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On a Generalized Recurrence for Bell Numbers

Jacob Katriel
Department of Chemistry
Technion - Israel Institute of Technology
Haifa 32000
Israel
jkatriel@technion.ac.il

Abstract

A novel recurrence relation for the Bell numbers was recently derived by Spivey, using a beautiful combinatorial argument. An algebraic derivation is proposed that allows straightforward q-deformation.

1 Introduction

Spivey [1] recently proposed a generalized recurrence relation for the Bell numbers,

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} B_k. \tag{1}$$

Here, B_n is the Bell number, $\left\{ {m \atop j} \right\}$ is the Stirling number of the second kind, and $\binom{n}{k}$ is a binomial coefficient. Recall that the Stirling numbers of the second kind can be defined via

$$x^k = \sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\} x(x-1)(x-2)\cdots(x-\ell+1)$$

and the Bell numbers are given by

$$B_k = \sum_{\ell=0}^{k} \left\{ {k \atop \ell} \right\}.$$

Spivey's derivation invokes a beautiful combinatorial argument.

An algebraic derivation of Spivey's identity is presented below, using the formalism due to Rota et al. [2] that was specifically developed for the present context by Cigler [3]. The relevance to the normal ordering problem of boson operators has been considered in Katriel [4, 5]. The basic ingredients of the Rota-Cigler formalism are also known as the Bargmann representation of the boson operator algebra [6]. The main advantage of the algebraic derivation is that it readily yields the more general q-deformed identity.

2 Algebraic derivation of Spivey's identity

We shall use the operator X of mutiplication by the variable x, and the q-derivative D

$$\begin{aligned} Xf(x) &= xf(x) \\ Df(x) &= \frac{f(qx)-f(x)}{x(q-1)}, \end{aligned}$$

that satisfy

$$DX - qXD = 1. \tag{2}$$

From the definition of the q-derivative it follows that

$$Dx^n = [n]_q x^{n-1}, \tag{3}$$

where $[n]_q = \frac{q^n-1}{q-1}$, and from Eq. (2) it follows that

$$DX^n = q^n X^n D + [n]_q X^{n-1}. \tag{4}$$

Eq. (4) can also be written in the form

$$(XD)X^n = X^n\Big([n]_q + q^n(XD)\Big), \tag{5}$$

that will be useful below. The q-exponential function $e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[n]_q!}$, where $[0]_q! = 1$ and $[n+1]_q! = [n]_q![n+1]_q$, satisfies

$$De_q(x) = e_q(x). \tag{6}$$

Repeated application of the q-commutation relation (2) yields

$$(XD)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q X^k D^k \tag{7}$$

where $\left\{ {0 \atop 0} \right\}_q = 1$ and

$$\left\{ {n+1 \atop k} \right\}_q = q^{k-1} \left\{ {n \atop k-1} \right\}_q + [k]_q \left\{ {n \atop k} \right\}_q .$$

The initial value and the recurrence relation are sufficient to identify the coefficients $\left\{ {n \atop k} \right\}_q$ as the q-Stirling numbers, hence the sum $B_n(q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q$ is the q-Bell number.

Applying both sides of the identity (7) to the q-exponential function we obtain

$$\frac{1}{e_q(x)}(XD)^n e_q(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q x^k \equiv B_n(q;x). \tag{8}$$

For $x = 1$ the q-Bell polynomial $B_n(q;x)$ reduces to the q-Bell number $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q$, and for $x = -1$ it reduces to the q-Rényi number [7] $R_n(q) = \sum_{k=0}^{n}(-1)^k \left\{ {n \atop k} \right\}_q$. Since $(XD)x^m = [m]_q x^m$ it follows that $(XD)^n x^m = [m]_q^n x^m$ and $(XD)^n e_q(x) = \sum_{k=0}^{\infty} \frac{[k]_q^n}{[k]_q!} x^k$, yielding the Dobinski formula for the q-Bell polynomial.

Using Eq. (7), then Eq. (5), and finally the binomial theorem we obtain

$$\begin{aligned} (XD)^{n+m} &= (XD)^n \sum_{j=0}^{m} \left\{ {m \atop j} \right\}_q X^j D^j \\ &= \sum_{j=0}^{m} \left\{ {m \atop j} \right\}_q X^j \Big([j]_q + q^j (XD) \Big)^n D^j \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} X^j (XD)^k D^j \end{aligned}$$

Applying this operator identity to the q-exponential function, using Eqs. (6) and (8), dividing by $e_q(x)$ and setting $x = 1$ we obtain

$$B_{n+m}(q) = \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} B_k(q).$$

Note that the binomial coefficient is not deformed. For $q = 1$ this identity reduces to Eq. (1).

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