.3) is very close to $p = 0$. This means that if σ changes – and we have evidence that it has been increasing in recent years – it has a very significant impact on the production (and income) of an economy.

Note also that not only Y is a general mean of order one, but so is a variable of central importance, income per person, denoted $y \equiv Y/L$. Indeed, dividing

[1]The derivation of the production function can be found in [1]

A Conjecture on General Means

Olivier de La Grandville and Robert M. Solow both sides of (1) by L we have (dropping the subscript):

$$y = Y/L = [\delta r^p + (1-\delta)]^{\frac{1}{p}},$$

a general mean of r and 1 of order p. The above conjecture gives the mathematical reason of the considerable impact that a change in the elasticity of substitution in any given economy may have both on income per person and its growth rate.

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