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Permutations and Combinations of Colored Multisets

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Abstract

Given positive integers m and n, let S_n^m be the m-colored multiset $\{1^m, 2^m, \ldots, n^m\}$, where i^m denotes m copies of i, each with a distinct color. This paper discusses two types of combinatorial identities associated with the permutations and combinations of S_n^m. The first identity provides, for $m \geq 2$, an $(m-1)$-fold sum for $\binom{mn}{n}$. The second type of identities can be expressed in terms of the Hermite polynomial, and counts color-symmetrical permutations of S_n^2, which are permutations whose underlying uncolored permutations remain fixed after reflection and a permutation of the uncolored numbers.

1 Introduction

The primary object of study in this paper is a certain class of multisets, namely the m-colored multisets $S_n^m = \{1^m, 2^m, \ldots, n^m\}$, where i^m denotes m copies of i with distinct colors. Where does S_n^m naturally arise in combinatorics? Suppose we want to count the permutations of

S_n^m of length n which consist of *distinct* integers. First, we ignore the colors of the integers and notice we have a set of n distinct uncolored numbers. There are $n!$ ways to arrange these n uncolored numbers. We then decide what color to paint such numbers. Since there are m colors, this color choice gives us a factor of m^n, for a total of $m^n n!$ such permutations. In [2], the authors studied properties of multifactorials $n!_m$, where for a positive integer m, $n!_m = n(n-m)!_m$, with $0!_m = 1$. From this recursive definition, we can show that $(mn)!_m = m^n n!$. Therefore, the m-colored multiset provides a combinatorial meaning for well-known multifactorial identity.

This observation leads us to wonder if m-colored multisets could provide nice combinatorial proofs for other well known identities. In the process of investigating this question, the authors found two general classes of identities that are readily proven in the context of S_n^m. The first type of identity provides a $(m-1)$-sum formula for the binomial coefficient $\binom{mn}{n}$. This result is Theorem 3. The second type of identity counts color-symmetrical permutations of S_n^2, where a ***color-symmetrical*** permutation of S_n^2 is a colored permutation whose underlying set partition structure is fixed via vertical reflection. The color-symmetrical permutation identities occur in Section 3 have connections with Hermite polynomials. The techniques used to prove the identities of Section 3 recall the methodology the first author used to enumerate symmetrically inequivalent two dimensional proper arrays [5].

2 Enumerating Combinations and Permutations

Let i, m, and n be positive integers. Let i^m denote m copies of i, with the property that each copy of i has a *distinct* color. Let S_n^m be the m***-colored multiset*** $\{1^m, 2^m, \ldots, n^m\}$. For examples,

$$S_4^2 = \{1, 1, 2, 2, 3, 3, 4, 4\}, \qquad \text{and} \qquad S_4^3 = \{1, 1, 1, 2, 2, 4, 3, 3, 3, 4, 4, 4\}.$$

Note that $|S_n^m| = mn$. If we consider different colored copies of i as distinct, the number of permutations of S_n^m is $(mn)!$. From this simple observation, we are able to derive an identity for $(mn)!$ involving an $(m-1)$-fold summation. We will now assume $m \geq 2$. The proof of this identity utilizes simple combinatorial arguments. We demonstrate the argument for the case of $m = 2$, and then state the general result for $m \geq 2$.

Consider $S_n^2 = \{1^2, 2^2, \ldots, n^2\}$, where each number occurs in red and blue. The number of permutations of S_n^2 is $(2n)!$. We think of a permutation of S_n^2 as a horizontal row of $2n$ elements arranged in $2n$ slots. Our analysis proceeds by carefully analyzing the manner in which we fill in the first half, or the first n slots, of the permutation. We say a number i, where $1 \leq i \leq n$, has a ***repetition*** in the first half if and only if both the red i and the blue i can be found there. Let s be the number of repetitions that occur in the first half of the permutation. Note that $0 \leq s \leq [\frac{n}{2}]$. For each s, the number of ways to fill the first n slots can be computed in three steps (see Figure 1).

1. Pick the locations, in the first half of the permutation, where the repetitions will occur. The number of ways to select s pairs of slots from the n slots is

$$\prod_{j=1}^{s} \frac{\binom{n-2j+2}{2}}{s!} = \frac{n!}{s!(n-2s)!2^s}.$$

2. Pick the numbers to fill these s pairs of slots. In order to do this, we first form an s-permutation from $D = \{1, 2, \ldots, n\}$. The k^{th} number in this s-permutation will fill the k^{th} pair of slots. For each pair of slots, we must decide if we want red first, then blue, or vice versa. Hence, the number of ways to fill these s pairs of slots is $\frac{n!}{(n-s)!}2^s$.

3. Fill the remaining $n - 2s$ spaces in the first half of the permutation with numbers that have not been used in Step 2. The number of ways to do this is $\frac{(n-s)!}{s!}2^{n-2s}$.

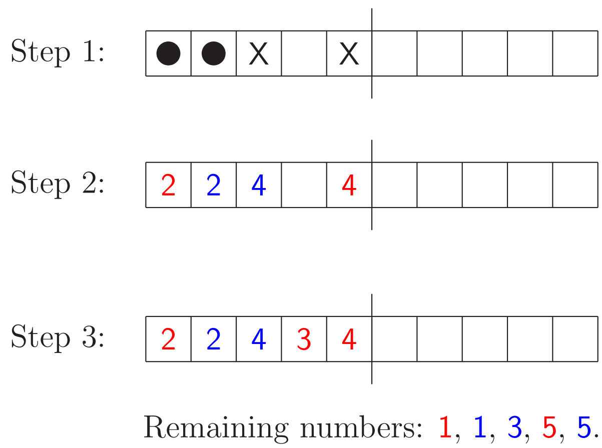

Figure 1: Constructing the first half of a permutation of S_5^2.

After completing these three steps, we have n colored numbers remaining. They fill the second half of the permutation in $n!$ ways. Varying s and combining the aforementioned combinatorial reasoning proves Lemma 1.

Lemma 1. *Let n be a nonnegative integer. Then,*

$$(2n)! = n! \sum_{s=0}^{[\frac{n}{2}]} \frac{n!n!2^{n-2s}}{s!s!(n-2s)!} = (n!)^2 \sum_{s=0}^{[\frac{n}{2}]} \binom{n}{s}\binom{n-s}{n-2s} 2^{n-2s}. \qquad (1)$$

Remark 2. Lemma 1 provides a combinatorial proof of Equation (3.99) in [1] since we may rewrite Equation (1) as follows.

$$\begin{aligned} \binom{2n}{n} &= \sum_{s=0}^{[\frac{n}{2}]} \frac{n!2^{n-2s}}{s!s!(n-2s)!} \\ &= \sum_{s=0}^{[\frac{n}{2}]} \frac{n!}{(2s)!(n-2s)!} \cdot \frac{(2s)!}{s!s!} 2^{n-2s} \\ &= \sum_{s=0}^{[\frac{n}{2}]} \binom{n}{2s}\binom{2s}{s} 2^{n-2s}. \end{aligned}$$

Here is an alternative argument for deriving Equation (1). Let $D = \{1, 2, \ldots, n\}$. The number of permutations of S_n^2 is $(n!)^2$ times the number of ways to select the n colored numbers which occur in the first half. Let s denote the number of numbers that occur twice, that is, both the red and blue copies are selected. We note that $0 \le s \le [\frac{n}{2}]$. There are $\binom{n}{s}$ ways to select these s numbers. From the remaining $n - s$ numbers in D, there are $\binom{n-s}{n-2s}$ ways to select $n - 2s$ uncolored numbers, where each uncolored number has a choice of two colors to chose from. Summing over s yields the right side of Equation (1).

We are now in a position to generalize Lemma 1. We will only describe how to generalize the second argument. Those readers interested in the generalization of the first argument may contact the authors. We think of a permutation of S_n^m as a horizontal arrangement of mn elements into mn slots. Subdivide, from left to right, these mn slots into m sections, each containing n slots. We analyze the number of repetitions that occur the first section, or the leftmost n slots, of the permutation.

We say a number has a ***repetition of type*** k if it appears exactly k times in the first section. Let s_k be the number of numbers of type k. The number of ways to choose n colored numbers to fill the first section of the permutation is clearly $\binom{mn}{n}$. Alternatively, we could choose these n colored numbers from $D = \{1, 2, \ldots, n\}$ as follows. First, we select the type m numbers, then the type $m - 1$ numbers, and so forth. There are $\binom{n}{s_m}$ ways to chose the type m numbers, each of which will use up all m colors. In general, after numbers of types m through $k + 1$ are chosen, we have $n - \sum_{i=k+1}^{m} s_i$ numbers left in D from which to select s_k numbers of type k. Each such number can be colored in $\binom{m}{k}$ ways. This argument proves the following theorem.

Theorem 3. *Let n be a positive integer. Let m be a positive integer, $m \ge 2$. Then,*

$$\binom{mn}{n} = \sum_{\substack{s_1, s_2, \ldots, s_m \ge 0 \\ s_1 + 2s_2 + \cdots + ms_m = n}} \prod_{k=1}^{m} \binom{n - \sum_{i=k+1}^{m} s_i}{s_k} \binom{m}{k}^{s_k}. \tag{2}$$

Two important non-trivial examples of Theorem 3 occur when $m = 3$ and $m = 4$. Let $m = 3$, and $s_3 = s$, and $s_2 = t$. Then, Equation (2) becomes

$$\binom{3n}{n} = \sum_{s=0}^{[\frac{n}{3}]} \sum_{t=0}^{[\frac{n-3s}{2}]} \binom{n}{3s + 2t} \binom{3s + 2t}{s, t, 2s + t} 3^{n-3s-t}.$$

If $m = 4$, Equation (2) becomes

$$\binom{4n}{n} = \sum_{s_4=0}^{[\frac{n}{4}]} \sum_{s_3=0}^{[\frac{n-4s_4}{3}]} \sum_{s_2=0}^{[\frac{n-4s_4-3s_3}{2}]} A \binom{n}{n - 4s_4 - 3s_3 - 2s_2} \binom{4s_4 + 3s_3 + 2s_2}{s_4, s_3, s_2, s_2 + 2s_3 + 3s_4},$$

where,

$$A = \frac{4^{n-4s_4-3s_3-2s_2} (4 \cdot 3 \cdot 2)^{s_3} (4 \cdot 3)^{s_2}}{(3!)^{s_3} (2!)^{s_2}}.$$

On closer inspection of the proof of Theorem 3, we see that argument describes how to select any l colored numbers from S_m^n. Thus, we have actually proven the next result.

Theorem 4. *Let n and m be positive integers. Let l be a positive integer such that $1 \le l \le mn$. Then,*

$$\binom{mn}{l} = \sum_{\substack{s_1, s_2, \ldots, s_m \ge 0 \\ s_1 + 2s_2 + \cdots + ms_m = l}} \prod_{k=1}^{m} \binom{n - \sum_{i=k+1}^{m} s_i}{s_k} \binom{m}{k}^{s_k}.$$

3 Color-Symmetrical Permutations of S_n^2

We now study a special kind of reflective symmetry in the permutations of S_n^2. Think of a permutation σ of S_n^2 as a way to fill the $2n$ squares of a $1 \times 2n$ rectangular array, and associate with it a partition of $\{1, 2, \ldots, 2n\}$ in the form $\pi(\sigma) = \{S_1, S_2, \ldots, S_n\}$ such that S_i is the 2-subset containing the two positions occupied by i. We call $\pi(\sigma)$ the ***set partition associated with*** σ. We say a permutation is ***color-symmetrical*** if, upon reflection about the vertical line through its middle, the resulting permutation has the same collection of 2-subsets in its associated set partition. For example, the permutation p_2 in Figure 2 is obtained from p_1 by reflection. Notice that

$$\pi(p_1) = \pi(p_2) = \big\{\{2,6\}, \{5,9\}, \{3,4\}, \{7,8\}, \{1,10\}\big\}.$$

Therefore p_1 is color-symmetrical, and, so is p_2.

Figure 2: Two color-symmetrical permutations of S_5^2.

Another way to understand this notion of color-symmetry is as follows. Let p_2 be the permutation obtain from p_1 via reflection. The reflection can be viewed as a function $\phi : S_n^2 \to S_n^2$ such that $\phi(p_1) = p_2$. We say that σ is color-symmetrical if, for each i, $\phi(i^2) = j^2$, for some j (recall that i^2 means the two copies of i colored red and blue). In other words, p_1 is color-symmetrical if one could obtain p_2 by renaming the colored numbers while preserving the associated set partition. Due to symmetry, if $\phi(i^2) = j^2$, we also $\phi(j^2) = i^2$. So in effect we are interchanging i^2 with j^2 in the reflection. For example, in the two permutations p_1 and p_2 in Figure 2, the numbers 1 and 2 are interchanged, and so are 3 and 4. Note that it is possible for $\phi(i^2) = i^2$, as are the two copies of 5 in Figure 2.

The idea of fixing set partition structure and interchanging labels was described by Ross Drewe in sequence A047974 of the OEIS [3]. The difference between our context, and that of Drewe's, is that our set label, namely the numbers, contain an extra parameter of color.

Our goal is to describe a formula for SS_n^2, the number of color-symmetrical permutations of S_n^2. Our strategy is to divide the $1 \times 2n$ rectangle into two halves, each with n squares. We fill the first half, or the leftmost n squares, with an arbitrary colored permutation of length n derived from the elements of S_n^2. We then use reflective symmetry to complete the second half of the $1 \times 2n$ rectangle. Since reflective symmetry *must* preserve the set partition structure, there are three types of numbers that can occur in the first half of the $1 \times 2n$ rectangle. These possibilities are the same possibilities the first author used to calculate H_{2m} in [5]. In particular,

- A number could map to a number that does *not* occur in the first half.
- A number could map to itself under reflection.
- A number could map to another number which also *appears* in the first half. If this happens, we say a ***first-half interchange*** (or simply FHI) occurs.

Figure 3 illustrates these three cases for a color-symmetric permutation of S_5^2.

Filling in the first half:	5	1	2	1	3					
Vertical reflection, with $1 \leftrightarrow 4$, $2 \leftrightarrow 3$, $5 \leftrightarrow 5$:	5	1	2	1	3	2	4	3	4	5

Figure 3: Constructing a color-symmetrical permutation of S_5^2.

Going through these three possibilities, we are able to derive a formula that counts color-symmetrical permutations of S_n^2, as follows.

1. Determine, in the first half, where the repetitions occur. Under reflection, a repetition must map to a different repetition in the second half. We arbitrarily assign a color scheme to each repetition. If s is the number of repetitions that occur in the first n squares, the number of ways to complete these s double repetitions in a color-symmetrical manner is
$$\frac{n!}{2^s s!(n-2s)!} \cdot \frac{n! 2^{2s}}{(n-2s)!}.$$
2. There are now $n-2s$ spaces in the first half that remain to be filled. We must fill these $n-2s$ spaces with a colored permutation that does not have any repetitions. The number of ways to do that is $(n-2s)!2^{n-2s}$.
3. Determine where the FHIs occur. Notice that any numbers that do not form a FHI pair must map to themselves. Let t be the number of FHI pairs that occur among the $n-2s$ non-repeating positions in the first n slots. The number of ways to place these t FHI pairs is $\frac{(n-2s)!}{2^t t!(n-2s-2t)!}$.

Lemma 5. *For $n \geq 1$,*
$$SS_n^2 = \sum_{s=0}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n-2s}{2}]} \frac{n!n!2^{n-s-t}}{s!t!(n-2s-2t)!},$$

By standard convolution arguments, it is easy to show that

$$\sum_{n=0}^{\infty} SS_n^2 \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} \sum_{\substack{r,s,t\geq 0\\ r+2s+2t=n}} \frac{(2x^2)^s}{s!} \cdot \frac{(2x^2)^t}{t!} \cdot \frac{(2x)^r}{r!} = e^{2x^2} \cdot e^{2x^2} \cdot e^{2x} = e^{4x^2+2x}. \tag{3}$$

We can write the right hand side of Equation (3) as

$$e^{4x^2+2x} = \sum_{k,j=0}^{\infty} \frac{(4x^2)^k}{k!} \cdot \frac{(2x)^j}{j!} = \sum_{k,j=0}^{\infty} \frac{(2x)^{2k+j}}{k!j!} = \sum_{j=0}^{\infty} \sum_{k=0}^{[\frac{j}{2}]} \frac{2^j}{k!(j-2k)!} x^j. \tag{4}$$

Comparing the coefficients in (3) and (4) yields the next result.

Lemma 6. *For $n \geq 1$,*

$$SS_n^2 = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!n!2^n}{k!(n-2k)!} = 2^n n! \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \binom{n-k}{k} k!. \tag{5}$$

We should note that the right sum of Equation (5) is reminiscent of $H_n(x)$, the Hermite polynomial of degree n, whose explicit formula is [1]

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{k} \binom{n-k}{k} k!(2x)^{n-2k}. \tag{6}$$

If we let $x = \frac{i}{2}$ in Equation (6), we obtain another representation for SS_n^2.

Corollary 7. *For $n \geq 1$,*

$$SS_n^2 = (-2i)^n n! H_n\left(\frac{i}{2}\right).$$

4 Three Variations on SS_n^2

When we defined the notion of a color-symmetrical permutation of S_n^2, we only concerned ourselves with fixing the set partition associated with the permutation. The color scheme for each part of the set partition was arbitrarily assigned, and hence played *no* role in determining whether a permutation of S_n^2 was color-symmetric. We now want to put restrictions on our color scheme.

The most natural restriction is to require colors to map to themselves whenever possible. Recall that for $S_n^2 = \{1^2, 2^2, \ldots, n^2\}$, we have one red i and one blue i for each i. The aforementioned color restriction would require that if the set partition structure maps i to j, where $1 \leq i < j \leq n$, we *also* require that a red i maps to a red j, and hence, a blue i maps to a blue j.

Note that there is no color restriction on i if the reflection maps i to itself. One possibility for color restriction involves the repetitions in the first half of the $1 \times 2n$ rectangle, since

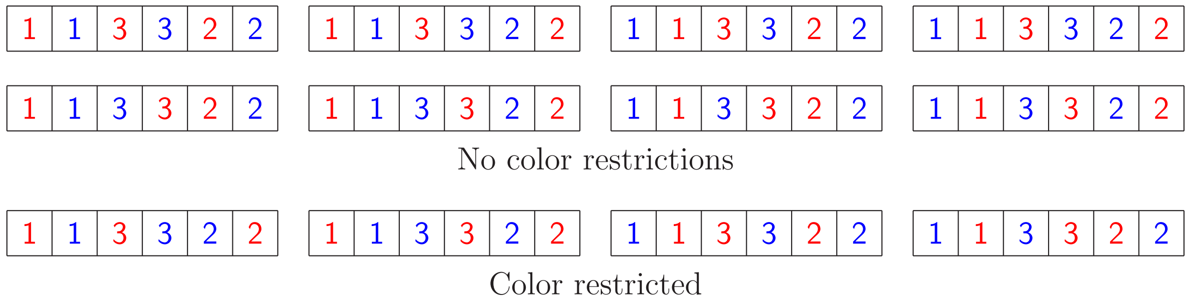

Figure 4: Color-symmetrical permutations of S_3^2 with and without color restrictions.

these repetitions have an arbitrarily assigned color scheme. Figure 4 illustrates how the color restriction (or lack thereof) applies to a subset of color-symmetrical permutation of S_3^2 which has repetitions in the first half.

The only change in the formula associated with Step 1 in Section 3 is that the factor 2^{2s} in the numerator of the second fraction becomes a 2^s. Note that Steps 2 and 3 stay the same. We find the following double sum formula for $\widehat{SS}_n^2$, the number of color-symmetrical permutations of S_n^2 whose double repetition blocks obey the color restriction.

Lemma 8. *For $n \geq 1$,*

$$\widehat{SS}_n^2 = \sum_{s=0}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n-2s}{2}]} \frac{n!n!2^{n-2s-t}}{s!t!(n-2s-2t)!}.$$

Using standard convolution arguments, we can show that

$$\sum_{n=0}^{\infty} \widehat{SS}_n^2 \frac{x^n}{(n!)^2} = e^{3x^2+2x}.$$

With the appropriate changes to the argument used to prove Lemma 6, we can prove the following result.

Lemma 9. *For $n \geq 1$,*

$$\widehat{SS}_n^2 = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!n!3^k 2^n}{2^{2k} k!(n-2k)!} = 2^n n! \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \binom{n-k}{k} \left(\frac{3}{4}\right)^k k!.$$

An alternative proof to Lemma 9, which could also prove Lemma 8, is as follows.

$$\begin{aligned}
\widehat{SS}_n^2 &= \sum_{k=0}^{[\frac{n}{2}]} \frac{2^{n-k}(n!)^2}{(n-2k)!} \sum_{\substack{s,t\geq 0\\ s+t=k}} \frac{1}{s!t!2^s} \\
&= \sum_{k=0}^{[\frac{n}{2}]} \frac{2^{n-k}(n!)^2}{k!(n-2k)!} \sum_{s=0}^{k} \binom{k}{s} \frac{1}{2^s} \\
&= \sum_{k=0}^{[\frac{n}{2}]} \frac{2^{n-k}(n!)^2}{k!(n-2k)!} \left(\frac{3}{2}\right)^k \\
&= 2^n n! \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k}\binom{n-k}{k}\left(\frac{3}{4}\right)^k .
\end{aligned}$$

We should also note that we can use the Hermite polynomial to calculate $\widehat{SS}_n^2$.

Corollary 10. *For $n \geq 1$,*

$$\widehat{SS}_n^2 = (-i\sqrt{3})^n n! H_n\left(\frac{i}{\sqrt{3}}\right).$$

The second way to place the color restriction is in a FHI pair. This is done in the following manner. Let $1 \leq i < j \leq n$. Suppose both i and j appear in the first n slots. Furthermore, suppose vertical reflection maps i to j. If we determine which color of i occurs in the first n slots, we have *also* determined the color scheme for j. We demonstrate in Figure 5 the color restriction for those color-symmetrical permutations of S_3^2 which have a FHI.

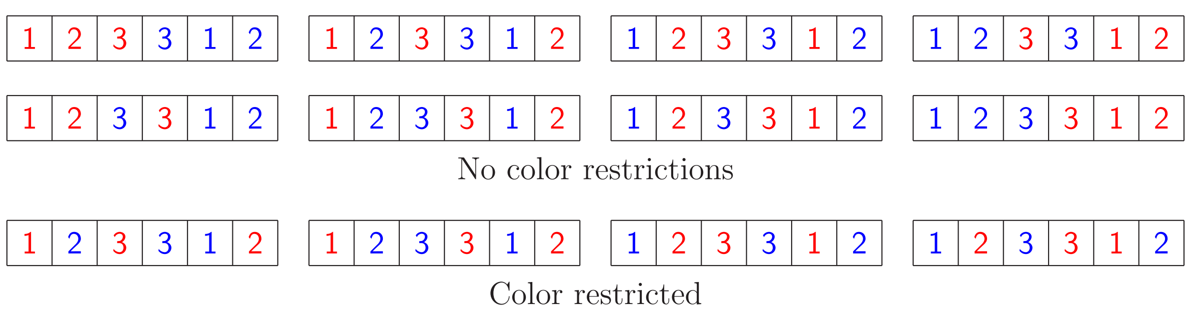

Figure 5: Color-symmetrical permutations of S_3^2 with 1 and 2 interchanged.

We construct an isomorphism between those color-symmetrical permutations of S_n^2 with color restriction on the repetitions and those with color restriction on the interchanges, as follows. For each pair of slots where a repetition occurs in the first half of the permutation, switch the second colored number with its reflective image. This operation changes a repetition into a FHI. This procedure is reversible, that is, it maps a FHI to a repetition .

For example, this isomorphism maps the first four permutations in Figure 4 to the first four permutations in Figure 5 (and vice versa). If we let $\widetilde{SS}_n^2$ be the number of color-symmetrical permutations of S_n^2 whose FHIs obey the color restriction, the previous reasoning leads to our next lemma.

Lemma 11. *Let $\widetilde{SS}_n^2$ be as previously defined. Then,*

$$\widetilde{SS}_n^2 = \widehat{SS}_n^2 = \sum_{s=0}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n-2s}{2}]} \frac{n!n!2^{n-2s-t}}{s!t!(n-2s-2t)!}. \tag{7}$$

We could also derive Equation (7) by carefully analyzing how we place the FHI pairs, and their associated color scheme, in the first half of the permutation. Details of this method are available upon request from the authors.

For the third variation, we will place the color restriction on *both* the repetitions and the FHI pairs. Combining the techniques for the previous two variations, we obtain the following result.

Lemma 12. *Let $\dddot{SS}_n^2$ be the number of color-symmetrical permutations of S_n^2 which have the color restriction on repetitions* and *the interchange pairs. Then,*

$$\dddot{SS}_n^2 = \sum_{s=0}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n-2s}{2}]} \frac{n!n!2^{n-2s-2t}}{s!t!(n-2s-2t)!}.$$

By applying standard convolution arguments, we find

$$\sum_{n=0}^{\infty} \dddot{SS}_n^2 \frac{x^n}{(n!)^2} = e^{2x^2+2x}.$$

The next result is obtained from an argument similar to that of Lemma 9.

Lemma 13. *For $n \geq 1$,*

$$\dddot{SS}_n^2 = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!n!2^{n-k}}{k!(n-2k)!} = 2^n n! \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k} \binom{n-k}{k} \left(\frac{1}{2}\right)^k k!. \tag{8}$$

Corollary 14. *For $n \geq 1$,*

$$\dddot{SS}_n^2 = (-i\sqrt{2})^n n! H_n \left(\frac{i}{\sqrt{2}}\right).$$

We end this section with a table that records the values of SS_n^2, $\widehat{SS}_n^2 = \widetilde{SS}_n^2$, and $\dddot{SS}_n^2$ for
$1 \leq n \leq 8$.

n	1	2	3	4	5	6	7	8
SS_n^2	2	24	336	9600	311040	15252480	840591360	61281239040
$\widehat{SS}_n^2$	2	20	264	6432	191040	8081280	401990400	25439016960
$\dddot{SS}_n^2$	2	16	192	3840	99840	3502080	149667840	7865946880

Table 1: Values for SS_n^2, $\widehat{SS}_n^2 = \widetilde{SS}_n^2$, and $\dddot{SS}_n^2$.

5 Closing Remarks and Open Questions

In this paper, we have discussed two classes of identities which are readily proven using colored permutations and colored combinations. The first such identity is, for $m \geq 2$, an $(m-1)$-fold sum for $\binom{mn}{n}$. The second class of identities enumerates the color-symmetrical permutations of the 2-colored multiset S_n^2. It is natural to ask for a generalization in S_n^m. Currently, the authors are working on the cases of $m = 3$ and $m = 4$. There are two basic differences between the $m = 2$ situation and the $m = 3$ case. When $m = 3$, no number can be mapped to itself under vertical reflection. Also, one must analyze the case of even n separately from odd n.

Another possible topic for future research involves arranging the colored permutations of S_n^m into an $m \times n$ rectangular grid, and then using a symmetry operation of the $m \times n$ rectangle to enumerate those colored permutations whose set partition structure is fixed with respect to this symmetry operation. Such an enumeration will rely on the techniques the first author used to count the symmetrical inequivalent $m \times n \times p$ letter representations [5].

Instead of using the colored multiset S_n^m, we could relate the problems to the set $[mn] = \{1, 2, \ldots, mn\}$ by matching the integer i colored k from S_n^m with the integer $(i-1)m + k$ from $[mn]$. Essentially we are grouping the integers from S_n^m according to their values, and within the same value, lining up the integers according to their colors.

Of course, we can also match the integer i colored k from S_n^m with the integer $(k-1)n+i$ from $[mn]$. This time, all the integers of the same color are grouped together, and within the same color, the integers are lined up in ascending order.

These two correspondences could lead to other questions. For instance, the symmetry in the permutations can be defined according to certain number-theoretic properties. These symmetries may be rather different from what we have discussed above. The options are plentiful, and we invite the readers to explore them.

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