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Multiple Binomial Transforms and Families of Integer Sequences

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Abstract

Based on the multiple binomial transforms introduced in this paper, the n-fold generating and generated sequences of a given integer sequence can be defined and a family of this integer sequence can be constructed. The family sets form a partition of the set of integer sequences. Special attention is paid to the recurrent integer sequences, which are produced by some linear and homogeneous recurrence relations or difference equations. For the recurrent integer sequences, a distinct rule to construct their families is obtained based on the linear difference calculus.

1 Introduction

We know that if for an integer sequence $a(t)$, $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, a function $g(x)$, $x \in \mathbb{R}$, has a power series form: $g(x) = \sum_{j=0}^{\infty} a(j)\frac{x^j}{j!}$, then $g(x)$ is called *exponential generating function* of the integer sequence $a(t)$ (see [1]). However, the generating function is a real function defined on real numbers $\mathbb{R}$, not another integer sequence. If the generating function itself still is an integer sequence, then we can continuously repeat such operation on the new integer sequence and generate a series of integer sequences, one after another.

In fact, we can accomplish this thinking, for we know that in sense of the calculus on time scales, the binomial coefficients $\{\binom{t}{j}, j = 0, 1, 2, \ldots\}$, are just the "polynomials on time scale $\mathbb{N}_0$,", like $\{\frac{x^j}{j!}, j = 0, 1, 2, \ldots\}$, which are the polynomials on time scale $\mathbb{R}$ (see [2]). Any integer sequence can be expanded to be a "binomial coefficient series" with integer coefficients (called *the discrete Taylor series*, or *the inverse binomial transform* [1, 2]), and on the other hand, any integer sequence also always can be "binomial expansion" coefficients of another integer sequence (called *the binomial transform* [3]). In this way, we tie one given integer sequence in with others by using successive binomial transforms or inverse binomial transforms (we call it the *Multiple Binomial Transform*), which form a series of new integer sequences and naturally compose a large family of the given integer sequence.

The successive binomial transforms or inverse binomial transforms have been used in some authors' work. For example, Spivey and Steil [5] used successive binomial transforms or inverse binomial transforms to prove the invariance of the Hankel transform of integer sequences under the falling k-binomial transform when k is an integer. The falling k-binomial transform is a new variation of the binomial transform introduced in [5].

In this paper, we give a special attention to the recurrent integer sequences (The integer geometric sequence, the Fibonacci sequence, and the Tribonacci sequence are some examples), and describe a distinct rule to construct their families based on the linear difference calculus. By the way, please note that in the text of this paper, for any integer sequence $a(t)$, we always regard $t \in \mathbb{N}_0$.

2 Multiple binomial transforms

Definition 1. [Multiple binomial transform] Let $a(t)$ be an integer sequence. Then we define the following transform $\phi_n(a)$ to be the n-fold binomial transform of $a(t)$: for $n = 1, 2, \ldots,$

$$b(t) = \phi_n(a) = \sum_{k_n=0}^{t} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} a(k_1) \binom{k_2}{k_1} \binom{k_3}{k_2} \cdots \binom{k_n}{k_{n-1}} \binom{t}{k_n}. \tag{1}$$

The integer sequence $b(t)$ is called the *image sequence* of $a(t)$ with respect to the n-fold binomial transform, denoted by $b = \phi_n(a)$. Conversely, the integer sequence $a(t)$ is called the *inverse image sequence* of $b(t)$, and can be denoted by $a = \phi_{-n}(b)$. For the case of $n = 0$, define $\phi_0 = 1$ (the identity transform).

Remark 2. We can see from (1), the $\phi_1(a)$ ($\phi_{-1}(a)$) is just well-known binomial (inverse binomial) transform of $a(t)$ (see [3]). $\phi_n(a)$ can be considered as n successive binomial transform ϕ_1 on $a(t)$: $\phi_n(a) = \underbrace{\phi_1(\phi_1(\cdots(\phi_1}_{n-fold}(a)))$, and $\phi_{-n}(a)$ can be considered as n successive inverse binomial transform ϕ_{-1} on $a(t)$: $\phi_{-n}(a) = \underbrace{\phi_{-1}(\phi_{-1}(\cdots(\phi_{-1}}_{n-fold}(a)))$

Proposition 3. *If defining successive two multiple binomial transforms on an integer sequence as a transform multiplication, then the set of the multiple binomial transform on integer sequences,* $\Phi(a) = \{\phi_n(a), n = 0, \pm 1, \pm 2, \ldots\}$ *(a is any integer sequence) is an Abelian transformation group. We can call it the binomial transformation group.*

Proof. From (1) we can see that for any two transforms, ϕ_n and ϕ_m, $(\phi_m \cdot \phi_n)(a) = \phi_m(\phi_n(a)) = \phi_{n+m}(a)$, and $(\phi_n \cdot \phi_m)(a) = \phi_n(\phi_m(a)) = \phi_{m+n}(a)$, that is, $\phi_n \cdot \phi_m = \phi_m \cdot \phi_n$ (the commutative law). For three successive transforms ϕ_n, ϕ_m and ϕ_p, we have that $(\phi_n \cdot \phi_m) \cdot \phi_p = \phi_{m+n} \cdot \phi_p = \phi_{p+m+n}$, and $\phi_n \cdot (\phi_m \cdot \phi_p) = \phi_n \cdot \phi_{p+m} = \phi_{p+m+n}$, that is, $(\phi_n \cdot \phi_m) \cdot \phi_p = \phi_n \cdot (\phi_m \cdot \phi_p)$ (the associative law). Besides for any n, noticing $\phi_0 = 1$, we have that $\phi_n \cdot \phi_0 = \phi_0 \cdot \phi_n = \phi_n$ and $\phi_{-n}\phi_n = \phi_n\phi_{-n} = \phi_0$. Hence, $(\Phi(a), \cdot)$ is an Abelian transformation group. □

Proposition 4. *Let $a(t)$ be an integer sequence. Then the terms of the inverse image sequence of $a(t)$ with respect to the binomial transform, $a^{(-1)}(t) = \phi_{-1}(a)$, are the discrete Taylor expansion coefficients of $a(t)$, that is,*

$$a^{(-1)}(t) = \triangle^t a(0) = \sum_{j=0}^{t} (-1)^j \binom{t}{j} a(t-j) = \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} a(j). \tag{2}$$

The inverse image sequence of $a(t)$ with respect to the n-fold binomial transform, denoted by $a^{(-n)}(t) = \phi_{-n}(a)$, can be calculated by n successive ϕ_{-1} transforms.

Proof. From Definition 1, we have that $a(t) = \sum_{k=0}^{t} a^{(-1)}(k)\binom{t}{k}$. On the other hand, the discrete Taylor expansion of $a(t)$ is that $a(t) = \sum_{k=0}^{t} \triangle^k a(0)\binom{t}{k}$ (see [2]). Hence, integer sequence $a^{(-1)}(t) - \triangle^t a(0)$ is the 1-generated sequences of the zero sequence $\omega(t) = 0, t = 0, 1, 2, \ldots$ (A000004 in [3]), that is, $\omega(t)$ itself. Therefore, $a^{(-1)}(t) = \triangle^t a(0)$. For $a^{(-n)}(t) = \phi_{-n}(a)$, from Definition 1 and Remark 2, we have that $\phi_{-n}(a) = \underbrace{\phi_{-1}(\phi_{-1}(\cdots(\phi_{-1}}_{n-fold}(a)))$. □

3 Families of integer sequences

Definition 5. [Generating and generated sequences] Let $a(t)$ be an integer sequence. Define the image sequence of the n-fold binomial transform of $a(t)$, denoted by $a^{(n)}(t)$, as the n-fold generating sequence of $a(t)$, and the inverse image sequence of the n-fold binomial transform of $a(t)$, denoted by $a^{(-n)}(t)$, as the n-fold generated sequence of $a(t)$, where $n = 1, 2, \ldots$. In case $n = 0$, $a^{(0)}(t) = a(t)$.

From Definition 1 and 5, and Proposition 4, we can get the following two Corollaries.

Corollary 6. *If integer sequence $a(t)$ is the n-fold generating (n-fold generated) sequence of integer sequence $b(t)$, then $b(t)$ is the n-fold generated (n-fold generating) sequence of $a(t)$.*

Corollary 7. *The terms of the n-fold generated integer sequence of an integer sequence $a(t)$ can be calculated as follows,*

$$a^{(-1)}(t) = \triangle^t a(0), a^{(-2)}(t) = \triangle^t a^{(-1)}(0), \ldots, a^{(-n)}(t) = \triangle^t a^{(-(n-1))}(0). \tag{3}$$

Remark 8. For the zero sequence $\omega(t)$, obviously, each of its n-generating and n-generated sequences ($n = 1, 2, \ldots$) is still the zero sequence itself.

Remark 9. For the unit pulse sequence $\delta(t)$: $\delta(0) = 1$, and $\delta(t) = 0$ for $t = 1, 2, \ldots$ (A000007 in [3]). $\delta^{(n)}(t) = n^t$ ($n = \pm 1, \pm 2, \ldots$). Hence, we may obtain some interesting identities, such as for $n = 1, 2, \ldots$,

$$\sum_{k_n=0}^{t} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \delta(k_1) \binom{k_2}{k_1} \binom{k_3}{k_2} \cdots \binom{k_n}{k_{n-1}} \binom{t}{k_n} = n^t,$$

and

$$\sum_{k_n=0}^{t}\sum_{k_{n-1}=0}^{k_n}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(-n)^{k_1}\binom{k_2}{k_1}\binom{k_3}{k_2}\cdots\binom{k_n}{k_{n-1}}\binom{t}{k_n}=\delta(t),$$

Furthermore, because the identical sequence $b(t)=1$ (A000012 in [3]) is the 1-generating sequence of $\delta(t)$, from the above identity we have that for $n=1,2,\ldots$,

$$\sum_{k_{n+1}}^{t}\sum_{k_n=0}^{k_{n+1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(-n)^{k_1}\binom{k_2}{k_1}\binom{k_3}{k_2}\cdots\binom{k_{n+1}}{k_n}\binom{t}{k_{n+1}}\equiv 1,$$

Definition 10. [Family of integer sequences] Let $a(t)$ be an integer sequence, and $a^{(n)}(t)$, $n=\pm1,\pm2,\ldots$ be respectively the n-fold generating and $n$fold -generated sequences of $a(t)$. Then we define the set $\{a^{(n)}(t),\ n=0,\pm1,\pm2,\ldots\}$ to be the *family* of $a(t)$, and denote it by $\mathcal{F}[a]$.

Proposition 11. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of integer sequences. Then $\mathcal{F}_1=\mathcal{F}_2$ iff set $\mathcal{F}_1\bigcap\mathcal{F}_2$ is nonempty.*

Proof. If $\mathcal{F}_1=\mathcal{F}_2$, obviously $\mathcal{F}_1\bigcap\mathcal{F}_2$ is nonempty. Conversely, if $\mathcal{F}_1\bigcap\mathcal{F}_2$ is a nonempty set and integer sequence $c(t)\in\mathcal{F}_1\bigcap\mathcal{F}_2$, then from Definition 10 and Corollary 6, an arbitrary element of $\mathcal{F}_1$ is a certain generation generating or generated sequence of $c(t)$, and hence it is also an element of $\mathcal{F}_2$. Vice versa. Hence, $\mathcal{F}_1=\mathcal{F}_2$. □

Corollary 12. *Let n be an arbitrary integer, and integer sequence $a^{(n)}$ be the n-generating or n-generated sequence of integer sequence a. If $\mathcal{F}_1[a]$ is family of a and $\mathcal{F}_2[a^{(n)}]$ is family of $a^{(n)}$, then $\mathcal{F}_1[a]=\mathcal{F}_2[a^{(n)}]$.*

Theorem 13 (Existence and Uniqueness)**.** *Every integer sequence belongs to one and only one family.*

Proof. We can directly construct a family of a given sequence by using Definitions 5 and 10. This is Existence proof. Uniqueness is a direct conclusion of Proposition 11. □

Remark 14. Theorem 13 implies that the family sets form a partition of the set of all integer sequences.

Remark 15. Most of the families of integer sequences are infinite sets. However, the family of zero sequence $\omega(t)$ is a one element set $\{\omega(t)\}$.

We now give an interesting property of the families of integer sequences as follows.

Theorem 16. *Let $\mathcal{F}[a]$ be the family of an integer sequence $a(t)$, and $a^{(n)}(t), n=0,\pm1,\pm2,\ldots,$ be all the elements of set $\mathcal{F}[a]$. Then all of the sequences, $a^{(n)}(t)$, have the same Hankel transform.*

Proof. We know from [4] that the Hankel transform is invariant under the binomial transform. Hence from they definitions, all of the sequences $a^{(n)}(t), n=0,\pm1,\pm2,\ldots,$ have the same Hankel transform. □

4 Families of recurrent integer sequences

An integer sequence $a(t)$ is called a recurrent integer sequence, if it is produced by a linear and homogeneous recursion formula as follows:

$$a(0) = a_0, a(1) = a_1, \ldots, a(p-1) = a_{p-1}, \quad \text{and} \quad a(t+p) = \sum_{k=1}^{p} \xi_k a(t+p-k),\ t = 0, 1, 2, \ldots, \tag{4}$$

where, all of a_j $(j = 0, 1, 2, \ldots, p-1)$ and ξ_k $(k = 1, 2, \ldots, p)$ are integers, and $\xi_p \neq 0$. The positive integer p is called the recurrence order of $a(t)$. For such a recurrent integer sequence $a(t)$, we know the following basic results.

The recursion formula (4) corresponds to a linear and homogeneous difference equation for $a(t)$. By using a basic relation in the difference equation theory (see [6]): $a(t+k) = \sum_{j=0}^{k} \binom{k}{j} \triangle^{k-j} a(t)$, $k = 1, 2, \ldots, p$ in (4), and then merging the coefficients of every identical order difference of $a(t)$, we can get that

$$\Delta^p a(t) + \sum_{k=1}^{p} \eta_k \Delta^{p-k} a(t) = 0, \tag{5}$$

where,

$$\eta_k = \binom{p}{k} - \sum_{j=1}^{k} \binom{p-j}{k-j} \xi_j. \tag{6}$$

Conversely, if the difference equation (5) is given, then by using another basic relation in the difference equation theory: $\triangle^k a(t) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a(t+k-j)$, $k = 1, 2, \ldots, p$, in (5), and merging the coefficients of every identical backward term of $a(t)$, we can obtain the corresponding recursion formula of $a(t)$ in the form of (4), where

$$\xi_k = (-1)^{k+1} \binom{p}{k} + \sum_{j=1}^{k} (-1)^{k-(j-1)} \binom{p-j}{k-j} \eta_j. \tag{7}$$

In this paper, we call the characteristic values of difference equation (5), that is, the roots λ_k $(k = 1, 2, \ldots, p)$ of characteristic equation $\lambda^p + \sum_{k=1}^{p} \eta_k \lambda^{p-k} = 0$, as Δ-*characteristic values* of $a(t)$, to avoid confusing with the roots of recurrence characteristic equation $\sigma^p - \sum_{k=1}^{p} \xi_k \sigma^{p-k} = 0$. (The later is often called by some authors as characteristic values of a recurrent sequence.)

According to the difference equation theory, the general term of $a(t)$ is

$$a(t) = \sum_{k=1}^{p} c_k (1 + \lambda_k)^t, \tag{8}$$

where coefficients $c_1, c_2, \ldots, c_p$ are determined by the initial conditions, which lead to the following linear algebraic equation:

$$\sum_{k=1}^{p} \lambda_k^j c_k = \triangle^j a(0) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} a_k, \quad j = 0, 1, \ldots, p-1. \tag{9}$$

For the n-generating (n-generated) sequence of a recurrent integer sequence $a(t)$, we can prove the following theorem.

Theorem 17. *Let $a(t)$ be a recurrent integer sequence defined in (4), and λ_k and c_k ($k = 1, 2, \dots, p$) be respectively the Δ-characteristic values and general term coefficients of $a(t)$, and $a^{(n)}(t)$ ($a^{(-n)}(t)$) be the n-fold generating (n-fold generated) sequence of $a(t)$, $n = 1, 2, \dots$. Then for $n = \pm 1, \pm 2, \dots$,*

(A1) the Δ-characteristic values of $a^{(n)}(t)$ are

$$\lambda_k^{(n)} = \lambda_k + n, \quad k = 1, 2, \dots, p, \tag{10}$$

(A2) the general term of $a^{(n)}(t)$ is

$$a^{(n)}(t) = \sum_{k=1}^{p} c_k(1 + n + \lambda_k)^t, \tag{11}$$

(A3) the p-th order linear and homogeneous difference equation for $a^{(n)}(t)$ is

$$\triangle^p a^{(n)}(t) + \sum_{k=1}^{p} \eta_k^{(n)} \triangle^{p-k} a^{(n)}(t) = 0, \tag{12}$$

with integer coefficients

$$\eta_k^{(n)} = (-n)^k \binom{p}{k} + \sum_{j=1}^{k} (-n)^{k-j} \binom{p-j}{k-j} \xi_j, \quad k = 1, 2, \dots, p. \tag{13}$$

(A4) the recursion formula of $a^{(n)}(t)$ is

$$a^{(n)}(t+p) = \sum_{k=1}^{p} \xi_k^{(n)} a^{(n)}(t+p-k), \tag{14}$$

with recursion coefficients

$$\xi_k^{(n)} = (-1)^{k-1} \binom{p}{k} - \sum_{j=1}^{k} (-1)^{k-j} \binom{p-j}{k-j} \eta_j^{(n)}, \quad k = 1, 2, \dots, p. \tag{15}$$

Proof. Using induction we get that $\triangle^j[c_k(1+\lambda_k)^t] = c_k\lambda_k^j(1+\lambda_k)^t$, ($k = 1, 2, \dots, p$), for $j = 0, 1, 2, \dots$. Hence, the general term of $a^{(-1)}(t)$ is that $a^{(-1)}(t) = \triangle^t a(0) = \sum_{k=1}^{p} c_k \lambda_k^t$. This means that $a^{(-1)}(t)$ has the Δ-characteristic values of $\lambda_k^{(-1)} = \lambda_k - 1$ and the general term coefficients are identical to the coefficients of $a(t)$, c_k, ($k = 1, 2, \dots, p$). From Corollary 6, we see that $a(t)$ is the 1-generated sequence of $a^{(1)}(t)$. Hence, $\lambda_k = \lambda_k^{(1)} - 1$, or $\lambda_k^{(1)} = \lambda_k + 1$, ($k = 1, 2, \dots, p$), and the general term coefficients are still c_k, ($k = 1, 2, \dots, p$). Using induction, we can see that for any fold generating (generated) sequence of $a(t)$, (A1) and (A2) hold.

From (5), we get that the Δ-characteristic polynomial of $a(t)$ is $a(\lambda) = \lambda^p + \sum_{k=1}^{p} \eta_k \lambda^{p-k} = \prod_{k=1}^{p}(\lambda - \lambda_k)$. Hence from (A1), for $n = \pm 1, \pm 2, \ldots$, the Δ-characteristic polynomials of $a^{(n)}(t)$ are $a^{(n)}(\lambda) = \prod_{k=1}^{p}[\lambda - (\lambda_k + n)] = \prod_{k=1}^{p}[(\lambda - n) - \lambda_k] = (\lambda - n)^p + \sum_{k=1}^{p} \eta_k (\lambda - n)^{p-k}$. Expanding $(\lambda - n)^j$ $(j = 1, 2, \ldots, p)$ and then merging the coefficients of the same power of λ, we get $a(\lambda) = \lambda^p + \sum_{k=1}^{p} \eta_k^{(n)} \lambda^{p-k}$ which leads to (12) and (13). This is (A3). Replacing η_k and ξ_k by $\eta_k^{(n)}$ and $\xi_k^{(n)}$ $(k = 1, 2, \ldots, p)$ in (7), we get (15), that is, (A4) holds. □

Remark 18. For the first order recurrent integer sequence: $a(0) = a_0$ and $a(t+1) = \xi a(t)$, $t = 0, 1, 2, \ldots$, the corresponding difference equation is $\triangle a - (\xi - 1)a = 0$, the Δ-eigenvalue of $a(t)$ is $\lambda_1 = \xi - 1$, the general term is $a(t) = a_0 \xi^t$. The Δ-eigenvalues of $a^{(n)}(t)$ are $\lambda_1^{(n)} = \xi - 1 + n$, and their general terms are $a^{(n)}(t) = a_0(\xi + n)^t$ $(n = \pm 1, \pm 2, \ldots,)$. Hence, the family of $a(t)$ is just the set of whole integer geometric sequences with identical initial value a_0. Noticing that $a^{(-\xi)}(t) = a_0 \delta(t)$, we see that the case of $a_0 = 1$, is the family of the unit pulse sequence $\delta(t)$ (Remark 9). Obviously, the case of $a_0 = 0$, is the family of the zero sequence $\omega(t)$ (Remarks 8).

Remark 19. For the second order recurrent integer sequence, we take the Fibonacci sequence (A000045 in [3]) as an example: $F(0) = 0$, $F(1) = 1$, and $F(t+2) = F(t+1) + F(t)$, $t = 0, 1, 2, \ldots$. Its difference equation is $\triangle^2 F + \triangle F - F = 0$, and the Δ-eigenvalues of $F(t)$ are $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, and the general term is $F(t) = \frac{1}{\sqrt{5}}[(\frac{1}{2} + \frac{\sqrt{5}}{2})^t - (\frac{1}{2} - \frac{\sqrt{5}}{2})^t]$. The Δ-eigenvalues of $F^{(n)}(t)$ are $\lambda_1^{(n)} = \frac{2n-1}{2} + \frac{\sqrt{5}}{2}$, and $\lambda_2^{(n)} = \frac{2n-1}{2} - \frac{\sqrt{5}}{2}$, $(n = \pm 1, \pm 2, \ldots)$. The general term of $F^{(n)}(t)$ is $F^{(n)}(t) = \frac{1}{\sqrt{5}}[(\frac{2n+1}{2} + \frac{\sqrt{5}}{2})^t - (\frac{2n+1}{2} - \frac{\sqrt{5}}{2})^t]$, and recurrence formula is $F^{(n)}(0) = 0$, $F^{(n)}(1) = 1$, and

$$F^{(n)}(t+2) = (2n+1)F^{(n)}(t+1) - (n^2 + n - 1)F^{(n)}(t), \tag{16}$$

where $t = 0, 1, 2, \ldots$. Table 1 lists some family elements of the Fibonacci sequence.

Remark 20. For the third order recurrent integer sequence, we take the Tribonacci sequence (A000073 in [3]) as an example: $T(0) = 0$, $T(1) = 0$, $T(2) = 1$, and $T(t+3) = T(t+2) + T(t+1) + T(t)$, $t = 0, 1, 2, \ldots$. Its difference equation is $\triangle^3 T + 2\triangle^2 T - 2T = 0$, and the corresponding Δ-eigenvalues, λ_1, λ_2, and λ_3, are the roots of the Δ-characteristic equation $\lambda^3 + 2\lambda^2 - 2 = 0$ (λ_1 is a real root, and λ_2, and λ_3 are a pair of conjugate complex roots). Its general term is $T(t) = c_1(1 + \lambda_1)^t + c_2(1 + \lambda_2)^t + c_3(1 + \lambda_3)^t$. The Δ-eigenvalues of $T^{(n)}(t)$ are $\lambda_k^{(n)} = \lambda_k + n$, $k = 1, 2, 3$, and the corresponding general term is $T^{(n)}(t) = c_1(1 + n + \lambda_1)^t + c_2(1 + n + \lambda_2)^t + c_3(1 + n + \lambda_3)^t$ for $n = \pm 1, \pm 2, \ldots$. All of $T^{(n)}(t)$ have the same first three terms $0, 0, 1$, and the same general term coefficients c_1, c_2, c_3. The recurrence formula is $T^{(n)}(0) = 0$, $T^{(n)}(1) = 0$, $T^{(n)}(2) = 1$ and

$$T^{(n)}(t+3) = (1+3n)T^{(n)}(t+2) + (1 - 2n - 3n^2)T^{(n)}(t+1) + (1 - n + n^2 + n^3)T^{(n)}(t), \tag{17}$$

where $t = 0, 1, 2, \ldots$. Table 2 lists some family elements of the Tribonacci sequence.

Table 1: Elements in family of the Fibonacci sequence ($F^{(n)}(t)$, $n = 0, \pm1, \pm2, \pm3, \pm4$)

Sequence	Initial values	Recurrence formula	Number in [3]
$F(t)$	$0, 1$	$F(t+2) = F(t+1) + F(t)$	$A000045$
$F^{(1)}(t)$	$0, 1$	$F^{(1)}(t+2) = 3F^{(1)}(t+1) - F^{(1)}(t)$	$A001906$
$F^{(2)}(t)$	$0, 1$	$F^{(2)}(t+2) = 5F^{(2)}(t+1) - 5F^{(2)}(t)$	$A093131$
$F^{(3)}(t)$	$0, 1$	$F^{(3)}(t+2) = 7F^{(3)}(t+1) - 11F^{(3)}(t)$	nil
$F^{(4)}(t)$	$0, 1$	$F^{(4)}(t+2) = 9F^{(4)}(t+1) - 19F^{(4)}(t)$	nil
$F^{(-1)}(t)$	$0, 1$	$F^{(-1)}(t+2) = -F^{(-1)}(t+1) + F^{(-1)}(t)$	$A039834$
$F^{(-2)}(t)$	$0, 1$	$F^{(-2)}(t+2) = -3F^{(-2)}(t+1) - F^{(-2)}(t)$	nil
$F^{(-3)}(t)$	$0, 1$	$F^{(-3)}(t+2) = -5F^{(-3)}(t+1) - 5F^{(-3)}(t)$	nil
$F^{(-4)}(t)$	$0, 1$	$F^{(-4)}(t+2) = -7F^{(-4)}(t+1) - 11F^{(-4)}(t)$	nil

Table 2: Elements in family of the Tribonacci sequence ($F^{(n)}(t)$, $n = 0, \pm1, \pm2, \pm3, \pm4$)

Sequence	Initial values	Recurrence formula	Number in [3]
$T(t)$	$0, 0, 1$	$T(t+3) = T(t+2) + T(t+1) + T(t)$	$A000073$
$T^{(1)}(t)$	$0, 0, 1$	$T^{(1)}(t+3) = 4T^{(1)}(t+2) - 4T^{(1)}(t+1) + 2T^{(1)}(t)$	$A115390$
$T^{(2)}(t)$	$0, 0, 1$	$T^{(2)}(t+3) = 7T^{(2)}(t+2) - 15T^{(2)}(t+1) + 11T^{(2)}(t)$	nil
$T^{(3)}(t)$	$0, 0, 1$	$T^{(3)}(t+3) = 10T^{(3)}(t+2) - 32T^{(3)}(t+1) + 34T^{(3)}(t)$	nil
$T^{(4)}(t)$	$0, 0, 1$	$T^{(4)}(t+3) = 13T^{(4)}(t+2) - 55T^{(4)}(t+1) + 77T^{(4)}(t)$	nil
$T^{(-1)}(t)$	$0, 0, 1$	$T^{(-1)}(t+3) = -2T^{(-1)}(t+2) + 2T^{(-1)}(t)$	nil
$T^{(-2)}(t)$	$0, 0, 1$	$T^{(-2)}(t+3) = -5T^{(-2)}(t+2) - 7T^{(-2)}(t+1) - T^{(-2)}(t)$	nil
$T^{(-3)}(t)$	$0, 0, 1$	$T^{(-3)}(t+3) = -8T^{(-3)}(t+2) - 20T^{(-3)}(t+1) - 14T^{(-3)}(t)$	nil
$T^{(-4)}(t)$	$0, 0, 1$	$T^{(-4)}(t+3) = -11T^{(-4)}(t+2) - 39T^{(-4)}(t+1) - 43T^{(-4)}(t)$	nil

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