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An Asymptotic Expansion for the Bernoulli Numbers of the Second Kind

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Abstract

In this paper we derive a complete asymptotic series for the Bernoulli numbers of the second kind, and provide a recurrence relation for the coefficients.

1 Introduction

The Bernoulli numbers of the second kind b_n (also known as the Cauchy numbers, Gregory coefficients or logarithmic numbers) are defined by the generating function

$$\frac{x}{\log(1+x)} = \sum_{n \geq 0} b_n x^n.$$

The first few are $b_0 = 1$, $b_1 = 1/2$, $b_2 = -1/12$, $b_3 = 1/24$, $b_4 = -19/720$. The numerators and denominators are given by Sloane's sequences A002206 and A002207. They are related to the generalized Bernoulli numbers [3, p. 596] by

$$b_n = -\frac{B_n^{(n-1)}}{(n-1)\, n!}.$$

The $B_n^{(t)}$'s are the coefficients of the exponential generating function $x^t / (e^x - 1)^t$. It is known [1] that

$$b_n = \int_0^1 \frac{t(t-1)\cdots(t-n+1)}{n!} dt,$$

which can be shown as follows:

$$\frac{x}{\log(1+x)} = \int_0^1 \exp(t\log(1+x))\,dt = \int_0^1 (1+x)^t\,dt = \int_0^1 \sum_{n\ge 0}\binom{t}{n} x^n dt$$
$$= \sum_{n\ge 0}\left(\int_0^1 \binom{t}{n} dt\right) x^n = \sum_{n\ge 0}\left(\int_0^1 \frac{t(t-1)\cdots(t-n+1)}{n!} dt\right) x^n.$$

Using the $s(n,k)$ Stirling numbers of the first kind (Sloane's A048994) defined by

$$t(t-1)\cdots(t-n+1) = \sum\nolimits_{k=0}^n s(n,k)\,t^k,$$

we immediately obtain the representation

$$b_n = \frac{1}{n!}\sum_{k=0}^n \frac{s(n,k)}{k+1}.$$

The investigation of the asymptotic behavior of these numbers was begun by Steffensen [4], who proved that

$$b_n \sim \frac{(-1)^{n+1}}{n\log^2 n} =: a_n$$

as $n \to +\infty$. However, the ratio b_n/a_n converges very slowly toward 1, as was pointed out by Davis [1], who derived better approximations including the following:

$$b_n \approx \frac{(-1)^{n+1}\,\Gamma(\xi_n+1)}{n\left(\log^2 n + \pi^2\right)},$$

where $0 < \xi_n < 1$ and Γ is the gamma function. The aim of this paper is to extend Steffensen's asymptotic approximation into a complete asymptotic expansion in terms of $1/\log n$.

2 The asymptotic expansion

Theorem 1. *The Bernoulli numbers of the second kind b_n have an asymptotic expansion of the form*

$$b_n \sim \frac{(-1)^{n+1}}{n\log^2 n}\sum_{k\ge 0}\frac{\beta_k}{\log^k n} \tag{1}$$

as $n \to +\infty$, where

$$\beta_k = (-1)^k \left[\frac{d^{k+1}}{ds^{k+1}}\left(\frac{1}{\Gamma(s)}\right)\right]_{s=0}. \tag{2}$$

Note that the main term of this asymptotic series is just Steffensen's approximation. Computing the first few coefficients β_k, our expansion takes the form

$$b_n \sim \frac{(-1)^{n+1}}{n\log^2 n}\left(1 - \frac{2\gamma}{\log n} - \frac{\pi^2 - 6\gamma^2}{2\log^2 n} + \frac{2\pi^2\gamma - 4\gamma^3 - 8\zeta(3)}{\log^3 n} + \cdots\right),$$

where γ is the Euler-Mascheroni constant and ζ is the Riemann zeta function.

In our proof we will use the following special case of Watson's lemma.

Lemma 2. *Let $g(s)$ be a function of the positive real variable s, such that*

$$g(s) = \sum_{k\geq 1} g_k s^k$$

as $s \to 0+$. Then for each nonnegative integer N

$$\int_0^{+\infty} g(s)\, e^{-ms} ds = \sum_{k=0}^{N-1} \frac{(k+1)! g_{k+1}}{m^{k+2}} + \mathcal{O}\left(\frac{1}{m^{N+2}}\right)$$

as $m \to +\infty$, provided that this integral converges throughout its range for all sufficiently large m.

For a more general version and proof see, e.g., Olver [2, p. 71] or Wong [5, p. 20]. We will also need sharp bounds for the ratio of two gamma functions.

Lemma 3. *For $n > 2$ and $0 \leq s \leq 1$ we have*

$$\frac{1}{n}\frac{1}{n^s} \leq \frac{\Gamma(n-s)}{\Gamma(n+1)} \leq \frac{1}{n}\frac{1}{n^s} + \frac{2}{n^2}\frac{s}{n^s}.$$

Proof of Lemma 3. Fix $n > 1$ and let

$$f_1(s) = -(s+1)\log n,$$
$$f_2(s) = \log \Gamma(n-s) - \log \Gamma(n+1)$$

for $0 \leq s \leq 1$. The function f_1 is affine while the function f_2 is convex (since $\log \Gamma$ is convex). Furthermore, $f_1'(0) = -\log n$, $f_2'(0) = -\psi(n)$, where $\psi := \Gamma'/\Gamma$ is the Digamma function. From the simple inequality $\psi(n) < \log n$ we see that $f_1'(0) < f_2'(0)$, hence

$$\frac{1}{n}\frac{1}{n^s} \leq \frac{\Gamma(n-s)}{\Gamma(n+1)}$$

holds for $n > 1$ and $0 \leq s \leq 1$. To prove the upper bound, we first show that

$$\frac{\Gamma(n+a)}{\Gamma(n+1)} \leq \frac{1}{n^{1-a}} \tag{3}$$

for $n \geq 1$ and $0 \leq a \leq 1$. Fix $n \geq 1$ and let

$$g_1(a) = \log \Gamma(n+a) - \log \Gamma(n+1),$$
$$g_2(a) = (a-1)\log n$$

for $0 \leq a \leq 1$. The function g_1 is convex while the function g_2 is affine. Since $g_1(0) = g_2(0) = -\log n$ and $g_1(1) = g_2(1) = 0$, the inequality (3) holds. From this it follows that for $n > 2$ and $0 \leq s \leq 1$

$$\frac{\Gamma(n-s)}{\Gamma(n+1)} = \frac{\Gamma(n+(1-s))}{(n-s)\Gamma(n+1)} \leq \frac{1}{n}\frac{1}{n^s\left(1-\frac{s}{n}\right)} \leq \frac{1}{n}\frac{1}{n^s} + \frac{2}{n^2}\frac{s}{n^s}.$$

□

Proof of Theorem 1. As shown by Steffensen,

$$b_n = \frac{(-1)^{n+1}}{\pi} \int_0^1 \Gamma(s+1)\sin(\pi s)\frac{\Gamma(n-s)}{\Gamma(n+1)}ds.$$

By Lemma 3 we find that

$$\begin{aligned} 0 \leq \int_0^1 \Gamma(s+1)\sin(\pi s)\frac{\Gamma(n-s)}{\Gamma(n+1)}ds - \frac{1}{n}\int_0^1 \Gamma(s+1)\sin(\pi s)e^{-ms}ds \\ \leq \frac{2}{n^2}\int_0^1 s\Gamma(s+1)\sin(\pi s)e^{-ms}ds \end{aligned}$$

where $m := \log n$. Hence, we conclude that

$$\begin{aligned} \left| b_n - \frac{(-1)^{n+1}}{\pi n}\int_0^1 \Gamma(s+1)\sin(\pi s)e^{-ms}ds \right| &\leq \frac{2}{\pi n^2}\int_0^1 s\Gamma(s+1)\sin(\pi s)e^{-ms}ds \\ &< \frac{2}{\pi n^2}\int_0^{+\infty} se^{-ms}ds = \frac{2}{\pi n^2 \log^2 n}. \end{aligned}$$

Thus, we derived the asymptotic formula

$$\begin{aligned} b_n &= \frac{(-1)^{n+1}}{\pi n}\int_0^1 \Gamma(s+1)\sin(\pi s)e^{-ms}ds + \mathcal{O}\left(\frac{1}{n^2\log^2 n}\right) \\ &= \frac{(-1)^{n+1}}{n}\int_0^1 \frac{s}{\Gamma(1-s)}e^{-ms}ds + \mathcal{O}\left(\frac{1}{n^2\log^2 n}\right) \end{aligned}$$

as $n \to +\infty$. Here we used the reflection formula $\Gamma(s+1)\sin(\pi s) = \pi s/\Gamma(1-s)$. The function $s/\Gamma(1-s)$ is analytic in the range $0 < s < 1$ (in fact, it is an entire function), let

$$\frac{s}{\Gamma(1-s)} = \sum_{k\geq 1}\gamma_k s^k.$$

Define the function $\Delta(s)$ in the positive real variable s by

$$\Delta(s) := \begin{cases} s/\Gamma(1-s), & \text{if } 0 < s < 1; \\ 0, & \text{if } s \geq 1. \end{cases}$$

Then our asymptotic formula becomes

$$b_n = \frac{(-1)^{n+1}}{n}\int_0^{+\infty}\Delta(s)e^{-ms}ds + \mathcal{O}\left(\frac{1}{n^2\log^2 n}\right).$$

The integral satisfies the conditions of Watson's Lemma and we obtain that for each non-negative integer N

$$\begin{aligned} b_n &= \frac{(-1)^{n+1}}{n}\left(\sum_{k=0}^{N-1}\frac{(k+1)!\gamma_{k+1}}{\log^{k+2} n} + \mathcal{O}\left(\frac{1}{\log^{N+2} n}\right)\right) + \mathcal{O}\left(\frac{1}{n^2\log^2 n}\right) \\ &= \frac{(-1)^{n+1}}{n\log^2 n}\left(\sum_{k=0}^{N-1}\frac{\beta_k}{\log^k n} + \mathcal{O}\left(\frac{1}{\log^N n}\right) + \mathcal{O}\left(\frac{1}{n}\right)\right) \end{aligned}$$

as $n \to +\infty$, where

$$\begin{aligned}\beta_k := (k+1)!\gamma_{k+1} &= \left[\frac{d^{k+1}}{ds^{k+1}}\left(\frac{s}{\Gamma(1-s)}\right)\right]_{s=0} \\ &= \left[\frac{d^{k+1}}{ds^{k+1}}\left(-\frac{1}{\Gamma(-s)}\right)\right]_{s=0} \\ &= (-1)^k\left[\frac{d^{k+1}}{ds^{k+1}}\left(\frac{1}{\Gamma(s)}\right)\right]_{s=0}.\end{aligned}$$

Since for every $N \geq 0$

$$\frac{1}{n} = o\left(\frac{1}{\log^N n}\right)$$

as $n \to +\infty$, we have proved the theorem. □

3 Recurrence for the coefficients β_k

Here we derive a recurrence formula for the coefficients β_k in the asymptotic expansion (1). Since the reciprocal of the Gamma function is an entire function, we can write it as a power series around 0, say

$$\frac{1}{\Gamma(s)} = \sum_{k\geq 1}\alpha_k s^k.$$

According to the formula for the Taylor coefficients and equation (2), we have

$$\alpha_k = \frac{1}{k!}\left[\frac{d^k}{ds^k}\left(\frac{1}{\Gamma(s)}\right)\right]_{s=0} = \frac{(-1)^{k-1}\beta_{k-1}}{k!}. \tag{4}$$

It is known that $\alpha_1 = 1$, $\alpha_2 = \gamma$ and

$$k\alpha_{k+1} = \gamma\alpha_k - \sum_{j=2}^{k}(-1)^j\zeta(j)\,\alpha_{k-j+1}$$

for $k \geq 2$ (cf. [3, p. 139]). This can be seen as follows. The Digamma function has the power series

$$\psi(s+1) = -\gamma + \sum_{k\geq 2}(-1)^k\zeta(k)\,s^{k-1}$$

(see, e.g., [3, p. 139]) and differentiating

$$\frac{1}{\Gamma(s+1)} = \frac{1}{s\Gamma(s)} = \sum_{k\geq 1}\alpha_k s^{k-1}$$

we find the power series for

$$-\frac{\Gamma'(s+1)}{\Gamma^2(s+1)} = -\frac{\psi(s+1)}{\Gamma(s+1)},$$

but this power series can be obtained by Cauchy multiplication of the two previous ones. In this way we get the recursion formula for the α_k's. From this recursion formula and (4) it follows that $\beta_0 = 1$, $\beta_1 = -2\gamma$ and

$$k\beta_k = -\gamma\,(k+1)\,\beta_{k-1} - \sum_{j=2}^{k} \binom{k+1}{j} j!\zeta\,(j)\,\beta_{k-j}$$

for $k \geq 2$.

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References

[1] H. T. Davis, The approximation of logarithmic numbers, *Amer. Math. Monthly* **64** (1957), 11–18.

[2] F. W. J. Olver, *Asymptotics and Special Functions.* A. K. Peters. Reprint, with corrections, of original Academic Press edition, 1974.

[3] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions.* Cambridge University Press, New York, 2010.

[4] J. F. Steffensen, On Laplace's and Gauss' summation-formulas, *Skandinavisk Aktuariettidskrift* (1924), 2–4.

[5] R. Wong, *Asymptotic Approximations of Integrals.* Academic Press. Reprinted with corrections by SIAM, Philadelphia, PA, 2001.

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(Concerned with sequences A002206, A002207, and A048994.)

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