

A VARIANT OF JESSEN'S INEQUALITY OF MERCER'S TYPE FOR SUPERQUADRATIC FUNCTIONS

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**Jessen's Inequality of Mercer's
Type and Superquadracity**

S. Abramovich, J. Barić
and J. Pečarić

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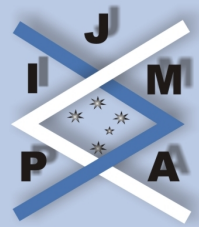
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Abstract:

A variant of Jessen's inequality for superquadratic functions is proved. This is a refinement of a variant of Jessen's inequality of Mercer's type for convex functions. The result is used to refine some comparison inequalities of Mercer's type between functional power means and between functional quasi-arithmetic means.



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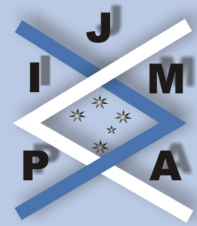
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1. Introduction

Let E be a nonempty set and L be a linear class of real valued functions $f : E \rightarrow \mathbb{R}$ having the properties:

L1: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

L2: $1 \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \rightarrow \mathbb{R}$ having the properties:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L, \alpha, \beta \in \mathbb{R}$ (A is linear);

A2: $f \in L, f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (A is isotonic).

The following result is Jessen's generalization of the well known Jensen's inequality for convex functions [10] (see also [12, p. 47]):

Theorem A. *Let L satisfy properties L1, L2 on a nonempty set E , and let φ be a continuous convex function on an interval $I \subset \mathbb{R}$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in I$, we have $A(g) \in I$ and*

$$\varphi(A(g)) \leq A(\varphi(g)).$$

Similar to Jensen's inequality, Jessen's inequality has a converse [7] (see also [12, p. 98]):

Theorem B. *Let L satisfy properties L1, L2 on a nonempty set E , and let φ be a convex function on an interval $I = [m, M]$, $-\infty < m < M < \infty$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in I$ so that $m \leq g(t) \leq M$ for all $t \in E$, we have*

$$A(\varphi(g)) \leq \frac{M - A(g)}{M - m} \cdot \varphi(m) + \frac{A(g) - m}{M - m} \cdot \varphi(M).$$

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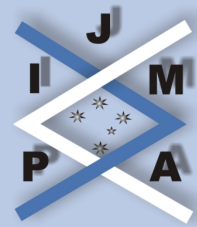
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Inspired by I.Gavrea's [9] result, which is a generalization of Mercer's variant of Jensen's inequality [11], recently, W.S. Cheung, A. Matković and J. Pečarić, [8] gave the following extension on a linear class L satisfying properties $L1, L2$.

Theorem C. *Let L satisfy properties $L1, L2$ on a nonempty set E , and let φ be a continuous convex function on an interval $I = [m, M]$, $-\infty < m < M < \infty$. If A is an isotonic linear functional on L with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m + M - g) \in L$ so that $m \leq g(t) \leq M$ for all $t \in E$, we have the following variant of Jessen's inequality*

$$(1.1) \quad \varphi(m + M - A(g)) \leq \varphi(m) + \varphi(M) - A(\varphi(g)).$$

In fact, to be more specific we have the following series of inequalities

$$(1.2) \quad \begin{aligned} &\varphi(m + M - A(g)) \\ &\leq A(\varphi(m + M - g)) \\ &\leq \frac{M - A(g)}{M - m} \cdot \varphi(M) + \frac{A(g) - m}{M - m} \cdot \varphi(m) \\ &\leq \varphi(m) + \varphi(M) - A(\varphi(g)). \end{aligned}$$

If the function φ is concave, inequalities (1.1) and (1.2) are reversed.

In this paper we give an analogous result for superquadratic function (see also different analogous results in [6]). We start with the following definition.

Definition A ([1, Definition 2.1]). *A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is **superquadratic** provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that*

$$(1.3) \quad \varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C(x)(y - x)$$

*for all $y \geq 0$. We say that f is **subquadratic** if $-f$ is a superquadratic function.*

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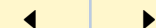
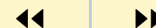
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For example, the function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in (0, 2]$.

Theorem D ([1, Theorem 2.3]). *The inequality*

$$f\left(\int g d\mu\right) \leq \int \left(f(g(s)) - f\left(\left|g(s) - \int g d\mu\right|\right)\right) d\mu(s)$$

holds for all probability measures μ and all non-negative μ -integrable functions g , if and only if f is superquadratic.

The following discrete version that follows from the above theorem is also used in the sequel.

Lemma A. *Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$ and let $\bar{x} = \sum_{r=1}^n \lambda_r x_r$ where $\lambda_r \geq 0$ and $\sum_{r=1}^n \lambda_r = 1$. Then*

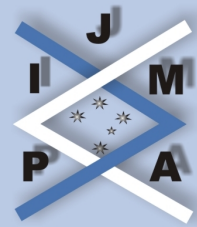
$$\sum_{r=1}^n \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^n \lambda_r f(|x_r - \bar{x}|).$$

In [3] and [4] the following converse of Jensen's inequality for superquadratic functions was proved.

Theorem E. *Let (Ω, A, μ) be a measurable space with $0 < \mu(r) < \infty$ and let $f : [0, \infty) \rightarrow \mathbb{R}$ be a superquadratic function. If $g : \Omega \rightarrow [m, M] \leq [0, \infty)$ is such that $g, f \circ g \in L_1(\mu)$, then we have*

$$\begin{aligned} \frac{1}{\mu(\Omega)} \int_{\Omega} f(g) d\mu &\leq \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) \\ &\quad - \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int_{\Omega} ((M - g) f(g - m) + (g - m) f(M - g)) d\mu, \end{aligned}$$

for $\bar{g} = \frac{1}{\mu(\Omega)} \int_{\Omega} g d\mu$.



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The discrete version of this theorem is:

Theorem F. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a superquadratic function. Let (x_1, \dots, x_n) be an n -tuple in $[m, M]^n$ ($0 \leq m < M < \infty$), and (p_1, \dots, p_n) be a non-negative n -tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Denote $\bar{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i$, then

$$\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \leq \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) \\ - \frac{1}{P_n (M - m)} \sum_{i=1}^n p_i [(M - x_i) f(x_i - m) + (x_i - m) f(M - x_i)].$$

In Section 2 we give the main result of our paper which is an analogue of Theorem C for superquadratic functions. In Section 3 we use that result to derive some refinements of the inequalities obtained in [8] which involve functional power means of Mercer's type and functional quasi-arithmetic means of Mercer's type.

2. Main Results

Theorem 2.1. Let L satisfy properties L1, L2, on a nonempty set E , $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a continuous superquadratic function, and $0 \leq m < M < \infty$. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $g \in L$ is such that $m \leq g(t) \leq M$, for all $t \in E$, and such that $\varphi(g)$, $\varphi(m+M-g)$, $(M-g)\varphi(g-m)$, $(g-m)\varphi(M-g) \in L$, then we have

$$\begin{aligned}
 & \varphi(m+M-A(g)) \\
 & \leq \frac{A(g)-m}{M-m} \varphi(m) + \frac{M-A(g)}{M-m} \varphi(M) \\
 & \quad - \frac{1}{M-m} [(A(g)-m)\varphi(M-A(g)) + (M-A(g))\varphi(A(g)-m)] \\
 (2.1) \quad & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\
 & \quad - \frac{1}{M-m} A((g-m)\varphi(M-g) + (M-g)\varphi(g-m)) \\
 & \quad - \frac{1}{M-m} [(A(g)-m)\varphi(M-A(g)) + (M-A(g))\varphi(A(g)-m)].
 \end{aligned}$$

If the function φ is subquadratic, then all the inequalities above are reversed.

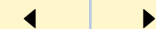
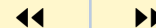
Proof. From Lemma A for $n = 2$, as well as from Theorem F, we get that for $0 \leq m \leq t \leq M$,

$$\begin{aligned}
 (2.2) \quad \varphi(t) & \leq \frac{M-t}{M-m} \varphi(m) + \frac{t-m}{M-m} \varphi(M) \\
 & \quad - \frac{M-t}{M-m} \varphi(t-m) - \frac{t-m}{M-m} \varphi(M-t).
 \end{aligned}$$



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Replacing t with $M + m - t$ in (2.2) it follows that

$$\begin{aligned} \varphi(M + m - t) &\leq \frac{t - m}{M - m} \varphi(m) + \frac{M - t}{M - m} \varphi(M) \\ &\quad - \frac{t - m}{M - m} \varphi(M - t) - \frac{M - t}{M - m} \varphi(t - m) \\ &= \varphi(m) + \varphi(M) - \left[\frac{t - m}{M - m} \varphi(M) + \frac{M - t}{M - m} \varphi(m) \right] \\ &\quad - \frac{t - m}{M - m} \varphi(M - t) - \frac{M - t}{M - m} \varphi(t - m). \end{aligned}$$

Since $m \leq g(t) \leq M$ for all $t \in E$, it follows that $m \leq A(g) \leq M$ and we have

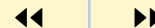
$$\begin{aligned} (2.3) \quad \varphi(m + M - A(g)) &\leq \varphi(m) + \varphi(M) - \left[\frac{A(g) - m}{M - m} \varphi(M) + \frac{M - A(g)}{M - m} \varphi(m) \right] \\ &\quad - \frac{A(g) - m}{M - m} \varphi(M - A(g)) - \frac{M - A(g)}{M - m} \varphi(A(g) - m). \end{aligned}$$

On the other hand, since $m \leq g(t) \leq M$ for all $t \in E$ it follows that

$$\begin{aligned} \varphi(g(t)) &\leq \frac{M - g(t)}{M - m} \varphi(m) + \frac{g(t) - m}{M - m} \varphi(M) \\ &\quad - \frac{M - g(t)}{M - m} \varphi(g(t) - m) - \frac{g(t) - m}{M - m} \varphi(M - g(t)). \end{aligned}$$

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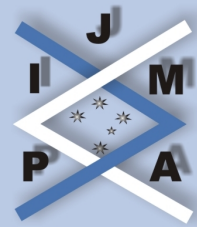


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Using functional calculus we have

$$(2.4) \quad A(\varphi(g)) \leq \frac{M - A(g)}{M - m} \varphi(m) \\ + \frac{A(g) - m}{M - m} \varphi(M) - \frac{1}{M - m} A((M - g(t))\varphi(g(t) - m)) \\ - \frac{1}{M - m} A((g(t) - m)\varphi(M - g(t))).$$

Using inequalities (2.3) and (2.4), we obtain the desired inequality (2.1).

The last statement follows immediately from the fact that if φ is subquadratic then $-\varphi$ is a superquadratic function. \square

Remark 1. If a function φ is superquadratic and nonnegative, then it is convex [1, Lema 2.2]. Hence, in this case inequality (2.1) is a refinement of inequality (1.1).

On the other hand, we can get one more inequality in (2.1) if we use a result of S. Banić and S. Varošaneć [5] on Jessen's inequality for superquadratic functions:

Theorem 2.2 ([5, Theorem 8, Remark 1]). *Let L satisfy properties L1, L2, on a nonempty set E , and let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a continuous superquadratic function. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $f \in L$ is nonnegative and such that $\varphi(f), \varphi(|f - A(f)|) \in L$, then we have*

$$(2.5) \quad \varphi(A(f)) \leq A(\varphi(f)) - A(\varphi(|f - A(f)|)).$$

If the function φ is subquadratic, then the inequality above is reversed.

Using Theorem 2.2 and some basic properties of superquadratic functions we prove the next theorem.

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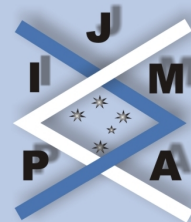
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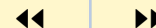
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Theorem 2.3. Let L satisfy properties $L1, L2$, on a nonempty set E , and let $\varphi : [0, \infty) \rightarrow \mathbb{R}$ be a continuous superquadratic function, and let $0 \leq m < M < \infty$. Assume that A is an isotonic linear functional on L with $A(1) = 1$. If $g \in L$ is such that $m \leq g(t) \leq M$, for all $t \in E$, and such that $\varphi(g), \varphi(m + M - g), (M - g)\varphi(g - m), (g - m)\varphi(M - g), \varphi(|g - A(g)|) \in L$, then we have

$$(2.6) \quad \begin{aligned} & \varphi(m + M - A(g)) \\ & \leq A(\varphi(m + M - g)) - A(\varphi(|g - A(g)|)) \end{aligned}$$

$$(2.7) \quad \begin{aligned} & \leq \frac{A(g) - m}{M - m} \varphi(m) + \frac{M - A(g)}{M - m} \varphi(M) \\ & \quad - \frac{1}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) \\ & \quad - A(\varphi(|g - A(g)|)) \end{aligned}$$

$$(2.8) \quad \begin{aligned} & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \frac{2}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) \\ & \quad - A(\varphi(|g - A(g)|)). \end{aligned}$$

If the function φ is subquadratic, then all the inequalities above are reversed.

Proof. Notice that $(m + M - g) \in L$. Since $m \leq g(t) \leq M$ for all $t \in E$, it follows that $m \leq m + M - g(t) \leq M$ for all $t \in E$. Applying (2.5) to the function $f = m + M - g$ we get

$$\begin{aligned} & \varphi(A(m + M - g)) \\ & = \varphi(m + M - A(g)) \\ & \leq A(\varphi(m + M - g)) - A(\varphi(|m + M - g - A(m + M - g)|)) \end{aligned}$$

$$\begin{aligned}
&= A(\varphi(m + M - g)) - A(\varphi(|m + M - g - m - M + A(g)|)) \\
&= A(\varphi(m + M - g)) - A(\varphi(|g - A(g)|)),
\end{aligned}$$

which is the inequality (2.6).

From the discrete Jensen's inequality for superquadratic functions we get for all $m \leq x \leq M$,

$$\begin{aligned}
(2.9) \quad \varphi(x) &\leq \frac{M-x}{M-m}\varphi(m) + \frac{x-m}{M-m}\varphi(M) \\
&\quad - \frac{M-x}{M-m}\varphi(x-m) - \frac{x-m}{M-m}\varphi(M-x).
\end{aligned}$$

Replacing x in (2.9) with $m + M - g(t) \in [m, M]$ for all $t \in E$, we have

$$\begin{aligned}
\varphi(m + M - g(t)) &\leq \frac{g(t) - m}{M - m}\varphi(m) + \frac{M - g(t)}{M - m}\varphi(M) \\
&\quad - \frac{g(t) - m}{M - m}\varphi(M - g(t)) - \frac{M - g(t)}{M - m}\varphi(g(t) - m).
\end{aligned}$$

Since A is linear, isotonic and satisfies $A(1) = 1$, from the above inequality it follows that

$$\begin{aligned}
(2.10) \quad A(\varphi(m + M - g)) &\leq \frac{A(g) - m}{M - m}\varphi(m) + \frac{M - A(g)}{M - m}\varphi(M) \\
&\quad - \frac{1}{M - m}A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)).
\end{aligned}$$

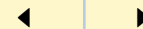
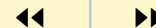
Adding $-A(\varphi(|g - A(g)|))$ on both sides of (2.10) we get

$$\begin{aligned}
(2.11) \quad A(\varphi(m + M - g)) - A(\varphi(|g - A(g)|)) \\
\leq \frac{A(g) - m}{M - m}\varphi(m) + \frac{M - A(g)}{M - m}\varphi(M)
\end{aligned}$$



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$$- \frac{1}{M-m} A((g-m)\varphi(M-g) + (M-g)\varphi(g-m)) - A(\varphi(|g-A(g)|)),$$

which is the inequality (2.7).

The right hand side of (2.11) can be written as follows

$$(2.12) \quad \varphi(m) + \varphi(M) - \frac{M-A(g)}{M-m} \varphi(m) - \frac{A(g)-m}{M-m} \varphi(M) \\ - \frac{1}{M-m} A((g-m)\varphi(M-g) + (M-g)\varphi(g-m)) - A(\varphi(|g-A(g)|)).$$

On the other hand, replacing x , in (2.9), with $g(t) \in [m, M]$, for all $t \in E$, we get

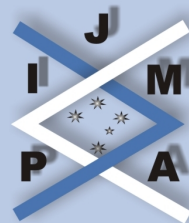
$$(2.13) \quad \varphi(g(t)) \leq \frac{M-g(t)}{M-m} \varphi(m) + \frac{g(t)-m}{M-m} \varphi(M) \\ - \frac{M-g(t)}{M-m} \varphi(g(t)-m) - \frac{g(t)-m}{M-m} \varphi(M-g(t)).$$

Applying the functional A on (2.13) we have

$$(2.14) \quad A(\varphi(g)) \leq \frac{M-A(g)}{M-m} \varphi(m) + \frac{A(g)-m}{M-m} \varphi(M) \\ - \frac{1}{M-m} A((M-g)\varphi(g-m) + (g-m)\varphi(M-g)),$$

The inequality (2.14) can be written as follows

$$- \frac{M-A(g)}{M-m} \varphi(m) - \frac{A(g)-m}{M-m} \varphi(M) \\ \leq -A(\varphi(g)) - \frac{1}{M-m} A((g-m)\varphi(M-g) + (M-g)\varphi(g-m)).$$



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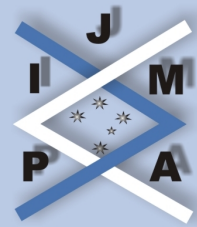
Using (2.12) we get

$$\begin{aligned} & \frac{A(g) - m}{M - m} \varphi(m) + \frac{M - A(g)}{M - m} \varphi(M) \\ & \quad - \frac{1}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) - A(\varphi(|g - A(g)|)) \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \frac{1}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) \\ & \quad - \frac{1}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) - A(\varphi(|g - A(g)|)) \\ & = \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \frac{2}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) - A(\varphi(|g - A(g)|)). \end{aligned}$$

Now, it follows that

$$\begin{aligned} & \frac{A(g) - m}{M - m} \varphi(m) + \frac{M - A(g)}{M - m} \varphi(M) \\ & \quad - \frac{1}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) - A(\varphi(|g - A(g)|)) \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \frac{2}{M - m} A((g - m)\varphi(M - g) + (M - g)\varphi(g - m)) - A(\varphi(|g - A(g)|)), \end{aligned}$$

which is the inequality (2.8). □



3. Applications

Throughout this section we suppose that:

- (i) L is a linear class having properties $L1, L2$ on a nonempty set E .
- (ii) A is an isotonic linear functional on L such that $A(1) = 1$.
- (iii) $g \in L$ is a function of E to $[m, M]$ ($0 < m < M < \infty$) such that all of the following expressions are well defined.

Let ψ be a continuous and strictly monotonic function on an interval $I = [m, M]$, ($0 < m < M < \infty$).

For any $r \in \mathbb{R}$, a power mean of Mercer's type functional

$$Q(r, g) := \begin{cases} [m^r + M^r - A(g^r)]^{\frac{1}{r}}, & r \neq 0 \\ \frac{mM}{\exp(A(\log g))}, & r = 0, \end{cases}$$

and a quasi-arithmetic mean functional of Mercer's type

$$\widetilde{M}_\psi(g, A) = \psi^{-1}(\psi(m) + \psi(M) - A(\psi(g)))$$

are defined in [8] and the following theorems are proved.

Theorem G. *If $r, s \in \mathbb{R}$ and $r \leq s$, then*

$$Q(r, g) \leq Q(s, g).$$

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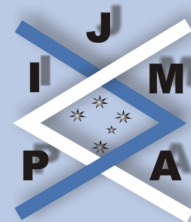
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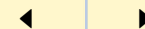
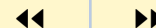
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Theorem H.

(i) If either $\chi \circ \psi^{-1}$ is convex and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is concave and χ is strictly decreasing, then

$$(3.1) \quad \widetilde{M}_\psi(g, A) \leq \widetilde{M}_\chi(g, A).$$

(ii) If either $\chi \circ \psi^{-1}$ is concave and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is convex and χ is strictly decreasing, then the inequality (3.1) is reversed.

Applying the inequality (2.1) to the adequate superquadratic functions we shall give some refinements of the inequalities in Theorems G and H. To do this, we will define following functions.

$$\begin{aligned} \diamond(m, M, r, s, g, A) &= \frac{1}{M^r - m^r} A((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}) \\ &\quad + \frac{1}{M^r - m^r} A((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}) \\ &\quad + \frac{1}{M^r - m^r} (A(g^r) - m^r) (M^r - A(g^r))^{\frac{s}{r}} \\ &\quad + \frac{1}{M^r - m^r} (M^r - A(g^r)) (A(g^r) - m^r)^{\frac{s}{r}}. \end{aligned}$$

and

$$\begin{aligned} &\diamond(m, M, \psi, \chi, g, A) \\ &= \frac{1}{\psi(M) - \psi(m)} A((\psi(M) - \psi(g))\chi(\psi^{-1}(\psi(g) - \psi(m)))) \\ &\quad + \frac{1}{\psi(M) - \psi(m)} A((\psi(g) - \psi(m))\chi(\psi^{-1}(\psi(M) - \psi(g)))) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\psi(M) - \psi(m)} (A(\psi(g)) - \psi(m)) \chi(\psi^{-1}(\psi(M) - A(\psi(g)))) \\
& + \frac{1}{\psi(M) - \psi(m)} (\psi(M) - A(\psi(g))) \chi(\psi^{-1}(A(\psi(g)) - \psi(m))).
\end{aligned}$$

Now, the following theorems are valid.

Theorem 3.1. *Let $r, s \in \mathbb{R}$.*

(i) *If $0 < 2r \leq s$, then*

$$(3.2) \quad Q(r, g) \leq [(Q(s, g))^s - \diamond(m, M, r, s, g, A)]^{\frac{1}{s}}.$$

(ii) *If $2r \leq s < 0$, then for $(Q(s, g))^s - \diamond(M, m, r, s, g, A) > 0$*

$$(3.3) \quad Q(r, g) \leq [(Q(s, g))^s - \diamond(M, m, r, s, g, A)]^{\frac{1}{s}},$$

where we used $\diamond(M, m, r, s, g, A)$ to denote the new function derived from the function $\diamond(m, M, r, s, g, A)$ by changing the places of m and M .

(iii) *If $0 < s \leq 2r$, then for $(Q(s, g))^s - \diamond(M, m, r, s, g, A) > 0$ the reverse inequality (3.2) holds.*

(iv) *If $s \leq 2r < 0$, then the reversed inequality (3.3) holds.*

Proof.

(i) It is given that

$$0 < m \leq g \leq M < \infty.$$

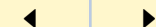
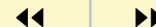
Since $0 < 2r \leq s$, it follows that

$$0 < m^r \leq g^r \leq M^r < \infty.$$



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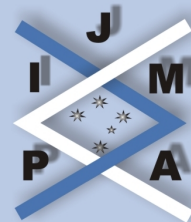


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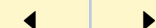
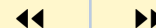
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Applying Theorem 2.1, or more precisely inequality (2.1) to the superquadratic function $\varphi(t) = t^{\frac{s}{r}}$ (note that $\frac{s}{r} \geq 2$ here) and replacing g, m and M with g^r, m^r and M^r , respectively, we have

$$\begin{aligned} & [m^r + M^r - A(g^r)]^{\frac{s}{r}} + \frac{1}{M^r - m^r} (A(g^r) - m^r) (M^r - A(g^r))^{\frac{s}{r}} \\ & \quad + \frac{1}{M^r - m^r} (M^r - A(g^r)) (A(g^r) - m^r)^{\frac{s}{r}} \\ & \leq m^s + M^s - A(g^s) \\ & \quad - \frac{1}{M^r - m^r} A((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}) \\ & \quad - \frac{1}{M^r - m^r} A((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}). \end{aligned}$$

i.e.

$$(3.4) \quad [Q(r, g)]^s \leq [Q(s, g)]^s - \diamond(m, M, r, s, g, A).$$

Raising both sides of (3.4) to the power $\frac{1}{s} > 0$, we get desired inequality (3.2).

(ii) In this case we have

$$0 < M^r \leq g^r \leq m^r < \infty.$$

Applying Theorem 2.1 or, more precisely, the reversed inequality (2.1) to the subquadratic function $\varphi(t) = t^{\frac{s}{r}}$ (note that now we have $0 < \frac{s}{r} \leq 2$) and replacing g, m and M with g^r, m^r and M^r , respectively, we get

$$\begin{aligned} & [M^r + m^r - A(g^r)]^{\frac{s}{r}} + \frac{1}{m^r - M^r} (A(g^r) - M^r) (m^r - A(g^r))^{\frac{s}{r}} \\ & \quad + \frac{1}{m^r - M^r} (m^r - A(g^r)) (A(g^r) - M^r)^{\frac{s}{r}} \end{aligned}$$

$$\begin{aligned} &\geq M^s + m^s - A(g^s) - \frac{1}{m^r - M^r} A((m^r - g^r)(g^r - M^r)^{\frac{s}{r}}) \\ &\quad - \frac{1}{m^r - M^r} A((g^r - M^r)(m^r - g^r)^{\frac{s}{r}}). \end{aligned}$$

Since $2r \leq s < 0$, raising both sides to the power $\frac{1}{s}$, it follows that

$$[M^r + m^r - A(g^r)]^{\frac{1}{r}} \leq [M^s + m^s - A(g^s) - \diamond(M, m, r, s, g, A)]^{\frac{1}{s}},$$

or

$$Q(r, g) \leq [(Q(s, g))^s - \diamond(M, m, r, s, g, A)]^{\frac{1}{s}}.$$

(iii) In this case we have $0 < \frac{s}{r} \leq 2$. Since $0 < m^r \leq g^r \leq M^r < \infty$, we can apply Theorem 2.1, or more precisely, the reversed inequality (2.1) to the subquadratic function $\varphi(t) = t^{\frac{s}{r}}$. Replacing g, m and M with g^r, m^r and M^r , respectively, it follows that

$$\begin{aligned} &[m^r + M^r - A(g^r)]^{\frac{s}{r}} \\ &\quad + \frac{1}{M^r - m^r} (A(g^r) - m^r) (M^r - A(g^r))^{\frac{s}{r}} \\ &\quad + \frac{1}{M^r - m^r} (M^r - A(g^r)) (A(g^r) - m^r)^{\frac{s}{r}} \\ &\geq m^s + M^s - A(g^s) \\ &\quad - \frac{1}{M^r - m^r} A((M^r - g^r)(g^r - m^r)^{\frac{s}{r}}) \\ &\quad - \frac{1}{M^r - m^r} A((g^r - m^r)(M^r - g^r)^{\frac{s}{r}}), \end{aligned}$$

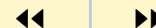
i.e.

$$(3.5) \quad [Q(r, g)]^s \geq [Q(s, g)]^s - \diamond(m, M, r, s, g, A).$$



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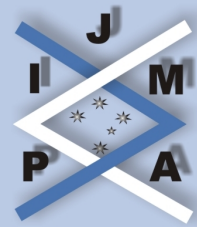


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Raising both sides of (3.5) to the power $\frac{1}{s} > 0$ we get

$$Q(r, g) \geq [(Q(s, g))^s - \diamond(m, M, r, s, g, A)]^{\frac{1}{s}}.$$

(iv) Since $r < 0$, from $0 < m \leq g \leq M < \infty$ it follows that $0 < M^r \leq g^r \leq m^r < \infty$. Now, we are applying Theorem 2.1 to the superquadratic function $\varphi(t) = t^{\frac{s}{r}}$, because $\frac{s}{r} \geq 2$ here, and analogous to the previous theorem we get

$$[Q(r, g)]^s \leq [Q(s, g)]^s - \diamond(M, m, r, s, g, A).$$

Raising both sides to the power $\frac{1}{s} < 0$ it follows that

$$Q(r, g) \geq [(Q(s, g))^s - \diamond(M, m, r, s, g, A)]^{\frac{1}{s}}. \quad \square$$

Theorem 3.2. Let $r, s \in \mathbb{R}$.

(i) If $0 < 2s \leq r$, then

$$(3.6) \quad Q(r, g) \geq [(Q(s, g))^r + \diamond(m, M, s, r, g, A)]^{\frac{1}{r}},$$

where we used $\diamond(m, M, s, r, g, A)$ to denote the new function derived from the function $\diamond(m, M, r, s, g, A)$ by changing the places of r and s .

(ii) If $2s \leq r < 0$, then

$$(3.7) \quad Q(r, g) \leq [(Q(s, g))^r + \diamond(M, m, s, r, g, A)]^{\frac{1}{r}}.$$

(iii) If $0 < r \leq 2s$, then the reversed inequality (3.6) holds.

(iv) If $r \leq 2s < 0$, then the reversed inequality (3.7) holds.

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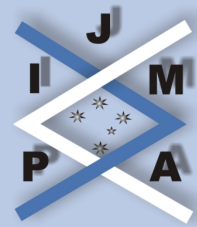
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Proof.

- (i) Applying inequality (2.1) to the superquadratic function $\varphi(t) = t^{\frac{r}{s}}$ (note that $\frac{r}{s} \geq 2$ here) and replacing g , m and M with g^s , m^s and M^s , ($0 < m^s \leq g^s \leq M^s < \infty$) respectively, we have

$$\begin{aligned} & [m^s + M^s - A(g^s)]^{\frac{r}{s}} \\ & + \frac{1}{M^s - m^s} (A(g^s) - m^s) (M^s - A(g^s))^{\frac{r}{s}} \\ & + \frac{1}{M^s - m^s} (M^s - A(g^s)) (A(g^s) - m^s)^{\frac{r}{s}} \\ & \geq m^r + M^r - A(g^r) \\ & - \frac{1}{M^s - m^s} A((M^s - g^s)(g^s - m^s)^{\frac{r}{s}}) \\ & - \frac{1}{M^s - m^s} A((g^s - m^s)(M^s - g^s)^{\frac{r}{s}}), \end{aligned}$$

i.e.

$$[Q(s, g)]^r \leq [Q(r, g)]^r - \diamond(m, M, s, r, g, A).$$

Raising both sides to the power $\frac{1}{r} > 0$, the inequality (3.6) follows.

- (ii) Since $s < 0$, we have $0 < M^s \leq g^s \leq m^s < \infty$ so the function \diamond will be of the form $\diamond(M, m, s, r, g, A)$. Since $0 < \frac{r}{s} \leq 2$, we will apply Theorem 2.1 to the subquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and, as in previous case, it follows that

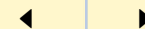
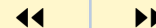
$$[Q(s, g)]^r + \diamond(M, m, s, r, g, A) \geq [Q(r, g)]^r.$$

Raising both sides to the power $\frac{1}{r} < 0$, the inequality (3.7) follows.



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(iii) Since $0 < \frac{r}{s} \leq 2$, we will apply Theorem 2.1 to the subquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and then it follows that

$$[Q(s, g)]^r + \diamond(m, M, s, r, g, A) \geq [Q(r, g)]^r.$$

Raising both sides to the power $\frac{1}{r} > 0$, we get

$$Q(r, g) \leq [(Q(s, g)]^r + \diamond(m, M, s, r, g, A)]^{\frac{1}{r}}.$$

(iv) Since $\frac{r}{s} \geq 2$, we will apply Theorem 2.1 to the superquadratic function $\varphi(t) = t^{\frac{r}{s}}$ and use the function $\diamond(M, m, s, r, g, A)$ instead of $\diamond(m, M, s, r, g, A)$. Then we get

$$[Q(s, g)]^r + \diamond(M, m, s, r, g, A) \leq [Q(r, g)]^r.$$

Raising both sides to the power $\frac{1}{r} < 0$, it follows that

$$Q(r, g) \geq [(Q(s, g)]^r + \diamond(M, m, s, r, g, A)]^{\frac{1}{r}}. \quad \square$$

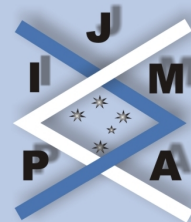
Remark 2. Notice that some cases in the last theorems have common parts. In some of them we can establish double inequalities. For example, if $0 < r \leq 2s$ and $0 < s \leq 2r$, then for $(Q(s, g)]^s - \diamond(M, m, r, s, g, A) > 0$

$$[(Q(s, g)]^r + \diamond(m, M, s, r, g, A)]^{\frac{1}{r}} \geq Q(r, g) \geq [(Q(s, g)]^s - \diamond(m, M, r, s, g, A)]^{\frac{1}{s}}.$$

Theorem 3.3. Let $\psi \in C([m, M])$ be strictly increasing and let $\chi \in C([m, M])$ be strictly monotonic functions.

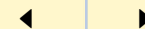
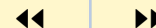
(i) If either $\chi \circ \psi^{-1}$ is superquadratic and χ is strictly increasing, or $\chi \circ \psi^{-1}$ is subquadratic and χ is strictly decreasing, then

$$(3.8) \quad \widetilde{M}_{\psi}(g, A) \leq \chi^{-1} \left(\chi \left(\widetilde{M}_{\chi}(g, A) \right) - \diamond(m, M, \psi, \chi, g, A) \right),$$



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(ii) If either $\chi \circ \psi^{-1}$ is subquadratic and χ is strictly increasing or $\chi \circ \psi^{-1}$ is superquadratic and χ is strictly decreasing, then the inequality (3.8) is reversed.

Proof. Suppose that $\chi \circ \psi^{-1}$ is superquadratic. Letting $\varphi = \chi \circ \psi^{-1}$ in Theorem 2.1 and replacing g, m and M with $\psi(g), \psi(m)$ and $\psi(M)$ respectively, we have

$$\begin{aligned} & \chi(\psi^{-1}(\psi(m) + \psi(M) - A(\psi(g)))) \\ & + \frac{1}{\psi(M) - \psi(m)} ((A(\psi(g)) - \psi(m)) \chi(\psi^{-1}(\psi(M) - A(\psi(g)))))) \\ & + \frac{1}{\psi(M) - \psi(m)} ((\psi(M) - A(\psi(g))) \chi(\psi^{-1}(A(\psi(g)) - \psi(m)))) \\ & \leq \chi(m) + \chi(M) - A(\chi(g)) \\ & - \frac{1}{\psi(M) - \psi(m)} A((\psi(M) - \psi(m)) \chi(\psi^{-1}(\psi(g) - \psi(m)))) \\ & - \frac{1}{\psi(M) - \psi(m)} A((\psi(g) - \psi(m)) \chi(\psi^{-1}(\psi(M) - \psi(g)))) , \end{aligned}$$

i.e.,

$$\begin{aligned} (3.9) \quad & \chi(\widetilde{M}_\psi(g, A)) \\ & \leq \chi(m) + \chi(M) - A(\chi(g)) - \diamond(m, M, \psi, \chi, g, A) \\ & \leq \chi \circ \chi^{-1}(\chi(m) + \chi(M) - A(\chi(g))) - \diamond(m, M, \psi, \chi, g, A) \\ & \leq \chi(\widetilde{M}_\chi(g, A)) - \diamond(m, M, \psi, \chi, g, A). \end{aligned}$$

If χ is strictly increasing, then the inverse function χ^{-1} is also strictly increasing and inequality (3.9) implies the inequality (3.8). If χ is strictly decreasing, then the inverse function χ^{-1} is also strictly decreasing and in that case the reverse of (3.9)

implies (3.8). Analogously, we get the reverse of (3.8) in the cases when $\chi \circ \psi^{-1}$ is superquadratic and χ is strictly decreasing, or $\chi \circ \psi^{-1}$ is subquadratic and χ is strictly increasing. \square

Remark 3. If the function ψ in Theorem 3.3 is strictly decreasing, then the inequality (3.8) and its reversal also hold under the same assumptions, but with m and M interchanged.

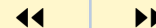
Remark 4. Obviously, Theorem 3.1 and Theorem 3.2 follow from Theorem 3.3 and Remark 3 by choosing $\psi(t) = t^r$ and $\chi(t) = t^s$, or vice versa.



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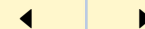
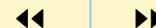
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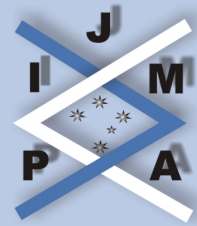
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