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Polynomials Associated with Reciprocation

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Abstract

Polynomials are defined recursively in various ways associated with reciprocation; e.g., $S_{n+1}(x)/T_{n+1}(x) = S_n(x)/T_n(x) \pm T_n(x)/S_n(x)$. Under certain conditions, the zeros of S_n interlace those of T_n. Identities for S_n, T_n, and related polynomials are derived, as well as recurrence relations and infinite sums involving roots of polynomials.

1 Introduction

A well-known problem [2] starts with the recurrence

$$x_{n+1} = x_n + 1/x_n, \tag{1}$$

given that $x_0 = 1$. The sequence $(1, 2, 5, 29, \ldots)$ thus determined is indexed in Sloane's *Online Encyclopedia of Integer Sequences* [4] as A073833. It is natural to ask what happens if the initial value 1 is replaced by an indeterminate x. The purpose of this paper is to respond to that question and related questions. For example, what if the recurrence is replaced by

$$x_{n+1} = x_n - 1/x_n, \tag{2}$$

as in A127814? The recurrence (2) leads to polynomials which are interesting because of the distribution of their zeros, as portended by the following definition: suppose $V = (v_1, v_2, \ldots, v_{m-1})$ and $W = (w_1, w_2, \ldots, w_m)$ are lists of numbers satisfying

$$w_1 < v_1 < w_2 < v_2 < \cdots < v_{m-1} < w_m;$$

then V *interlaces* W.

Throughout this paper, except where otherwise stipulated, the letter c denotes an arbitrary nonzero complex number.

2 The recurrence $S_{n+1}/T_{n+1} = (1/c)(S_n/T_n - T_n/S_n)$

Define polynomials $S_1 = S_1(x) = x$, $T_1 = T_1(x) = 1$, and

$$S_{n+1} = S_n^2 - T_n^2, \quad T_{n+1} = cS_nT_n, \tag{3}$$

so that

$$S_{n+1}/T_{n+1} = (1/c)(S_n/T_n - T_n/S_n), \tag{4}$$

and for $n \geq 2$,

$$\begin{aligned} T_n &= c^{n-1}S_1S_2\cdots S_{n-1}, \\ S_{n+1} &= (S_n - c^{n-1}S_1S_2\cdots S_{n-1})(S_n + c^{n-1}S_1S_2\cdots S_{n-1}). \end{aligned} \tag{5}$$

If $c = 1$, then (4) is the recurrence (2) with $x_n = S_n/T_n$. For $c > 0$ and $n \geq 1$, let $\mathcal{Z}_n$ be the list of zeros of S_n in increasing order, so that

$$\mathcal{Z}_1 = (0), \quad \mathcal{Z}_2 = (-1, 1),$$

and $\mathcal{Z}_3$ is the ordered list consisting of the numbers $(\pm c \pm \sqrt{c^2+4})/2$; e.g., if $c = 1$, then $\mathcal{Z}_3 = (-\varphi, -1/\varphi, 1/\varphi, \varphi)$, where $\varphi = (1+\sqrt{5})/2$, the golden ratio.

If V and W are ordered lists, we shall employ the set-union symbol $\cup$ for the *ordered* union of the merged list formed by the numbers in V and W, thus: $V \cup W$. Note that S_n has degree 2^{n-1}, that T_n has degree $2^{n-1} - 1$, and that $\bigcup_{k=1}^{n} \mathcal{Z}_k$ lists the zeros of T_n.

Theorem 1. *Suppose that c is a nonzero real number and $n \geq 1$. Then $\bigcup_{k=1}^{n} \mathcal{Z}_k$ interlaces $\mathcal{Z}_{n+1}$.*

Proof. First, suppose that $c > 0$. Clearly $\mathcal{Z}_1$ interlaces $\mathcal{Z}_2$. It will be helpful to denote the zeros of S_n in increasing order by $r_{n,i}$ where $i = 1, 2, \ldots, 2^{n-1}$. To complete a two-part first induction step, we shall show that $\mathcal{Z}_1 \cup \mathcal{Z}_2$ interlaces $\mathcal{Z}_3$. The function $f_2 = S_2/S_1$ is continuous and rises strictly from $-\infty$ to ∞ on each of the intervals $(-\infty, r_{11})$ and (r_{11}, ∞). Therefore there exist unique numbers r_{31}, r_{32} such that

$$r_{31} < r_{21} < r_{32} < r_{11}, \quad f_2(r_{31}) = -c, \quad f_2(r_{32}) = c \tag{6}$$

and r_{33}, r_{34} such that

$$r_{11} < r_{33} < r_{22} < r_{34}, \quad f_2(r_{33}) = -c, \quad f_2(r_{34}) = c. \tag{7}$$

Thus, r_{31}, r_{33} are the zeros of $S_2 + cS_1$, and r_{32}, r_{34} are the zeros of $S_2 - cS_1$. Now $S_3 = (S_2 + cS_1)(S_2 - cS_1)$, so that (6) and (7) imply that $\mathcal{Z}_1 \cup \mathcal{Z}_2$ interlaces $\mathcal{Z}_3$. As a general induction hypothesis, assume for $n \geq 3$ that $\bigcup_{k=1}^{n-2} \mathcal{Z}_k$ interlaces $\mathcal{Z}_{n-1}$ and that $\bigcup_{k=1}^{n-1} \mathcal{Z}_k$ interlaces $\mathcal{Z}_n$. Write

$$\begin{aligned} \bigcup_{k=1}^{n-2} \mathcal{Z}_k &= (\rho_{11}, \rho_{12}, \ldots, \rho_{1m}), \quad \text{where } m = 2^{n-2} - 1, \\ \bigcup_{k=1}^{n-1} \mathcal{Z}_k &= (\rho_{21}, \rho_{11}, \rho_{22}, \rho_{12}, \ldots, \rho_{2m}, \rho_{1m}, \rho_{2,m+1}). \end{aligned}$$

The function $f_n = S_n/(S_1 S_2 \cdots S_{n-1})$ is continuous and strictly increasing on each of the intervals

$$(-\infty, \rho_{11}), (\rho_{11}, \rho_{12}), \ldots, (\rho_{1,m-1}, \rho_{1m}), \tag{8}$$

with infinite limits (as in the argument above for f_2.) Let ρ_{31}, ρ_{32} be the unique numbers satisfying

$$\rho_{31} < \rho_{21} < \rho_{32} < \rho_{11}, \quad f_n(\rho_{31}) = -c, \quad f_2(\rho_{32}) = c.$$

Likewise applying the intermediate value theorem to the remaining intervals in (8), we conclude that $\bigcup\limits_{k=1}^{n} \mathcal{Z}_k$ interlaces $\mathcal{Z}_{n+1}$.

Now suppose that $c < 0$. The recurrences (3) show that the only exponents of c that occur in the polynomials S_n are even. Therefore, these polynomials and their zeros are identical to those already considered. ∎

Theorem 2. *If $n \geq 1$ and $r \in \mathcal{Z}_n$, then $(cr \pm \sqrt{c^2r^2+4})/2 \in \mathcal{Z}_{n+1}$.*

Proof. The proposition clearly holds for $n = 1$. The zeros of S_2 are -1 and 1; let r_1 be either of these, and let r_2 be any number satisfying $r_1 = (1/c)(r_2 - 1/r_2)$, so that

$$r_1 = \frac{r_2^2 - 1}{cr_2} = \frac{S_2(r_2)}{T_2(r_2)}.$$

Taking $r_1 = -1$ gives $S_2(r_2) + T_2(r_2) = 0$ and taking $r_1 = 1$ gives $S_2(r_2) - T_2(r_2) = 0$. Consequently,

$$S_3(r_2) = (S_2(r_2) + T_2(r_2))(S_2(r_2) - T_2(r_2)),$$

which implies that the 4 numbers r_2 are the zeros of S_3. Continuing with arbitrary zeros r_1, r_2 of S_2, let r_3 be any number satisfying $r_2 = (1/c)(r_3 - 1/r_3)$, so that $r_2 = S_2(r_3)/T_2(r_3)$. Then

$$\begin{aligned} r_1 &= \frac{1}{c}(r_2 - \frac{1}{r_2}) = \frac{1}{c}(\frac{S_2(r_3)}{cr_3} - \frac{cr_3}{S_2(r_3)}) \\ &= \frac{S_2^2(r_3) - c^2r_3^2}{cS_2(r_3)T_2(r_3)} = \frac{S_3(r_3)}{T_3(r_3)}. \end{aligned}$$

Taking $r_1 = -1$ gives $S_3(r_3) + T_3(r_3) = 0$ and taking $r_1 = 1$ gives $S_3(r_2) - T_3(r_3) = 0$, so that the 8 numbers r_3 are the zeros of S_4. This inductive procedure shows that the zeros ρ of S_{n+1} arise from the zeros r of S_n by the rule $r = (1/c)(\rho - 1/\rho)$, and solving this for ρ finishes the proof. ∎

Theorem 3. *If $n \geq 1$, then*

$$S_{n+1}(x) = (cx)^{2^{n-1}} S_n(\frac{x}{c} - \frac{1}{cx}). \tag{9}$$

More generally, if $k = 2, 3, \ldots, n+1$, then

$$S_{n+1} = T_k^{2^{n-k+1}} S_{n-k+2}(\frac{S_k}{T_k}). \tag{10}$$

Proof. First, we prove (9). The assertion clearly holds for $n = 1$. Assume that $n > 1$ and let $m = 2^{n-1}$. Denote the zeros of S_n by r_k for $k = 1, 2, \ldots, m$. By Theorem 2,

$$\begin{aligned} S_{n+1}(x) &= \prod_{h=1}^{m}(x - \frac{cr_h - \sqrt{c^2 r_h^2 + 4}}{2})(x - \frac{cr_h + \sqrt{c^2 r_h^2 + 4}}{2}) \\ &= \prod_{h=1}^{m}(x^2 - cr_h x - 1) \\ &= \prod_{h=1}^{m} x(x - \frac{1}{x} - cr_h) \\ &= (cx)^m S_n(\frac{x}{c} - \frac{1}{cx}). \end{aligned}$$

In case $k = 2$, equation (10) is essentially a restatement of (9), just proved. For $k > 2$, equations (10) will now be proved by induction on k. Assume for arbitrary k satisfying $2 \le k \le n$ that

$$S_{n+1} = T_{k-1}^{2^{n-k+2}} S_{n-k+3}(\frac{S_{k-1}}{T_{k-1}}). \tag{11}$$

Substitute S_k/T_k for x in (9) with n replaced by $n - k + 3$:

$$S_{n-k+3}(\frac{S_{k-1}}{T_{k-1}}) = (\frac{cS_{k-1}}{T_{k-1}})^{2^{n-k+1}} S_{n-k+2}(\frac{S_k}{T_k}),$$

so that by (11),

$$\begin{aligned} S_{n+1} &= T_{k-1}^{2^{n-k+2}} (\frac{cS_{k-1}}{T_{k-1}})^{2^{n-k+1}} S_{n-k+2}(\frac{S_k}{T_k}) \\ &= (cT_{k-1}S_{k-1})^{2^{n-k+1}} S_{n-k+2}(\frac{S_k}{T_k}) \\ &= {T_k}^{2^{n-k+1}} S_{n-k+2}(\frac{S_k}{T_k}). \end{aligned}$$

∎

Using Theorems 1-3, it is easy to establish the following properties of the polynomials and zeros:

1. $S_n(-x) = S_n(x)$ for $n \ge 2$, and if $r \in \mathcal{Z}_n$ then $-r \in \mathcal{Z}_n$ for $n \ge 1$.
2. $x^{2^{n-1}} S_n(1/x) = S_n(x)$ for $n \ge 3$, and if $r \in \mathcal{Z}_n$ then $1/r \in \mathcal{Z}_n$ for $n \ge 2$.
3. Suppose that $c > 0$, and let r_n denote the greatest zero of S_n. Then (r_n) is a strictly increasing sequence, and

$$\lim_{n\to\infty} r_n = \begin{cases} (1-c)^{-1/2}, & \text{if } 0 < c < 1; \\ \infty, & \text{if } c \ge 1. \end{cases}$$

An outline of a proof follows. Of course, $r_n = (cr_{n-1} + \sqrt{c^2 r_{n-1}^2 + 4})/2$, so that (r_n) is strictly increasing. If $0 < c < 1$, then, as is easily proved, $r_n < (1-c)^{-1/2}$ for all n, so that a limit r

exists; since $r = (cr + \sqrt{c^2r^2+4})/2$, we have $r = (1-c)^{-1/2}$. On the other hand, supposing $c \geq 1$, if (r_n) were bounded above, then the previous argument would give $r = (1-c)^{-1/2}$, but this is not a real number. Therefore $r_n \to \infty$.

4. If $n \geq 3$, then

$$S_n = S_{n-1}^2 + c^2 S_{n-1} S_{n-2}^2 - c^2 S_{n-2}^4. \tag{12}$$

To prove this recurrence, from (5) we obtain both

$$c^{2n-4}(S_1 S_2 \cdots S_{n-3})^2 S_{n-2}^2 = S_{n-1}^2 - S_n$$

and

$$c^{2n-6}(S_1 S_2 \cdots S_{n-3})^2 = S_{n-2}^2 - S_{n-1},$$

and (12) follows by eliminating $(S_1 S_2 \cdots S_{n-3})^2$.

In the case $c = 1$, the recurrence (12) is used (e.g., [1]) to define the Gorškov-Wirsing polynomials, for which the initial polynomials are $2x-1$ and $5x^2-5x+1$ rather than x and x^2-1.

3 The polynomials $V_n = S_n - c^{n-1} S_1 S_2 \cdots S_{n-1}$

Equation (5) shows that $n \geq 2$, the polynomial S_{n+1} factors. We use one of factors to define a sequence of polynomials

$$V_n = V_n(x) = S_n - c^{n-1} S_1 S_2 \cdots S_{n-1}, \tag{13}$$

so that

$$S_n(x) = V_{n-1}(x) V_{n-1}(-x), \tag{14}$$

$$2S_n(x) = V_n(x) + V_n(-x). \tag{15}$$

Suppose that $n \geq 3$. Substitute $n-1$ for n in (13) and then multiply both sides of the result by cS_{n-1} to obtain

$$cV_{n-1} S_{n-1} = cS_{n-1}^2 - c^{n-1} S_1 S_2 \cdots S_{n-1},$$

so that by (13),

$$V_n = S_n - cS_{n-1}^2 + cV_{n-1} S_{n-1},$$

and by (14), and (15) with $n-1$ substituted for n,

$$\begin{aligned} V_n(x) \quad &= \quad V_{n-1}(x) V_{n-1}(-x) - c\left(\frac{V_{n-1}(x) + V_{n-1}(-x)}{2}\right)^2 \\ &\quad + cV_{n-1}(x) \frac{V_{n-1}(x) + V_{n-1}(-x)}{2}. \end{aligned}$$

Thus, we have the following recurrence for the polynomials V_n :

$$V_n = V_{n-1}(x) V_{n-1}(-x) + \frac{c}{4}[V_{n-1}^2(x) - V_{n-1}^2(-x)],$$

for $n \geq 3$. Another recurrence for these polynomials stems directly from (9) and (14):

$$V_{n+1}(x) = (cx)^{2^{n-1}} V_n(\frac{x}{c} - \frac{1}{cx}),$$

for $n \geq 2$.

For the remainder of this section, assume that $c > 0$. For $n \geq 2$, let $\mathcal{Z}_n^+$ denote the ordered list of zeros of V_n Since $\mathcal{Z}_n^+ \subset \mathcal{Z}_n$, we need only observe that in the rule given by Theorem 2 for forming zeros, the half of the numbers in $\mathcal{Z}_n$ that descend from $r = 1$ in $\mathcal{Z}_1^+$ are the numbers that comprise $\mathcal{Z}_n^+$. That is, if $n \geq 2$and $r \in \mathcal{Z}_n^+$ then

$$(cr \pm \sqrt{c^2r^2 + 4})/2 \in \mathcal{Z}_{n+1}^+.$$

For $n \geq 3$, no number in $\mathcal{Z}_n^+$ is rational, so that V_n is irreducible over the rational integers.

Let r_n denote the greatest zero of S_n, and also of V_n. Then

$$r_{n+1} = (cr_n + \sqrt{c^2r_n^2 + 4})/2,$$

and from this recurrence easily follows

$$r_{n+1} - 1/r_{n+1} = cr_n, \tag{16}$$

of which the left-hand side is the distance from the least positive zero of S_{n+1} to the greatest.

4 The case $c = 1$

We turn now to the case that $c = 1$; that is, the immediate generalization of (2) to the case that the initial value is $S_1(x) = x$. The first four polynomials S_n and V_n are as shown here:

n	S_n	V_n
1	x	
2	$x^2 - 1$	$x - 1$
3	$x^4 - 3x^2 + 1$	$x^2 - x - 1$
4	$x^8 - 7x^6 + 13x^4 - 7x^2 + 1$	$x^4 - x^3 - 3x^2 + x + 1$

Arrays of coefficients for S_n are indexed [4] as A147985 and A147990, and for T_n as A147986. The polynomials V_n are related by the equation $V_n(x) = U_n(-x)$ to polynomials U_n presented at A147989.

As mentioned in Section 2, the greatest zero r_n of S_n grows without bound as $n \to \infty$. In order to discuss r_n in some detail, define

$$z(x) = (x + \sqrt{x^2 + 4})/2,$$

so that $r_1 = 1$ and $r_n = z(r_{n-1})$ for $n \geq 2$. The sequence (r_n) has some interesting properties arising from (16). For example, if x is the positive number satisfying $1 + 1/x = x$, then $x = r_2 = (1 + \sqrt{5})/2$, and inductively, if x is the positive number satisfying

$$\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} + \frac{1}{x} = x, \tag{17}$$

then $x = r_{n+1}$. Equation (17) shows how the numbers r_n arise naturally without reference to polynomials. Since

$$r_{n+1} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} + \frac{1}{r_{n+1}},$$

we have $\sum_{k=1}^{\infty} \frac{1}{r_k} = \infty$.

Theorem 4. *If $n \geq 1$, and r_n is the greatest zero of S_n, then*

$$\sqrt{2n} - 1 < r_{n+1} < \sqrt{2n+1}.$$

Proof. Taking $c = 1$ in (16) and squaring give

$$\frac{1}{r_{n+1}^2} = r_n^2 - r_{n+1}^2 + 2,$$

whence

$$\begin{aligned} \sum_{k=2}^{n+1} \frac{1}{r_k^2} &= r_1^2 - r_{n+1}^2 + 2n, \\ r_{n+1}^2 &= 1 + 2n - \sum_{k=2}^{n+1} \frac{1}{r_k^2} \\ &< 1 + 2n, \end{aligned}$$

so that $r_{n+1} < \sqrt{2n+1}$.

We turn next to an inductive proof that

$$\sqrt{2n} - 1 < r_{n+1} \tag{18}$$

for all n. This is true for $n = 1$, and we assume it true for arbitrary n and wish to prove that $r_{n+2} > \sqrt{2n+2} - 1$. We begin with the easily proved inequality

$$4n^2 + 2n + 1 + 4n\sqrt{2n} + 4n + 2\sqrt{2n} < (2n + 2\sqrt{2n} + 1)(2n + 2).$$

Taking the square root of both sides,

$$2n + \sqrt{2n} + 1 < (\sqrt{2n} + 1)(\sqrt{2n+2}),$$

so that

$$2n < (\sqrt{2n} + 1)(\sqrt{2n+2} - 1).$$

Expanding and adding appropriate terms to both sides,

$$2n - 2\sqrt{2n+2} + 5 > 4(2n+2) + 2n + 1 - 4\sqrt{2n}\sqrt{2n+2} - 4\sqrt{2n+2} + 2\sqrt{2n}.$$

Taking the square root of both sides,

$$\sqrt{2n - 2\sqrt{2n} + 5} > 2\sqrt{2n+2} - \sqrt{2n} - 1.$$

Equivalently,

$$\sqrt{2n} - 1 + \sqrt{2n - 2\sqrt{2n} + 5} > 2\sqrt{2n+2} - 2,$$

so that by the induction hypothesis (18),

$$r_{n+1} + \sqrt{2n - 2\sqrt{2n} + 5} > 2\sqrt{2n+2} - 2. \tag{19}$$

The inequality (18), after squaring, adding 4, and taking square roots, gives

$$\sqrt{r_{n+1}^2 + 4} > \sqrt{2n - 2\sqrt{2n} + 5}.$$

In view of (19), therefore,

$$r_{n+2} = \frac{r_{n+1} + \sqrt{r_{n+1}^2 + 4}}{2} > \sqrt{2n+2} - 1.$$

∎

Theorem 4 implies that $\lim_{n\to\infty} r_n/\sqrt{n} = \sqrt{2}$ and that

$$\frac{1}{2n+1} < \frac{1}{r_{n+1}^2} < \frac{1}{(\sqrt{2n}-1)^2}.$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{r_n^2} = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{n r_n} < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{2^{n-1} r_n} < \infty.$$

The second and third sums are approximately 2.26383447 and 1.518737247. For more digits of the latter, see A154310.

5 The case $c = 2$

It is easy to prove that there is exactly one choice of $c > 0$ in (3) for which the resulting polynomial $S_n(x) + iT_n(x)$ has the form

$$(x + a + bi)^{2^n}$$

for some real a and b and all $n \geq 2$. The unique values are $c = 2$ and $(a, b) = (0, 1)$. In this case, the first three polynomials S_n, T_n, V_n are as shown here:

n	S_n	T_n	V_n
1	x	1	
2	$x^2 - 1$	$2x$	$x^2 - 2x - 1$
3	$x^4 - 6x^2 + 1$	$4x^3 - 4x$	$x^4 - 4x^3 - 6x^2 + 4x + 1$

Arrays of coefficients for S_n and T_n are included as subarrays [4] of arrays closely associated with Pascal's triangle. Specifically, for S_n see A096754, A135670, and A141665; for T_n, see A095704 and A135685. In the same way, coefficients for V_n can be read from A108086, modified in accord with the identity $V_n(x) = U_n(-x)$.

The fact that the zeros of T_n interlace those of S_n is an example of Theorem 1. However, in this case, one can also appeal to the Hermite-Biehler theorem: *if S and T are nonconstant polynomials with real coefficients, then the polynomials S and T have interlacing zeros if and only if all the zeros of the polynomial $S + iT$ lie either in the upper half-plane or the lower half-plane.* For a discussion of this theorem and related matters, see Rahman and Schmeisser [3, pp. 196–209].

6 The case $c = 2i$

Suppose that S_n is a square for some n, and write $S_n(x) = H_n^2(x)$. Then

$$S_n(\frac{x}{c} - \frac{1}{cx}) = H_n^2(\frac{x}{c} - \frac{1}{cx}).$$

By (9),

$$S_{n+1}(x) = (cx)^{2^{n-1}} H_n^2(\frac{x}{c} - \frac{1}{cx}), \tag{20}$$

which implies that S_{n+1} is a square. It is easy to show that the only nonzero choices of c for which S_3 is a square are $\pm 2i$. Equation (20) gives the recurrence

$$H_{n+1}(x) = (2ix)^{2^{n-2}} H_n(\frac{i}{2x} - \frac{ix}{2}),$$

which implies

$$\left|H_{n+1}(e^{i\theta})\right| = 2^{\deg H_n} \left|H_n(\sin\theta)\right|$$

for all real θ. Another recurrence, which follows readily from (12), is

$$H_n = 2H_{n-2}^4 - H_{n-1}^2.$$

The first four of these polynomials are as follows:

$H_3(x) = x^2 + 1$
$H_4(x) = x^4 - 6x^2 + 1$
$H_5(x) = x^8 + 20x^6 - 26x^4 + 20x^2 + 1$
$H_6(x) = x^{16} - 88x^{14} + 92x^{12} - 872x^{10} + 1990x^8 - 872x^6 + 92x^4 - 88x^2 + 1$

Coefficients for the polynomials H_7 and H_8 are given at A154308.

7 The recurrence $P_{n+1}/Q_{n+1} = (1/c)(P_n/Q_n + Q_n/P_n)$

We return now to the recurrence (1), with initial value $x_0 = P_1 = P_1(x) = x$. Taking $Q_1 = Q_1(x) = 1$ leads to sequences P_n and Q_n defined by

$$P_n = P_{n-1}^2 + cQ_{n-1}^2 \quad \text{and} \quad Q_n = cP_{n-1}Q_{n-1}.$$

The properties of these polynomials are analogous to those of the polynomials S_n and T_n already discussed. Indeed,

$$P_n(x) = S_n(ix) \tag{21}$$

for $n \geq 2$, so that the zeros of P_n are ir, where r ranges through $\mathcal{Z}_n$, and, if $c > 0$, we have interlaced lists of zeros on the imaginary axis. The recurrence (12) holds without change; that is, for $n \geq 3$, we have

$$P_n = P_{n-1}^2 + P_{n-1}P_{n-2}^2 - P_{n-2}^4.$$

Putting $x = ie^{i\theta}$ in (9) and applying (21) lead to

$$\left|P_{n+1}(e^{i\theta})\right| = |c|^{\deg P_n} \left|P_n(\frac{2}{c}\cos\theta)\right|$$

for all real θ.

For $c = 1$, coefficient arrays are given for the polynomials P_n and Q_n are indexed as A147987 and A147988, respectively.

8 Concluding remarks

The author is grateful to a referee for pointing out various properties associated with polynomials discussed in this paper — properties which may be worth further study. For example, the interlacing of zeros in Theorem 1 implies that for fixed n, the polynomials T_n and S_n are consecutive members of some sequence of orthogonal polynomials. A consequence of Theorem 2 is that there exists Euclidean straightedge-and-compass constructions for the zeros of T_n and S_n. The manner in which S_{n+1} arises from the argument of S_n in (9) is similar to the Joukowski transform. Indeed (9) can be written as

$$e^{-i2^{n-1}\theta}S_{n+1}(e^{i\theta}) = c^{2^{n-1}}S_n(\frac{2i}{c}\sin\theta),$$

so that, apart from a constant, the modulus of S_{n+1} on the unit circle is the modulus of S_n on a line segment. This has consequences for estimates, such as the derivative estimates of the Bernstein-Markov type.

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