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Minimal r-Complete Partitions

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Abstract

A minimal r-complete partition of an integer m is a partition of m with as few parts as possible, such that all the numbers $1, \ldots, rm$ can be written as a sum of parts taken from the partition, each part being used at most r times. This is a generalization of M-partitions (minimal 1-complete partitions). The number of M-partitions of m was recently connected to the binary partition function and two related arithmetic functions. In this paper we study the case $r \geq 2$, and connect the number of minimal r-complete partitions to the $(r+1)$-ary partition function and a related arithmetic function.

1 Introduction

Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a partition of the natural number m into $n+1$ parts λ_i arranged in non-decreasing order,

$$m = \lambda_0 + \lambda_1 + \cdots + \lambda_n, \qquad 1 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n.$$

The sum of the parts is called the *weight* of the partition and is denoted by $|\lambda|$, while $n+1$ is the *length* of the partition.

MacMahon [3], [4, pp. 217–223] calls the partition λ of weight m *perfect* if each positive integer less than m can be written in a *unique* way as a sum of distinct parts λ_i. Park [6] calls λ a *complete partition* of m if the representation property is maintained, while the uniqueness constraint is dropped. (O'Shea [5] calls this a *weak M-partition*.) Prior to Park's paper, infinite complete sequences had been introduced by Hoggatt and King [2], and studied by Brown [1].

Park [7] generalized the notion of a complete partition to *r-complete partitions* for a positive integer r. The partition $\lambda = (\lambda_0, \ldots, \lambda_n)$ of m is r-complete if each integer w in the interval $0 \leqslant w \leqslant rm$ can be written as

$$w = \alpha_0\lambda_0 + \cdots + \alpha_n\lambda_n \qquad \text{with} \quad 0 \leqslant \alpha_i \leqslant r. \tag{1}$$

Clearly, "complete" is the same as "1-complete". An r-complete partition is also $(r+1)$-complete.

We call an r-complete partition of m of minimal length a *minimal r-complete partition* of m. O'Shea [5] uses the term *M-partition* in place of minimal complete partition. He showed that for half the numbers m, the number of M-partitions of m is equal to the number of binary partitions of $2^{n+1} - 1 - m$, where $n = \lfloor \log_2 m \rfloor$. (In a binary partition, all parts are powers of 2.) O'Shea's partial enumeration formula was completed by us in [8].

In this paper we connect the minimal r-complete partition function (for $r \geqslant 2$) to the $(r+1)$-ary partition function and a related arithmetic function. (In an $(r+1)$-ary partition, all parts are powers of $r+1$.) In Section 2 we state our results. In Section 3 we consider a characterization of minimal r-partitions, and in Section 4 we prove our main result using (truncated) polynomials and formal power series.

2 Statement of Results

Let $f(k)$ be the $(r+1)$-ary partition function, that is, the number of partitions of k into powers of $r+1$. For the generating function $F(x)$ we have

$$F(x) = \sum_{k=0}^{\infty} f(k)x^k = \prod_{i=0}^{\infty} \frac{1}{1 - x^{(r+1)^i}}.$$

We also define the auxiliary arithmetic function $g(k)$ as follows:

$$G(x) = \sum_{k=0}^{\infty} g(k)x^k = \sum_{j=0}^{\infty} \frac{x^{(r+1)^j - 1}}{1 - x^{2(r+1)^j}} F(x^{(2r+1)(r+1)^j}) \prod_{i=0}^{j} \frac{1}{1 - x^{(r+1)^i}}.$$

A straightforward verification shows that the following functional equations hold:

$$F(x) = \frac{1}{1-x} F(x^{r+1}), \tag{2}$$

$$G(x) = \frac{x^r}{1-x} G(x^{r+1}) + \frac{1}{(1-x)(1-x^2)} F(x^{2r+1}). \tag{3}$$

These functional equations give simple recurring relations for fast computation of $f(k)$ and $g(k)$. We adopt the convention that $g(k) = 0$ if k is not a non-negative integer.

Theorem 2.1. *Let $r \geqslant 2$, and let $a_r(m)$ be the number of minimal r-complete partitions of m. Then*

$$a_r(m) = f\left(\frac{1}{r}\left((r+1)^{n+1} - 1\right) - m\right) - g\left(\frac{1}{r}\left((2r+1)(r+1)^{n-1} - 1\right) - 1 - m\right),$$

where $n = \lfloor \log_{r+1}(rm) \rfloor$.

Corollary 2.1. *We have*

$$a_r(m) = f\left(\frac{1}{r}\left((r+1)^{n+1}-1\right) - m\right)$$

if $\frac{1}{r}((2r+1)(r+1)^{n-1}-1) \leqslant m \leqslant \frac{1}{r}((r+1)^{n+1}-1)$.

The case $r = 1$ is not covered by Theorem 2.1. This case is slightly different from $r \geqslant 2$, as an additional arithmetic function is required in the description of $a_1(m)$; see [8, Theorem 2]. The expression for $a_r(m)$ in Theorem 2.1 is, however, valid for $r = 1$ if $2^n + 2^{n-3} - 4 \leqslant m \leqslant 2^{n+1} - 1$. In particular, Corollary 2.1 remains valid if $r = 1$, a result due to O'Shea [5].

Some of the sequences appearing above can be found in Sloane's *On-Line Encyclopedia of Integer Sequences* [9]. For perfect partitions, see sequence A002033; for $a_1(m)$, see A100529. The sequences A000123, A018819, A0005704, A0005705, A0005706 give the first several values of $f(k)$ for $r = 1, 1, 2, 3$, and 4, respectively. In addition, sequence A117115 gives the 53 first values of $g(k)$ for $r = 1$, and A117117 gives the 53 first values of the additional arithmetic function required in the description of $a_1(m)$.

3 Completeness

The following lemma is due to Park [7], with partial results by Brown [1] and Park [6].

Lemma 3.1. *The partition* $\lambda = (\lambda_0, \ldots, \lambda_n)$ *is* r*-complete if and only if* $\lambda_0 = 1$ *and*

$$\lambda_i \leqslant 1 + r(\lambda_0 + \cdots + \lambda_{i-1}) \quad \textit{for} \quad i = 1, 2, \ldots, n. \tag{4}$$

The necessity of the conditions $\lambda_0 = 1$ and (4) is clear, and the sufficiency follows by induction on n; see the proof of Theorem 2.2 in [7].

Suppose that $\lambda = (\lambda_0, \ldots, \lambda_n)$ is an r-complete partition of m. Then (1) must be solvable for $rm + 1$ values of w. Since the right hand side attains at most $(r+1)^{n+1}$ distinct values, we have $rm + 1 \leqslant (r+1)^{n+1}$. Alternatively, by Lemma 3.1, $\lambda_i \leqslant (r+1)^i$ for $i = 0, 1, \ldots, n$, so that $rm \leqslant (r+1)^{n+1} - 1$. In any case, we have $\lfloor \log_{r+1}(rm) \rfloor \leqslant n$, cf. [7, Proposition 2.4].

On the other hand, for a given m, let $n = \lfloor \log_{r+1}(rm) \rfloor$. Order the $n+1$ positive integers $1, r+1, (r+1)^2, \ldots, (r+1)^{n-1}, k = m - \frac{1}{r}((r+1)^n - 1)$ in increasing order $1 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n$. We have $1 \leqslant k \leqslant (r+1)^n$, and it follows that λ is a minimal r-complete partition of m.

Lemma 3.2. *Let* λ *be an* r*-complete partition of weight* m *and length* $n+1$*. Then* λ *is minimal if and only if*

$$n = \lfloor \log_{r+1}(rm) \rfloor. \tag{5}$$

We have shown that if $\lambda = (\lambda_0, \ldots, \lambda_n)$ is a partition of weight m with $\lambda_0 = 1$, then λ is a minimal r-complete partition if and only if (4) and (5) hold.

4 Generating functions

In order to determine the number $a_r(m)$ of minimal r-complete partitions of weight m, we first consider the number $q_n(m)$ of r-complete partitions of weight m and length $n+1$. By Lemma 3.2, we know that such an r-complete partition is minimal if and only if $\frac{1}{r}((r+1)^n - 1) + 1 \leqslant m \leqslant \frac{1}{r}((r+1)^{n+1} - 1)$. Thus

$$a_r(m) = q_n(m) \quad \text{if} \quad \frac{1}{r}\left((r+1)^n - 1\right) + 1 \leqslant m \leqslant \frac{1}{r}\left((r+1)^{n+1} - 1\right). \tag{6}$$

For the generating function $Q_n(x)$ of $q_n(m)$, we have

$$Q_n(x) = \sum_{m=n+1}^{(1/r)((r+1)^{n+1}-1)} q_n(m)x^m = \sum_{\lambda} x^{|\lambda|}, \tag{7}$$

where we sum over the λ satisfying $1 = \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n$ and (4).

We change parameters by setting $\mu_i = (r+1)^i - \lambda_i$ for $i = 0, 1, \ldots, n$. Then the constraints, necessary for λ being r-complete, become $\mu_0 = 0$, and

$$r(\mu_0 + \cdots + \mu_{i-1}) \leqslant \mu_i \leqslant r(r+1)^{i-1} + \mu_{i-1} \quad \text{for } i = 1, \ldots, n. \tag{8}$$

Moreover,

$$|\lambda| = \frac{1}{r}((r+1)^{n+1} - 1) - |\mu|, \tag{9}$$

for $|\mu| = \mu_0 + \cdots + \mu_n$. For a fixed n, we are interested in the number of solutions λ of $|\lambda| = m$ for each m in the interval $\frac{1}{r}((r+1)^n - 1) + 1 \leqslant m \leqslant \frac{1}{r}((r+1)^{n+1} - 1)$, that is, the number of solutions μ of $|\mu| = k$ for each k in the interval $0 \leqslant k \leqslant (r+1)^n - 1$.

We write

$$R_n(x) = \sum_{k \geqslant 0} r_n(k)x^k = \sum_{\mu} x^{|\mu|}, \tag{10}$$

where we sum over the μ satisfying $\mu_0 = 0$ and (8). We are interested in the coefficients $r_n(k)$ for $k < (r+1)^n$. Therefore we shall on some occasions truncate polynomials and formal power series under consideration. We shall use the order symbol $O(x^N)$ for truncation of order N. Thus, if we write

$$\sum_k b(k)x^k = \sum_k c(k)x^k + O(x^N),$$

then $b(k) = c(k)$ for all $k < N$.

Let $n \geqslant 2$. It simplifies notations to "sum" over $\mu_0 = 0$. We have

$$R_n(x) = \sum_{\mu_0} \cdots \sum_{\mu_n} x^{\mu_0 + \cdots + \mu_n},$$

where the innermost sum is

$$\sum_{\mu_n = r(\mu_0 + \cdots + \mu_{n-1})}^{r(r+1)^{n-1} + \mu_{n-1}} x^{\mu_0 + \cdots + \mu_n} = x^{(r+1)(\mu_0 + \cdots + \mu_{n-1})} \frac{1 - x^{r(r+1)^{n-1} + 1 - r(\mu_0 + \cdots + \mu_{n-1}) + \mu_{n-1}}}{1 - x}.$$

Now, we have

$$R_n(x) = \frac{1}{1-x} R_{n-1}(x^{r+1}) - \frac{x^{r(r+1)^{n-1}+1}}{1-x} \sum_{\mu_0} \cdots \sum_{\mu_{n-1}} x^{\mu_0+\cdots+\mu_{n-2}+2\mu_{n-1}}.$$

We repeat this process once, and obtain

$$R_n(x) = \frac{1}{1-x} R_{n-1}(x^{r+1}) - \frac{x^{r(r+1)^{n-1}+1}}{(1-x)(1-x^2)} R_{n-2}(x^{2r+1}) + \frac{x^{r(r+3)(r+1)^{n-2}+3}}{(1-x)(1-x^2)} \sum_{\mu_0} \cdots \sum_{\mu_{n-2}} x^{\mu_0+\cdots+\mu_{n-3}+3\mu_{n-2}},$$

so that

$$R_n(x) = \frac{1}{1-x} R_{n-1}(x^{r+1}) - \frac{x^{r(r+1)^{n-1}+1}}{(1-x)(1-x^2)} R_{n-2}(x^{2r+1}) + O(x^{(r+1)^n}) \tag{11}$$

for $n \geqslant 2$.

By (2) and (3), we have

$$\begin{aligned} F(x) &= \frac{1}{1-x} + O(x^{r+1}), \\ G(x) &= \frac{1}{(1-x)(1-x^2)} + O(x^r). \end{aligned} \tag{12}$$

Moreover, $R_0(x) = 1$, and

$$R_1(x) = 1 + x + \cdots + x^r = \frac{1-x^{r+1}}{1-x} = F(x) + O(x^{r+1}),$$

so we may write

$$R_1(x) = F(x) - x^{r+1} G(x) + O(x^{r+1}). \tag{13}$$

Putting $n = 2$ in (11), we get

$$R_2(x) = \frac{1}{1-x} R_1(x^{r+1}) - \frac{x^{r(r+1)+1}}{(1-x)(1-x^2)} R_0(x^{2r+1}) + O(x^{(r+1)^2}),$$

and using (13), we obtain

$$R_2(x) = \frac{1}{1-x} F(x^{r+1}) - \frac{x^{r(r+1)+1}}{(1-x)(1-x^2)} + O(x^{(r+1)^2}).$$

Hence, by (2) and (12), we have

$$R_2(x) = F(x) - x^{r(r+1)+1} G(x) + O(x^{(r+1)^2}).$$

We claim that if $r \geqslant 2$ and $n \geqslant 1$, then

$$R_n(x) = F(x) - x^{r(r+1)^{n-1}+1} G(x) + O(x^{(r+1)^n}). \tag{14}$$

To prove this, we use induction on n. We have just seen that the claim is valid for $n = 1$ and $n = 2$. Suppose that (14) holds for n replaced by $n-1$ and by $n-2$ for some $n \geqslant 3$. Using (11) and the induction hypotheses, we obtain

$$\begin{aligned} R_n(x) = {} & \frac{1}{1-x}\Big(F(x^{r+1}) - x^{r(r+1)^{n-1}+r+1} G(x^{r+1}) + O(x^{(r+1)^n})\Big) \\ & - \frac{x^{r(r+1)^{n-1}+1}}{(1-x)(1-x^2)}\Big(F(x^{2r+1}) - x^{(2r+1)(r(r+1)^{n-3}+1)} G(x^{2r+1}) + O(x^{(2r+1)(r+1)^{n-2}})\Big) \\ & + O(x^{(r+1)^n}). \end{aligned}$$

We find that

$$R_n(x) = \frac{1}{1-x} F(x^{r+1}) - \frac{x^{r(r+1)^{n-1}+r+1}}{1-x} G(x^{r+1}) - \frac{x^{r(r+1)^{n-1}+1}}{(1-x)(1-x^2)} F(x^{2r+1}) + O(x^{(r+1)^n}),$$

and, using the functional equations (2) and (3), (14) follows.

We are now ready to conclude the proof of Theorem 2.1. By (10) and (9), we have

$$R_n(x) = \sum_\mu x^{|\mu|} = \sum_\lambda x^{(1/r)((r+1)^{n+1}-1)-|\lambda|}.$$

Moreover, by (7),

$$R_n(x) = x^{(1/r)((r+1)^{n+1}-1)} Q_n(x^{-1}) = \sum_{m=n+1}^{(1/r)((r+1)^{n+1}-1)} q_n(m) x^{(1/r)((r+1)^{n+1}-1)-m}.$$

Hence,

$$R_n(x) = \sum_{k \geqslant 0} r_n(k) x^k = \sum_{k=0}^{(1/r)((r+1)^{n+1}-1)-n-1} q_n\left(\frac{1}{r}\big((r+1)^{n+1}-1\big) - k\right) x^k;$$

that is,

$$r_n(k) = q_n\left(\frac{1}{r}\big((r+1)^{n+1}-1\big) - k\right). \tag{15}$$

For $n \geqslant 1$, we have by (14),

$$r_n(k) = f(k) - g(k - r(r+1)^{n-1} - 1) \qquad \text{for} \quad 0 \leqslant k \leqslant (r+1)^n - 1.$$

Setting $k = \frac{1}{r}\left((r+1)^{n+1}-1\right) - m$ and using (15) and (6), we get Theorem 2.1. By inspection, the theorem also holds for $n = 0$.

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