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MEROMORPHIC FUNCTIONS THAT SHARE ONE VALUE WITH THEIR DERIVATIVES

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ABSTRACT. In this paper, we deal with the problems of uniqueness of meromorphic functions that share one finite value with their derivatives and obtain some results that improve the results given by Rainer Brück and Qingcai Zhang.

Key words and phrases: Meromorphic functions, Uniqueness, Sharing values.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, a meromorphic function will mean meromorphic in the finite complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We say that f and g share ∞ CM provided that $1/f$ and $1/g$ share 0 CM. It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna Theory, as found in [3, 6].

Let $f(z)$ be a meromorphic function. It is known that the hyper-order of $f(z)$, denoted by $\sigma_2(f)$, is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 1996, R. Brück posed the following conjecture (see [1]).

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Conjecture 1.1. *Let f be a non-constant entire function such that the hyper-order $\sigma_2(f)$ of f is not a positive integer and $\sigma_2(f) < +\infty$. If f and f' share a finite value a CM, then*

$$\frac{f'-a}{f-a} = c,$$

where c is nonzero constant.

In [1], Brück proved that the conjecture holds when $a = 0$. In 1998, Gundersen and Yang [2] proved that the conjecture is true when f is of finite order. In 1999, Yang [4] confirmed that the conjecture is also true when f' is replaced by $f^{(k)}$ $(k \geq 2)$ and f is of finite order.

In 1996, Brück obtained the following result.

Theorem A ([1])**.** *Let f be a nonconstant entire function. If f and f' share the value 1 CM, and if $N\left(r, \frac{1}{f'}\right) = S(r,f)$, then*

$$\frac{f'-1}{f-1} \equiv c$$

for a non-zero constant c.

In 1998, Q. Zhang proved the next two results in [7].

Theorem B. *Let f be a nonconstant meromorphic function. If f and f' share the value 1 CM, and if*

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1))T(r,f'), \quad \left(0 < \lambda < \frac{1}{2}\right),$$

then

$$\frac{f'-1}{f-1} \equiv c$$

for some non-zero constant c.

Theorem C. *Let f be a nonconstant meromorphic function, k be a positive integer. If f and $f^{(k)}$ share the value 1 CM, and if*

$$2\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{f^{(k)}}\right) < (\lambda + o(1))T(r,f^{(k)}), \quad (0 < \lambda < 1),$$

then

$$\frac{f^{(k)}-1}{f-1} \equiv c$$

for some non-zero constant c.

The above results suggest the following question: What results can be obtained if the condition that f and f' share the value 1 CM is replaced by the condition that f and f' share the value 1 IM?

In this paper, we obtained the following results.

Theorem 1.2. *Let f be a nonconstant meromorphic function, if f and f' share the value 1 IM, and if*

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1))T(r,f') \qquad \left(0 < \lambda < \frac{1}{4}\right),$$

then

$$\frac{f'-1}{f-1} \equiv c$$

for some non-zero constant c.

Corollary 1.3. *Let f be a nonconstant entire function. If f and f' share the value 1 IM, and if*

$$\overline{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1))T(r, f'), \qquad \left(0 < \lambda < \frac{1}{4}\right),$$

then

$$\frac{f' - 1}{f - 1} \equiv c$$

for some non-zero constant c.

Theorem 1.4. *Let f be a nonconstant meromorphic function, k be a positive integer. If f and $f^{(k)}$ share the value 1 IM, and if*

$$(3k + 6)\overline{N}(r, f) + 5N\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f^k), \quad (0 < \lambda < 1),$$

then

$$\frac{f^{(k)} - 1}{f - 1} \equiv c$$

for some non-zero constant c.

Corollary 1.5. *Let f be a nonconstant entire function. If f and $f^{(k)}$ share the value 1 IM, and if*

$$N\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f), \quad \left(0 < \lambda < \frac{1}{10}\right),$$

then

$$\frac{f^{(k)} - 1}{f - 1} \equiv c$$

for some non-zero constant c.

Theorem 1.6. *Let f be a nonconstant meromorphic function, k be a positive integer. If f and $f^{(k)}$ share the value $a \neq 0$ CM, and satisfy one of the following conditions,*

(i) $\delta(0, f) + \Theta(\infty, f) > \frac{4k}{2k+1}$,

(ii) $\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f), \quad \left(0 < \lambda < \frac{2}{2k+1}\right)$,

(iii) $\left(k + \frac{1}{2}\right)\overline{N}(r, f) + \frac{3}{2}N\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f), \quad (0 < \lambda < 1)$.

Then $f \equiv f^{(k)}$.

Theorem 1.7. *Let f be a nonconstant meromorphic function. If f and f' share the value $a \neq 0$ IM, and if*

$$\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f), \quad \left(0 < \lambda < \frac{2}{3}\right),$$

then $f \equiv f'$.

2. SOME LEMMAS

Lemma 2.1 ([7]). *Let f be a nonconstant meromorphic function, k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) < N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f), \tag{2.1}$$

$$N\left(r, \frac{f^{(k)}}{f}\right) < k\overline{N}(r, f) + k\overline{N}\left(r, \frac{1}{f}\right) + S(r, f), \tag{2.2}$$

$$N\left(r, \frac{f^{(k)}}{f}\right) < k\overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.3}$$

Suppose that f and g share the value a IM, and let z_0 be a a-point of f of order p, a a-point of $f^{(k)}$ of order q. We denote by $N_L\left(r, \frac{1}{f^{(k)}-a}\right)$ the counting function of those a-points of $f^{(k)}$ where $q > p$.

Lemma 2.2. *Let f be a nonconstant meromorphic function. If f and $f^{(k)}$ share the value 1 IM, then*

$$\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) < \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}(r, f) + S(r, f). \tag{2.4}$$

Lemma 2.3 ([7])**.** *Let f be a nonconstant meromorphic function, k be a positive integer. If f and $f^{(k)}$ share the value 1 IM, then*

$$T(r, f) < 3T(r, f^{(k)}) + S(r, f),$$

specially if f is a nonconstant entire function, then

$$T(r, f) < 2T(r, f^{(k)}) + S(r, f).$$

3. PROOF OF THEOREM 1.2

Let $N_{1)}\left(r, \frac{1}{f-a}\right)$ denote the counting function of the simple zeros of $f-a$, $\overline{N}_{(2}\left(r, \frac{1}{f-a}\right)$ denote the counting function of the multiple a-points of f. Each point in these counting functions is counted only once. We denote by $N_2\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$, where a simple zero is counted once and a multiple zero is counted twice. It follows that

$$N_2\left(r, \frac{1}{f-a}\right) = N_{1)}\left(r, \frac{1}{f-a}\right) + 2\overline{N}_{(2}\left(r, \frac{1}{f-a}\right). \tag{3.1}$$

Set

$$F = \frac{f'''}{f''} - \frac{2f''}{f'-1} - \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right).$$

We suppose that $F \not\equiv 0$. By the lemma of logarithmic derivatives, we have

$$m(r, F) = S(r, f) \tag{3.2}$$

and

$$N(r, F) \le \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}_{(2}\left(r, \frac{1}{f'-1}\right) + \overline{N}_0\left(r, \frac{1}{f''}\right) + S(r, f), \tag{3.3}$$

where $\overline{N}_{(2}\left(r, \frac{1}{f'-1}\right)$ denotes the counting function of multiple 1-points of f', and each 1-point is counted only once; $\overline{N}_0\left(r, \frac{1}{f''}\right)$ denotes the counting functions of f'' which are not the zeros of f' and $f'-1$.

Since f and f' share the value 1 IM, we know that $f-1$ has only simple zeros. If $f'-1$ also has only simple zeros, then f and f' share the value 1 CM, and Theorem 1.2 follows by the conclusion of Theorem B.

Now we assume that $f'-1$ has multiple zeros. By calculation, we know that the common simple zeros of $f-1$ and $f'-1$ are the zeros of F; we denote by $N_E^{1)}\left(r,\frac{1}{f-1}\right)$ the counting function of common simple zeros of $f-1$ and $f'-1$. It follows that

$$N_E^{1)}\left(r,\frac{1}{f-1}\right) \le N\left(r,\frac{1}{F}\right) \le T(r,F) = N(r,F)+S(r,f). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{f-1}\right) \le{}& \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right) \\ &+\overline{N}_{(2}\left(r,\frac{1}{f'-1}\right)+\overline{N}_0\left(r,\frac{1}{f''}\right)+S(r,f). \end{aligned} \tag{3.5}$$

Notice that

$$\overline{N}\left(r,\frac{1}{f'-1}\right) = N_E^{1)}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f'-1}\right). \tag{3.6}$$

By the second fundamental theorem, we have

$$T(r,f') < \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{f'-1}\right)-\overline{N}_0\left(r,\frac{1}{f''}\right)+S(r,f). \tag{3.7}$$

From Lemma 2.2,

$$\overline{N}_{(2}\left(r,\frac{1}{f'-1}\right) = \overline{N}_L\left(r,\frac{1}{f'-1}\right) < \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+S(r,f). \tag{3.8}$$

Combining (3.5), (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned} T(r,f') &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+N_E^{1)}\left(r,\frac{1}{f-1}\right) \\ &\qquad +\overline{N}_L\left(r,\frac{1}{f'-1}\right)-\overline{N}_0\left(r,\frac{1}{f''}\right)+S(r,f) \\ &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+2\overline{N}_L\left(r,\frac{1}{f'-1}\right)+S(r,f) \\ &\le 4\overline{N}(r,f)+4\overline{N}\left(r,\frac{1}{f'}\right)+S(r,f), \end{aligned}$$

which contradicts the condition of Theorem 1.2. Therefore, we have $F\equiv 0$. By integrating twice, we have

$$\frac{1}{f-1} = \frac{A}{f'-1}+B,$$

where $A\neq 0$ and B are constants.

We distinguish the following three cases.

Case 1. If $B\neq 0,-1$, then

$$f = \frac{(B+1)f'+(A-B-1)}{Bf'+(A-B)},$$

$$f' = \frac{(B-A)f+(A-B-1}{Bf-(B+1)},$$

and so

$$\overline{N}\left(r, \frac{1}{f' + \frac{A-B}{B}}\right) = \overline{N}(r, f)$$

By the second fundamental theorem

$$\begin{aligned} T(r, f') &< \overline{N}(r, f') + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{f' + \frac{A-B}{B}}\right) + S(r, f) \\ &< 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f'}\right) + S(r, f), \end{aligned}$$

which contradicts the assumption of Theorem 1.2.

Case 2. If $B = -1$, then

$$f = \frac{A}{-f' + (A-1)}$$

and so

$$\overline{N}\left(r, \frac{1}{f' - (A+1)}\right) = \overline{N}(r, f).$$

We also get a contradiction by the second fundamental theorem.

Case 3. If $B = 0$, it follows that

$$\frac{f' - 1}{f - 1} = A,$$

and the proof of Theorem 1.2 is thus complete.

4. Proof of Theorem 1.4

Let

$$F = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} - \frac{f''}{f'} + 2\frac{f'}{f-1}$$

We suppose that $F \not\equiv 0$. Since the common zeros (with the same multiplicities) of $f-1$ and $f^{(k)} - 1$ are not the poles of F, and the common simple zeros of $f - 1$ and $f^{(k)} - 1$ are the zeros of F, we have

$$N_E^{1)}\left(r, \frac{1}{f-1}\right) \le N\left(r, \frac{1}{F}\right) \le T(r, F) = N(r, F) + S(r, f). \tag{4.1}$$

and

$$\begin{aligned} N(r, F) \le{}& \overline{N}(r, f) + \overline{N}_L\left(r, \frac{1}{f-1}\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)} - 1}\right) \\ &+ \overline{N}_{(2}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}_0\left(r, \frac{1}{f'}\right) + \overline{N}_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \end{aligned} \tag{4.2}$$

where $\overline{N}_0\left(r, \frac{1}{f^{(k+1)}}\right)$ denotes the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f^{(k)}$ and $f^{(k)} - 1$, $\overline{N}_0\left(r, \frac{1}{f'}\right)$ denotes the counting function of the zeros of f' which are not the zeros of f. Since

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)=\overline{N}\left(r,\frac{1}{f-1}\right)$$
$$=N_E^{1)}\left(r,\frac{1}{f-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right),\tag{4.3}$$

we obtain from (4.1), (4.2) and (4.3) that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)&\le\overline{N}(r,f)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\&\quad+\overline{N}\left({}_0r,\frac{1}{f'}\right)+2\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)\\&\quad+\overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\&\le\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f'}\right)+2\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)\\&\quad+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f),\end{aligned}$$

where $\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)$ is the counting function of common multiple zeros of $f-1$ and $f^{(k)}-1$, each point is counted once. By the second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}T(r,f^{(k)})&\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-\overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\&\le\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f'}\right)+2\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)\\&\quad+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\&\le 2\overline{N}(r,f)+N\left(r,\frac{1}{f^{(k)}}\right)+2\overline{N}\left((r,\frac{1}{f'}\right)\\&\quad+2\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+2\overline{N}(r,f)+S(r,f)\\&\le(3k+6)\overline{N}(r,f)+5N\left(r,\frac{1}{f}\right)+S(r,f),\end{aligned}$$

which contradicts the assumption of Theorem 1.2. Hence $F\equiv 0$.

By integrating twice, we get

$$\frac{1}{f-1}=\frac{C}{f^{(k)}-1}+D,$$

where $C\neq 0$ and D are constants. By arguments similar to the proof of Theorem 1.2, Theorem 1.4 follows.

Remark 4.1. Let f be a non-constant entire function. Then we obtain from Lemma 2.3 that

$$\frac{1}{2}T(r,f) \leq T(r,f^{(k)}) + S(r,f).$$

By Theorem 1.4, Corollary 1.5 holds.

5. PROOF OF THEOREM 1.6 AND THEOREM 1.7

Suppose that $f \not\equiv f^{(k)}$. Let

$$F = \frac{f}{f^{(k)}}.$$

Then

$$T(r,F) = m\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{F}\right) = N\left(r,\frac{f^{(k)}}{f}\right) + S(r,f). \tag{5.1}$$

Since f and $f^{(k)}$ share the value $a \neq 0$ CM, we have

$$N\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{1}{f-f^{(k)}}\right) \leq N\left(r,\frac{1}{F-1}\right) \leq T(r,F) + O(1). \tag{5.2}$$

By the lemma of logarithmic derivatives and the second fundamental theorem, we obtain

$$m\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f-a}\right) < m\left(r,\frac{1}{f^{(k)}}\right) + S(r,f), \tag{5.3}$$

and

$$T\left(r,f^{(k)}\right) < \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}(r,f^{(k)}) + \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + S(r,f), \tag{5.4}$$

from (5.4), we have

$$m\left(r,\frac{1}{f^{(k)}}\right) < \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + S(r,f). \tag{5.5}$$

Combining with (5.1), (5.2), (5.3), (5.4), (5.5) and (2.2) of Lemma 2.1, we obtain

$$\begin{aligned}
2T(r,f) &\leq m\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-a}\right) + S(r,f)\\
&\leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-a}\right) + S(r,f)\\
&\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + 2N\left(r,\frac{1}{f-a}\right) + S(r,f)\\
&\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + 2N\left(r,\frac{f^{(k)}}{f}\right) + S(r,f)\\
&\leq \overline{N}(r,f) + 2(k\overline{N}(r,f) + k\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + S(r,f)\\
&\leq (2k+1)\overline{N}(r,f) + (2k+1)N\left(r,\frac{1}{f}\right) + S(r,f),
\end{aligned}$$

which contradicts the assumptions (i) and (ii) of Theorem 1.6. Hence $f \equiv f^{(k)}$. Similarly, by the above inequality and (2.3) of Lemma 2.1, and suppose that (iii) is satisfied, then we get a contradiction if $f \not\equiv f^{(k)}$, and we complete the proof of Theorem 1.6.

Remark 5.1. For a nonconstant meromorphic function f, if f and f' share the value $a \neq 0$ IM and $f \not\equiv f^{(k)}$, since a a-point of f is not a zero of f', we know that $f - a$ has only simple zeros, and we have

$$N\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{F-1}\right) \leq T(r, F) + O(1),$$

where $F = f/f'$. By the arguments similar to the proof of Theorem 1.6, Theorem 1.7 follows.

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