Journal of Integer Sequences, Vol. 10 (2007),
Article 07.1.4

Complementary Equations

Clark Kimberling
Department of Mathematics
University of Evansville
1800 Lincoln Avenue
Evansville, IN 47722
USA
ck6@evansville.edu

Abstract

Increasing sequences $a()$ and $b()$ that partition the sequence of positive integers are called complementary sequences, and equations that explicitly involve both $a()$ and $b()$ are called complementary equations. This article surveys several families of such equations, including $b(n) = a(jn) \pm r$, $b(n) = a(jn) + kn$, $b(n) = f(a(n))$, and $b(n) = a(b(n-1)) + qn + r$.

1 Introduction

Under the assumption that sequences a and b partition the sequence $N = (1, 2, 3, \ldots)$ of positive integers, the designation *complementary equations* applies to equations such as

$$b(n) = a(a(n)) + 1$$

in much the same way that the designations *functional equations*, *differential equations*, and *Diophantine equations* apply elsewhere. Indeed, complementary equations can be regarded as a class of Diophantine equations.

Various pairs of complementary sequences, such as Beatty sequences ([1, 21]), have been widely studied, as evidenced by many entries in the *Online Encyclopedia of Integer Sequences*[19]. In particular, complementary sequences have been discussed extensively by Fraenkel in connection with Beatty sequences, spectra of numbers, and combinatorial games; see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. See also [16, 18]. However, in the literature the recognition of a single equation involving both sequences tends to occur only parenthetically. The purpose of this paper is to recognize classes of such equations explicitly.

Throughout, the symbols a and b denote strictly increasing complementary sequences; i.e., every number in N is $a(n)$ or $b(n)$ for some n in N, and no term of a is also a term of b. An *ordinary complementary equation* (OCE) is defined by the form

$$b(n) = f(\widehat{a}(n), \widehat{b}(n), n), \tag{1}$$

where

$$\widehat{a}(n) = (a\,(1), a\,(2)\,, \ldots a(p(n))), \quad a(p(n)) < b(n),$$
$$\widehat{b}(n) = (b\,(1), b\,(2)\,, \ldots b(q(n))), \quad b(q(n)) < b(n).$$

That is, the nth term of the complement, b, of a, is determined as a function, f, of n and terms of a and b that are previously defined. (It is common and convenient to use time-suggestive descriptors such as *previously*, but we note that inductive definitions do not, in fact, depend on time.)

In some cases, an OCE (1) forces $a(1) = 1$, but, as in Example 3 below, this need not be the case. By decreeing an initial value, either $a(1) = 1$ or else $b(1) = 1$, the OCE must then have a unique solution b, or equivalently, a.

An equation involving both a sequence and its complement and which cannot be represented in the form (1) is a *partial complementary equation* (PCE). Typically, a PCE determines only a part of a solution; which is to say that there may be many solutions, as in Example 2 just below.

Example 1. The OCE $b(n) = a(a(n)) + 1$ has unique solution $a(n) = \lfloor n\tau \rfloor$, where $\tau = (1 + \sqrt{5})/2$. The complement is given by $b(n) = \lfloor n\tau \rfloor + n$, which is also the unique solution of the OCE $b(n) = a(n) + n$, an equation discussed more generally in section 4.

Example 2. Every solution of the equation

$$b(n) = b(n-1)a(n+1) \tag{2}$$

has $a(1) = 1$. However, the number 2 could be either $a(2)$ or $b(1)$, so that (2) is a PCE.

Example 3. The OCE

$$b(n) = a(n-1) + b(n-1) \tag{3}$$

for $n \geq 2$, with initial condition $b(1) = 1$, has as solution the Hofstadter sequence A005228:

$$b = (1, 3, 7, 12, 18, 26, 35, 45, \ldots)$$

with complement $a =$ A030124.

Example 4. Bode, Harborth, and Kimberling [3] discuss the equation

$$b(n) = a(n-1) + a(n-2)$$

with prescribed initial terms $a(1)$ and $a(2)$.

Example 5. Another PCE is

$$a(b(n)) - b(a(n)) = 1.$$

Solutions include $b(n) = 2n$ and $b(n) = \lfloor n\tau \rfloor$.

In the sequel, we shall solve four types of OCEs: $b(n) = a(jn) \pm r$, $b(n) = a(jn) + kn$, $b(n) = f(a(n))$, and the PCE $b(n) = a(b(n-1)) + qn + r$.

2 The step sequence of an OCE

Suppose an OCE (1) is given. Because a and b are complementary, when jointly ranked they form the sequence N. The joint ranking has the form

$$\begin{aligned}
&a(1), \ldots, a(u_1), b\,(1)\,, \ldots, b(v_1),\\
&a(u_1+1), \ldots, a(u_2), b\,(v_1+1)\,, \ldots, b(v_2),\\
&a(u_2+1), \ldots, a(u_3), b\,(v_2+1)\,, \ldots, b(v_3), \ldots,
\end{aligned}$$

where the numbers u_i and v_i are nonnegative integers. Note that

$$\begin{aligned}
1 &= a(1) < b(1), \text{ if } u_1 > 0;\\
1 &= b(1) < a(1), \text{ if } u_1 = 0.
\end{aligned}$$

Each $n \geq v_1 + 1$ has a unique representation

$$n = v_{i-1} + r, \quad \text{where } 1 \leq r \leq v_i - v_{i-1},$$

where $i - 1 = \max\{m : n \geq v_m\}$.

Define the *step sequence* $s = (s(2), s(3), \ldots)$ by

$$s(n) = \begin{cases} u_i - u_{i-1} + 1, & \text{if } r = 1;\\ 1, & \text{if } r > 1.\end{cases}$$

Then b is clearly given by

$$b(n) = \begin{cases} u_1 + 1, & \text{if } n = 1;\\ b(n-1) + s(n), & \text{if } n \geq 2.\end{cases} \tag{3}$$

In many cases, we shall see, every $b(n)$ is immediately preceded and followed by a term of a, so that $v_i = i$ for all $i \geq 1$, and

$$s(n) = u_n - u_{n-1} + 1$$

for $n \geq 2$. In the sequel, we shall concentrate on sequences b of this kind.

3 The equations $b(n) = a(jn) \pm r$

Consider the equation

$$b(n) = a(jn) + r, \tag{4}$$

where $1 \le r \le j$. In order to solve this OCE, we find inductively that

$$a(n) = \begin{cases} n, & \text{if } 1 \le n < j + r; \\ n + 1, & \text{if } j + r \le n < 2j + r; \\ n + 2, & \text{if } 2j + r \le n < 3j + r; \\ \vdots & \vdots \\ n + q, & \text{if } qj + r \le n < (q + 1)j + r. \\ \vdots & \vdots \end{cases}$$

For example, we move from $a(n) = n$ to $a(n) = n + 1$ when $n = j + r$ in order to make room for $b(1) = j + r$. The inequality for which $a(n) = n + q$ is equivalent to

$$\frac{n - j - r}{j} < q \le \frac{n - r}{j},$$

so that $q = \left\lfloor \frac{n - r}{j} \right\rfloor$ and $a(n) = n + \left\lfloor \frac{n - r}{j} \right\rfloor$. Thus, for $n = 1$, we have $a(jn) = j$. Replacing n by jn gives

$$a(jn) = jn + \left\lfloor \frac{jn - r}{j} \right\rfloor = jn + n - 1.$$

To conclude, we have

$$a(jn) + r = jn + n - 1 + r$$

for all $n \ge 1$, so that the solution of (4) is given by

$$b(n) = (j + 1)n + r - 1.$$

The same method applies to the OCE

$$b(n) = a(jn) - r,$$

where $1 \le r \le j - 1$, giving the solution

$$b(n) = (j + 1)n - r.$$

4 The equation $b(n) = a(jn) + kn$

Suppose r and s are positive irrational numbers satisfying Beatty's equation [2]:

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{5}$$

Then the sequences a and b given by $a(n) = \lfloor nr \rfloor$ and $b(n) = \lfloor ns \rfloor$ are a pair of complementary sequences known as Beatty sequences ([21], [19], [1]).

The OCE

$$b(n) = a(n) + kn, \tag{6}$$

where k is a positive integer, occurs in Stolarsky [21] where it is solved by means of shift operators, related to morphisms and continued fractions ([1], [21]) and also closely related to the step sequences of section 2. Sequences satisfying (6) were also studied by Fraenkel [10]. The solution of (6) is given by the Beatty sequences

$$a(n) = \lfloor rn \rfloor, \qquad b(n) = \lfloor sn \rfloor, \tag{7}$$

where

$$r = 1 + \frac{\sqrt{k^2+4} - k}{2} \quad \text{and} \quad s = 1 + \frac{\sqrt{k^2+4} + k}{2}. \tag{8}$$

We wish to generalize Stolarsky's result to certain equations of the form

$$b(n) = a(jn) + kn; \tag{9}$$

specifically, we seek positive integers j and k for which the solution is a pair of Beatty sequences. Write

$$r = \frac{m + \sqrt{p}}{j}, \tag{10}$$

where m and p are rational numbers and $\sqrt{p}$ is irrational. Equation (5) then leads to

$$s = \frac{j\sqrt{p} + p + jm - m^2}{p - (m-j)^2}. \tag{11}$$

The desired equations

$$\lfloor sn \rfloor = \lfloor jrn \rfloor + kn$$

are equivalent to

$$sn - \delta_n = jrn - \varepsilon_n + kn, \tag{12}$$

where the fractional parts δ_n and ε_n satisfy

$$0 < \delta_n = sn - \lfloor sn \rfloor < 1 \quad \text{and} \quad 0 < \varepsilon_n = jrn - \lfloor jrn \rfloor < 1$$

for all n. Dividing both sides of (12) by n and taking the limit as $n \to \infty$ gives

$$s = jr + k.$$

Thus the coefficient $j/(p-(m-j)^2)$ of $\sqrt{p}$ on the right side of (11) must equal the coefficient of $\sqrt{p}$ in jr, which, by (10) is 1, so that

$$j = p - (m-j)^2,$$

which implies

$$j = \frac{2m - 1 \pm \sqrt{4(p-m)+1}}{2}.$$

In order that j be an integer, $\sqrt{4(p-m)+1}$ must be an odd integer:

$$\sqrt{4(p-m)+1} = 2q-1,$$

so that

$$p = q^2 - q + m.$$

Substituting into (11) and simplifying gives

$$s = q + \sqrt{p}.$$

Thus, for given m and q for which $q^2 - q + m$ is a nonsquare (below, we shall show that it is always a nonsquare), we put

$$\begin{aligned} j &= m + q - 1, \\ k &= q - m, \\ r &= \frac{m + \sqrt{q^2 - q + m}}{j}, \\ s &= q + \sqrt{q^2 - q + m}, \end{aligned}$$

and have the solution (7) of the equation (9).

Instead of starting with m and q, we can start with j and k to produce

$$\begin{aligned} q &= \frac{j+k+1}{2}, \\ m &= \frac{j-k+1}{2}, \\ \sqrt{p} &= \frac{\sqrt{(j+k+1)^2 - 4k}}{2}. \end{aligned}$$

It is this latter representation of p that we now use to show that $\sqrt{p}$ is irrational for all positive integers j and k. It suffices to show that $(j+k+1)^2 - 4k$ is a nonsquare. Let $M = j + k + 1$, and note that for $k = 0$ and $k = M - 1$ we have $M^2 - 4k$ taking the values M^2 and $(M-2)^2$, respectively. There is only one square between those numbers, namely $(M-1)^2$. Therefore, if $M^2 - 4k$ is a square for some k satisfing $1 \le k \le M - 2$, then that value of k must satisfy

$$M^2 - 4k = (M-1)^2. \tag{13}$$

However, (13) implies $4k = 2M - 1$, a number that is both even and odd. As there is no such number, $(j+k+1)^2 - 4k$ is not a square for any positive integers j and k.

Examples using Beatty-pair solutions of (9) are now easy to write out, as suggested by a table:

j	1	1	2	1	2	3	1	2	3	4
k	1	2	1	3	2	1	4	3	2	1
p	5/4	2	3	13/4	17/4	21/4	5	6	2	8

In connection with heap games, Fraenkel [13] considers the extension of (6) to the OCE

$$b(n) = ja(n) + kn,$$

where j and k are positive integers. For small values of j and k, solutions a and b include the pairs (A045671, A045672), (A045681, A045682), and (A045749, A045750), and (A045774, A045775).

Example 6. Taking $j = 1$ and leaving k arbitrary in (9) gives (8).

Example 7. Taking $j = k$ gives the OCE $b(n) = a(jn) + jn$ with solution (7) using

$$r = 1 + \frac{\sqrt{4j^2+1}}{2j} \quad \text{and} \quad r = \frac{2j+1+\sqrt{4j^2+1}}{2}.$$

5 The OCE of a dispersion, $b(n) = f(a(n))$

In this section, rather than starting with an OCE, we start with a certain kind of array consisting of all the positive integers, and we derive an OCE from it. Suppose f and g are strictly increasing complementary sequences and that $g(1) = 1$. The dispersion, $D(f) = \{d(i,j)\}_{i,j\geq 1}$ of f is defined [17] as the array having first column given by $d(i,1) = g(i)$ and subsequent columns given inductively by

$$d(i,j) = f(d(i,j-1)).$$

We shall see next that the general dispersion $D(f)$ is naturally associated with the OCE

$$b(n) = f(a(n)), \tag{14}$$

and that the dispersion provides a solution to this equation.

Note first that no member of column 1 of $D(f)$ is an image of f, so that the terms of column 1 belong to the sequence a. Next, every member m of column 2 is of the form $f(j)$ where j is in a, so that m is in b. Consequently, each m' in column 3 satisfies $m' = f(m)$ for some m in sequence b. Therefore, every term of column 3 is in a; otherwise, if m' were in b, then m would be in a, a contradiction. This shows that every term of column 3 is in a.

Continuing inductively, we conclude that the terms of the odd numbered columns of $D(f)$ are the terms of a, so that a is the ordered union of all the odd numbered columns. Likewise, b is the ordered union of all the even numbered columns. In $D(f)$, every positive integer occurs exactly once (see [17] for a proof), so that every positive integer is in a or b, which confirms that these are complementary sequences.

Example 8. Let $f(n) = 2n$ and $g(n) = 2n - 1$. The associated OCE is $b(n) = 2a(n)$. The northwest corner of the dispersion $D(f)$ is

$$\begin{array}{ccccccc} 1 & 2 & 4 & 8 & 16 & 32 & \cdots \\ 3 & 6 & 12 & 24 & 48 & 96 & \\ 5 & 10 & 20 & 40 & 80 & 160 & \\ 7 & 14 & 28 & 56 & 112 & 224 & \\ \vdots & & & & & & \ddots \end{array}$$

so that a is the ordered sequence of numbers

$$(2i+1)2^{2j}, \qquad i \geq 0, \ j \geq 0,$$

this being A003159, with complement $b =$ A036554, described as the numbers whose binary representation ends in an odd number of zeros.

As suggested by Example 8, the OCE

$$b(n) = ka(n),$$

for $k \geq 2$, has solution a the ordered union of the numbers

$$(ki+r)k^{2j}, \quad 1 \leq r \leq k-1, \ i \geq 0, \ j \geq 0,$$

with b the ordered union of the numbers $(ki+r)k^{2j+1}$.

Example 9. Let $f(n) = 2n+1$ for $n \geq 1$, let $g(1) = 1$ and $g(n) = 2n$ for $n \geq 2$. The associated OCE is $b(n) = 2a(n) + 1$. The northwest corner of the dispersion $D(f)$ is

$$\begin{array}{ccccccc} 1 & 3 & 7 & 15 & 31 & 63 & \cdots \\ 2 & 5 & 11 & 23 & 47 & 95 & \\ 4 & 9 & 19 & 39 & 79 & 159 & \\ 6 & 13 & 27 & 55 & 111 & 223 & \\ \vdots & & & & & & \ddots \end{array}$$

so that a is the ordered sequence of numbers

$$2^{2j+1} - 1 \quad \text{and} \quad (2i+1)2^{2j} - 1, \qquad i \geq 1, \ j \geq 0,$$

and b is the ordered sequence of numbers

$$2^{2j+2} - 1 \quad \text{and} \quad (2i+1)2^{2j+1} - 1, \qquad i \geq 1, \ j \geq 0.$$

It is natural to ask what OCEs are associated with well-known dispersions. In the next examples, we answer this question for the Wythoff array, the Wythoff difference array, the Stolarsky array, and the inverse Wythoff array.

Example 10. Column 1 of the Wythoff array $W = \{w(i,j)\}$ is given [20] by

$$w(i,1) = \lfloor \lfloor i\tau \rfloor \tau \rfloor,$$

and the ordered complement of column 1, written as an increasing sequence, is given by

$$f(n) = \lfloor (n+1)\tau \rfloor - 1.$$

The associated OCE is therefore

$$b(n) = \lfloor (a(n)+1)\tau \rfloor - 1.$$

Its solution a is the ordered union of odd numbered columns of W, so that a is simply the lower Wythoff sequence, A000201. The complement, b, is the rest of W, which as an increasing sequence is the upper Wythoff sequence, A001950. Initial terms are given by

$$a = (1, 3, 4, 6, 8, 9, 11, 12, \ldots), \qquad b = (2, 5, 7, 10, 13, 15, 18, 20, \ldots).$$

Fraenkel and Kimberling [12] discuss Example 10 in greater detail.

Example 11. The Wythoff difference array, $D = \{d(i, j)\}$, given by A080164, is the dispersion of the upper Wythoff sequence, which, when written in increasing order, is given by

$$f(n) = \lfloor (\tau + 1)n \rfloor .$$

The associated OCE is therefore

$$b(n) = \lfloor (\tau + 1)a(n) \rfloor .$$

Its solution from columns of D is given by initial terms as follows:

$$a = (1, 3, 4, 5, 6, 8, 9, 11, 12, \ldots), \qquad b = (2, 7, 10, 13, 15, 20, \ldots).$$

Example 12. The inverse Wythoff array, $X = \{x(i, j)\}$, has first column given by

$$x(i, 1) = s(n) = \begin{cases} 1, & \text{if } i = 1; \\ \lfloor i\tau \rfloor - 1, & \text{if } i > 1 \end{cases}$$

and ordered complement of column 1 given by

$$f(n) = \lfloor (n + 1)\tau \rfloor + n.$$

(The definition of the inverse of a dispersion is given in [17]). The associated OCE is

$$b(n) = \lfloor (a(n) + 1)\tau \rfloor + a(n).$$

Its solution from columns of X is given by initial terms as follows:

$$a = (1, 2, 3, 5, 7, 8, 10, 11, 12, 13, \ldots), \qquad b = (4, 6, 9, 14, 19, 22, 27, 30, 33, 35, \ldots).$$

6 The form $b(n) = a(b(n - 1)) + qn + r$

We begin with the OCE

$$b(n) = a(b(n - 1)) + 1. \tag{15}$$

If $a(1) = 1$, the solution is

$$b(n) = \frac{n^2 + n}{2} + 1,$$

whereas if $b(1) = 1$, the solution is

$$b(n) = \frac{n^2 + n}{2}. \tag{16}$$

We shall prove the latter, starting with a lemma closely related to (3).

Lemma. *Suppose a and b satisfy (15) and the initial condition* $b(1) = 1$. *Then*

$$a(m+1) = \begin{cases} a(m)+2, & \text{if } m = b(k) \text{ for some } k; \\ a(m)+1, & \text{otherwise.} \end{cases}$$

Proof. Because $b(1) = 1$, we have $a(1) \geq 2$, so that $b(2) \geq 3$, by (15). As a first step in an induction proof, we therefore have $b(2) - b(1) \geq 2$. Assume for arbitrary $k \geq 2$ that $b(k) - b(k-1) \geq 2$. Then, using (15),

$$\begin{aligned} b(k+1) - b(k) &= a(b(k)) - a(b(k-1)) \\ &\geq b(k) - b(k-1) \geq 2. \end{aligned}$$

Thus, for every $k \geq 2$, the numbers $b(k)-1$ and $b(k)+1$ are terms of the sequence a. As every positive integer is in exactly one of the sequences a and b, we have $a\,(m+1) = a\,(m) + 2$ if $a(m)$ is in b and $a\,(m+1) = a\,(m) + 1$ otherwise. ■

Now, we shall prove that (15), with the intial condition $b(1) = 1$, implies (16). By (15),

$$b(2) = a(b(1)) + 1 = a(1) + 1 \geq 3,$$

so that 2 cannot be $b(2)$ and must therefore be $a(1)$. Then $b(2) = a(b(1)) + 1 = a(1) + 1 = 3$. As a two-part induction hypothesis, assume for arbitrary $k \geq 1$ these two equations:

$$b(k) = \frac{k(k+1)}{2}, \tag{17}$$

$$a(b(k)) = b(k) + k. \tag{18}$$

Then

$$\begin{aligned} b(k+1) &= a(b(k)) + 1 \\ &= b(k) + k + 1 \text{ by (18)} \\ &= \frac{(k+1)(k+2)}{2} \text{ by (17),} \end{aligned}$$

and it remains to be proved that

$$a(b(k+1)) = b(k+1) + k + 1. \tag{19}$$

We have

$$\begin{aligned} a(b(k)+1) &= a(b(k)) + 2 \text{ by the lemma} \\ &= b(k) + k + 2 \text{ by (18).} \end{aligned}$$

Also by the lemma,

$$\begin{aligned} a(b(k)+2) &= a(b(k)+1) + 1 \\ a(b(k)+3) &= a(b(k)+2) + 1 = a(b(k)+1) + 2 \\ &\vdots \\ a(b(k)+k) &= a(b(k)+k-1) + 1 = a(b(k)+1) + k - 1 \\ a(b(k)+k+1) &= a(b(k)+k) + 1 = a(b(k)+1) + k. \end{aligned} \tag{20}$$

Now,

$$\begin{aligned} a(b(k)+1) &= a(b(k))+2 \text{ by the lemma} \\ &= b(k+1)+1 \text{ by (15),} \end{aligned}$$

so that

$$a(b(k)+1)+k = b(k+1)+k+1,$$

which by (20) implies

$$a(b(k)+k+1) = b(k+1)+k+1,$$

and the desired (19) now follows from the fact, already proved, that $b(k)+k+1 = b(k+1)$.

(Note that (18) is a PCE; one solution is given by (17); another, by $b(n) = 3n-2$.)

Similar inductive proofs can be given for various OCEs, including the following:

$$\begin{aligned} \text{Equation:} &\quad b(n) = a(b(n-1)) + r, \text{ where } r \geq 1 \\ \text{Initial value:} &\quad b(1) = 1 \\ \text{Solution:} &\quad b(n) = (r-1)(n-1) + \frac{n(n+1)}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :} &\quad b(n) = a(b(n-1)) + r, \text{ where } r \geq 1 \\ \text{Initial value :} &\quad a(1) = 1 \\ \text{Solution :} &\quad b(n) = rn + \frac{n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :} &\quad b(n) = a(b(n-1)) + qn, \text{ where } q \geq 1 \\ \text{Initial value :} &\quad b(1) = 1 \\ \text{Solution :} &\quad b(n) = \frac{q(n^2+n+2)+n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :} &\quad b(n) = a(b(n-1)) + qn, \text{ where } q \geq 1 \\ \text{Initial value :} &\quad a(1) = 1 \\ \text{Solution :} &\quad b(n) = \frac{q(n^2+n)+n^2-n+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :} &\quad b(n) = a(b(n-1)) + qn + r, \text{ where } q \geq 1,\ r \geq 1 \\ \text{Initial value :} &\quad b(1) = 1 \\ \text{Solution :} &\quad b(n) = \frac{q(n^2+n+2)+n^2+(r+1)n-2r+2}{2}. \end{aligned}$$

$$\begin{aligned} \text{Equation :} &\quad b(n) = a(b(n-1)) + qn + r, \text{ where } q \geq 1,\ r \geq 1 \\ \text{Initial value :} &\quad a(1) = 1 \\ \text{Solution :} &\quad b(n) = \frac{q(n^2+n)+n^2+(2r-1)n+2}{2}. \end{aligned}$$

To summarize, if $q \geq 0$ and $r \geq 0$ and q and r are not both 0, and if an initial value, either $a(1) = 1$ or $b(1) = 1$ is assumed, then the equation

$$b(n) = a(b(n-1)) + qn + r$$

holds for a unique second-degree polynomial in n. Several special cases are tabulated here, along with three examples in which $r < 0$.

$b(n) - a(b(n-1) =$	**Initial**	**Solution**	**Name**
1	$a(1) = 1$	$(n^2 + n + 2)/2$	A000124, Hogben's c. p. nos.
2	$a(1) = 1$	$(n^2 + 3n + 2)/2$	A000217, triangle numbers
n	$b(1) = 1$	n^2	A000290, square numbers
$2n + 2$	$a(1) = 1$	$(3n^2 + 5n + 2)/2$	A000326, pentagonal numbers
$3n + 2$	$a(1) = 1$	$2n^2+3n + 1$	A000384, hexagonal numbers
$2n + 1$	$a(1) = 1$	$2n^2+2n + 1$	A001844, centered square nos.
$n + 1$	$b(1) = 1$	$n^2+n - 1$	A028387
$n - 1$	$b(1) = 1$	$n^2-n + 1$	A002061, central polygonal nos.
$3n - 1$	$a(1) = 1$	$2n^2+1$	A058331
$3n - 2$	$b(1) = 1$	$2n^2-1$	A000384, hexagonal numbers

References

[1] J.-P. Allouche and J. Shallit, *Automatic Sequences*, Cambridge University Press, 2003.

[2] S. Beatty, Problem 3173, *Amer. Math. Monthly* **33** (1926) 159; **34** (1927) 159.

[3] J.-P. Bode, H. Harborth, and C. Kimberling, Complementary Fibonacci sequences, forthcoming.

[4] A. S. Fraenkel, The bracket function and complementary sets of integers, *Canadian J. Math.* **21** (1969), 6–27.

[5] A. S. Fraenkel, Complementing and exactly covering sequences, *J. Combinatorial Theory Ser. A* **14** (1973), 8–20.

[6] A. S. Fraenkel, A Characterization of exactly covering congruences, *Discrete Math.* **4** (1973), 359–366.

[7] A. S. Fraenkel and I. Borosh, A generalization of Wythoff's game, *J. Combinatorial Theory Ser. A* **5** (1973), 175–191.

[8] A. S. Fraenkel, Further characterizations and properties of exactly covering congruences, *Discrete Math.* **12** (1975), 93–100.

[9] A. S. Fraenkel, Complementary systems of integers, *Amer. Math. Monthly* **84** (1977), 114–115.

[10] A. S. Fraenkel, How to beat your Wythoff games' opponents on three fronts, *Amer. Math. Monthly* **89** (1982), 353–361.

[11] A. S. Fraenkel, Wythoff games, continued fractions, cedar trees and Fibonacci searches, *Theoret. Comput. Sci.* **29** (1984), 49–73.

[12] A. S. Fraenkel and C. Kimberling, Generalized Wythoff arrays, shuffles and interspersions, *Discrete Math.* **126** (1994), 137–149.

[13] A. S. Fraenkel, Heap games, numeration systems and sequences, *Ann. Combinatorics* **2** (1998), 197–210.

[14] A. S. Fraenkel and D. Krieger, The structure of complementary sets of integers: a 3-shift theorem, *Int. J. Pure Appl. Math.* **10** (2004), 1–49.

[15] R. K. Guy, Max and Mex Sequences, *Unsolved Problems in Number Theory,* 2nd ed., Springer-Verlag, New York (1994), 227–228, Problem 27.

[16] A. Holshouser and H. Reiter, A generalization of Beatty's theorem, *Southwest J. Pure Appl. Math.* (2001), 24–29.

[17] C. Kimberling, Interspersions and dispersions, *Proc. Amer. Math. Soc.* **117** (1993), 313–321.

[18] A. McD. Mercer, Generalized Beatty sequences, *Internat. J. Math. Sci.* **1** (1978), 525–528.

[19] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*.

[20] N. J. A. Sloane, Classic Sequences In *The On-Line Encyclopedia of Integer Sequences*: (Part 1:) The Wythoff Array and The Para-Fibonacci Sequence.

[21] K. B. Stolarsky, Beatty sequences, continued fractions, and certain shift operators, *Canadian Math. Bull.* **19** (1976) 473–482.

2000 *Mathematics Subject Classification*: Primary 11B37.
Keywords: Beatty sequence, complementary equation, complementary sequences, dispersion, inverse, polygonal numbers, Stolarsky array, Wythoff array, Wythoff difference array, Wythoff sequences.

(Concerned with sequences A000124 A000201 A000217 A000290 A000326 A000384 A001844 A001950 A002061 A003159 A005228 A028387 A036554 A045671 A045672 A045681 A045749 A045750 A045774 A045775 A058331 and A080164.)

Received May 16 2006; Revised versions received July 26 2006; October 11 2006. Published in *Journal of Integer Sequences*, December 30 2006.

Return to Journal of Integer Sequences home page.