Journal of Integer Sequences, Vol. 14 (2011),
Article 11.3.6

Enumeration of the Partitions of an Integer into Parts of a Specified Number of Different Sizes and Especially Two Sizes

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Abstract

A partition of a non-negative integer n is a way of writing n as a sum of a nondecreasing sequence of parts. The present paper provides the number of partitions of an integer n into parts of a specified number of different sizes. We establish new formulas for such partitions with particular interest to the number of partitions of n into parts of two sizes. A geometric application is given at the end of this paper.

[1]Both authors were supported by the laboratory LAID3.

1 Introduction and definitions

Let n and k be integers. A partition of n into k parts is an integral solution of the system

$$\begin{cases} n = n_1 + \cdots + n_k; \\ 1 \le n_1 \le \cdots \le n_k. \end{cases}$$

Euler was the first to undertake the problem of counting an integer's partitions. Since then, mathematicians have been more and more interested in integer partitions and their fascinating properties. In fact, in the theory of integer partitions, various restrictions on the nature of partitions are often considered. One may require that the n_i's be distinct, odd or even, or that n must be split into exactly k parts, etc. More on integer partitions can be found in [1, 2, 3, 4, 5] and [7, 8, 9].

Here, we are interested in partitions of the number n into parts of precisely s different sizes. Extending prior results, we derive several identities linking this kind of partitions to the number of divisors $\tau(n)$. In addition, we obtain new recurrence formulas to count the number of such partitions and a new identity to count the number of partitions of an integer into two sizes of parts.

Let $t(n,k,s)$ be the number of partitions of n into k parts of precisely s different sizes, $k = s, \ldots, n - \frac{s(s-1)}{2}$, it is an integral solution of the system

$$\begin{cases} n = a_1\ n_1 + \cdots + a_s\ n_s; \\ 1 \le n_1 < \cdots < n_s; \\ a_1 + \cdots + a_s = k; \\ a_1, \ldots, a_s \ge 1. \end{cases} \tag{1}$$

The total number of partitions of n into s different sizes of parts is denoted $t(n,s)$ (see A002133 for $t(n,2)$).

If s is specified, then $t(n,k,s) = 0$ if $k \le s-1$ and either $k > n - \frac{s(s-1)}{2}$ or $n < \max\{k, \frac{s(s+1)}{2}\}$. Then we have

$$t(n,s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} t(n,k,s) = \sum_{k \ge 1} t(n,k,s). \tag{2}$$

For instance, if $s = 1$, then $k \ge 1$, $n \ge k$, and

$$t(n,k,1) = \begin{cases} 1, & \text{if } n \text{ is a multiple of } k; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Therefore

$$\sum_{n \ge k} t(n,k,1)\ q^n = \frac{q^k}{1-q^k}.$$

Also, it is easy to see that

$$t(n,2,2) = \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$. So, we have

$$\begin{aligned}\sum_{n\geq k} t(n,2,2)\ q^n &= q^3+q^4+2q^5+2q^6+3q^7+3q^8+\cdots\\ &= \frac{q^3}{1-q}+\frac{q^5}{1-q}+\frac{q^7}{1-q}+\cdots\\ &= \frac{q^3}{(1-q)(1-q^2)}.\end{aligned}$$

P. A. MacMahon [6] was the first mathematician interested in this kind of partitions. Also, Emeric Deutsch studied the number of partitions of n into exactly two odd sizes of parts (see A117955) and the number of partitions of n into exactly two sizes of parts, one odd and one even (see A117956).

2 Preliminary results

Throughout the remainder of the paper, let $\tau(k)$ and let $\tau_{d\downarrow}(k)$ be respectively the number of positive divisors of k and the number of positive divisors of k less than or equal to d.

In this section we state some recurrence formulas involving the number $t(n,k,s)$. The main identity of the present work is based on the following results:

Theorem 1. *Let n, k and s be integers. For $k \geq s \geq 2$, $n \geq k + \frac{s(s-1)}{2}$ and $n \geq \max\{k, \frac{s(s+1)}{2}\}$, we have*

$$t(n,k,s) = \sum_{i=1}^{\lfloor \frac{2n-s(s-1)}{2k} \rfloor} \sum_{j=1}^{k-s+1} t(n-ki, k-j, s-1), \tag{4}$$

and

$$t(n,k,2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \tau_{k-1\downarrow}(n-ki). \tag{5}$$

Proof. Note that every part n_i in System (1), for $i = 2,\ldots,s$, can be written as $n_i = n_1 + d_i$, $d_i \geq 1$. Considering n_1 and a_1 as parameters, System (1) can be rewritten as follows:

$$\begin{cases} n - kn_1 = a_2\ d_2 + \cdots + a_s\ d_s;\\ 1 \leq d_2 < \cdots < d_s;\\ a_2 + \cdots + a_s = k - a_1;\\ a_1, \ldots, a_s \geq 1. \end{cases} \tag{6}$$

Hence, we get

$$t(n,k,s) = \sum_{n_1 \in \mathcal{N}} \sum_{a_1 \in \mathcal{A}} t(n - kn_1, k - a_1, s - 1),$$

where $\mathcal{N}$ and $\mathcal{A}$ are the sets containing the values of n_1 and a_1 respectively.

The smallest values of n_1 and a_1 is 1. The largest value of n_1 is found by setting $a_{i+1} = 1$ and $d_{i+1} = i$, for $i = 1, \ldots, s-1$, in the first equation of System (6). Then, we get

$$1 \leq n_1 \leq \left\lfloor \frac{2n - s(s-1)}{2k} \right\rfloor.$$

Setting $a_i = 1$, for $i = 2, \ldots, s$ in the third equation of System (6), one can see that the largest value of a_1 is $k - s + 1$.

To prove (5) we apply (4) with $s = 2$,

$$t(n,k,2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \sum_{j=1}^{k-1} t(n - ki, j, 1).$$

So by (3) we get

$$\sum_{j=1}^{k-1} t(n - ki, j, 1) = \tau_{k-1\downarrow}(n - ki).$$

This implies (5). □

Theorem 1 allows the easy recovery of known identities such as,

$$t(n,2,2) = \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{7}$$

Also, it allows to deduce some new values for $t(n,k,2)$. For instance, for $k = 3 \ldots 6$, we have

Corollary 2. *For $n \geq 3$, we have*

$$t(n,3,2) = \begin{cases} \dfrac{n-3}{3} + \left\lfloor \dfrac{n-3}{6} \right\rfloor, & \text{if } n \equiv 0 \pmod 3; \\[2ex] \dfrac{n-1}{3} + \left\lfloor \dfrac{n-1}{6} \right\rfloor, & \text{if } n \equiv 1 \pmod 3; \\[2ex] \dfrac{n-2}{3} + \left\lfloor \dfrac{n+1}{6} \right\rfloor, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$$t(n,4,2)=\begin{cases}\dfrac{n-4}{2}+\left\lfloor\dfrac{n-4}{12}\right\rfloor, & \textit{if} \;\; n\equiv 0 \pmod 4;\\[2ex] \dfrac{n-1}{4}+\left\lfloor\dfrac{n-1}{12}\right\rfloor, & \textit{if} \;\; n\equiv 1 \pmod 4;\\[2ex] \dfrac{n-2}{2}+\left\lfloor\dfrac{n+2}{12}\right\rfloor, & \textit{if} \;\; n\equiv 2 \pmod 4;\\[2ex] \dfrac{n-3}{4}+\left\lfloor\dfrac{n+5}{12}\right\rfloor, & \textit{if} \;\; n\equiv 3 \pmod 4.\end{cases}$$

$$t(n,5,2)=\begin{cases}\dfrac{n-5}{5}+\left\lfloor\dfrac{n-5}{10}\right\rfloor+\left\lfloor\dfrac{n-5}{15}\right\rfloor+\left\lfloor\dfrac{n-5}{20}\right\rfloor, & \textit{if} \;\; n\equiv 0 \pmod 5;\\[2ex] \dfrac{n-1}{5}+\left\lfloor\dfrac{n-1}{10}\right\rfloor+\left\lfloor\dfrac{n+4}{15}\right\rfloor+\left\lfloor\dfrac{n-1}{20}\right\rfloor, & \textit{if} \;\; n\equiv 1 \pmod 5;\\[2ex] \dfrac{n-2}{5}+\left\lfloor\dfrac{n+3}{10}\right\rfloor+\left\lfloor\dfrac{n-2}{15}\right\rfloor+\left\lfloor\dfrac{n+3}{20}\right\rfloor, & \textit{if} \;\; n\equiv 2 \pmod 5;\\[2ex] \dfrac{n-3}{5}+\left\lfloor\dfrac{n-3}{10}\right\rfloor+\left\lfloor\dfrac{n+2}{15}\right\rfloor+\left\lfloor\dfrac{n+7}{20}\right\rfloor, & \textit{if} \;\; n\equiv 3 \pmod 5;\\[2ex] \dfrac{n-4}{5}+\left\lfloor\dfrac{n+1}{10}\right\rfloor+\left\lfloor\dfrac{n+1}{15}\right\rfloor+\left\lfloor\dfrac{n+11}{20}\right\rfloor, & \textit{if} \;\; n\equiv 4 \pmod 5.\end{cases}$$

$$t(n,6,2) = \begin{cases} \dfrac{n-6}{2} + \left\lfloor \dfrac{n-6}{12} \right\rfloor + \left\lfloor \dfrac{n-6}{30} \right\rfloor, & \textit{if} \;\; n \equiv 0 \pmod 6; \\ \dfrac{n-1}{6} + \left\lfloor \dfrac{n-1}{30} \right\rfloor, & \textit{if} \;\; n \equiv 1 \pmod 6; \\ \dfrac{n-2}{3} + \left\lfloor \dfrac{n-2}{12} \right\rfloor + \left\lfloor \dfrac{n+4}{30} \right\rfloor, & \textit{if} \;\; n \equiv 2 \pmod 6; \\ \dfrac{n-3}{3} + \left\lfloor \dfrac{n+9}{30} \right\rfloor, & \textit{if} \;\; n \equiv 3 \pmod 6; \\ \dfrac{n-4}{3} + \left\lfloor \dfrac{n+2}{12} \right\rfloor + \left\lfloor \dfrac{n+14}{30} \right\rfloor, & \textit{if} \;\; n \equiv 4 \pmod 6; \\ a \\ \dfrac{n-5}{6} + \left\lfloor \dfrac{n+19}{30} \right\rfloor, & \textit{if} \;\; n \equiv 5 \pmod 6. \end{cases}$$

Proof. The results follow immediately from Theorem 1. □

Corollary 3. *For $n \geq 3$ and $\lceil \frac{n+1}{2} \rceil \leq k \leq n-1$, we have*

$$t(n,k,2) = \tau(n-k).$$

Proof. On the one hand, the sum in (5) is reduced to one element if $1 \leq \frac{n-1}{k} < 2$, i.e.,

$$\frac{n-1}{2} < k \leq n-1.$$

On the other hand, $\tau_{k-1\downarrow}(n-k) = \tau(n-k)$, if $k-1 \geq n-k$, i.e.,

$$k \geq \frac{n+1}{2} \, .$$

Hence the result follows. □

Remark 4. From Corollary 3, for $n \geq 2k+1$ and $k \geq 1$, we have

$$t(n, n-k, 2) = \tau(k).$$

For instance, for $n \geq 27$, we get

$$t(n, n-13, 2) = \tau(13) = 2.$$

3 Main identity

The aim of this section is to derive an explicit formula for $t(n,k,2)$. Before giving the next Theorem, we introduce some notation. Let

- $\varphi_i(j) = \begin{cases} 1, & \text{if } j \equiv 0 \pmod{i}; \\ 0, & \text{otherwise.} \end{cases}$

- $\chi_k(i,j) = \begin{cases} 0, & \text{if } i \neq 0 \text{ and } \gcd(k,j) \neq 1 \text{ and } \gcd(i,j) = 1; \\ 1, & \text{otherwise.} \end{cases}$

- $W_k = [W_k(i,j)]$, $0 \le i \le k-1$, $1 \le j \le k-1$ be a matrix, whose elements are given by

$$W_k(i,j) = \begin{cases} d, & \text{if } i \in I_{k,j}(d) \text{ and } \chi_k(i,j) = 1; \\ j, & \text{otherwise.} \end{cases}$$

where, $0 \le d \le \dfrac{j}{\gcd(k,j)} - 1$ and

$$I_{k,j}(d) = \left\{ i = \left(\left\lfloor \frac{dk-1}{j} \right\rfloor + a \right) j - dk \ / \ 1 \le a \le \left\lfloor \frac{(d+1)k-1}{j} \right\rfloor - \left\lfloor \frac{dk-1}{j} \right\rfloor \right\}.$$

Remark 5. The construction of matrix W_k is special, it is filled column by column as follows:

1. Case $\chi_k(i,j) = 1$
   Each value of the parameter d generates some values of the parameter a, which in return produce the values of the lines i, this process allows to define the elements of the concerned lines.

2. Case $\chi_k(i,j) = 0$
   The empty elements are replaced by the number j of the column.

For example, for $k = 6$, we get

$$W_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 4 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 2 & 0 & 4 & 2 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 3 & 4 & 0 \end{bmatrix}.$$

The formulas of Corollary 2 are special cases that motivate the following generalization.

Theorem 6. *For* $n \ge 3$, $n \equiv i \pmod{k}$, $2 \le k \le n-1$, *we have*

$$t(n,k,2) = \begin{cases} \sum_{j=1}^{k-1} \left\lfloor \frac{\gcd(k,j)}{kj}(n-k) \right\rfloor, & \textit{if } i=0; \\ \sum_{j=1}^{k-1} \chi_k(i,j) \left\lfloor 1+\gcd(k,j)\frac{n-i-k-kW_k(i,j)}{kj} \right\rfloor, & \textit{otherwise.} \end{cases}$$

Proof. **Case 1.** $n = kl$, i.e., $i = 0$. Using Theorem 1, we get

$$\begin{aligned} t(n,k,2) &= \sum_{h=1}^{l-1} \tau_{k-1\downarrow}(kh) \\ &= \sum_{j=1}^{k-1}\sum_{h=1}^{l-1} \varphi_j(kh). \end{aligned}$$

The divisors of kh which are multiples of j are of the form $\frac{kjd_h}{\gcd(k,j)}$. Then

$$\begin{aligned} t(n,k,2) &= \sum_{j=1}^{k-1} \left\lfloor \frac{\gcd(k,j)}{j}(l-1) \right\rfloor \\ &= \sum_{j=1}^{k-1} \left\lfloor \frac{\gcd(k,j)}{kj}(n-k) \right\rfloor. \end{aligned}$$

Case 2. $n = kl + i$, $1 \le i \le k-1$. Using Theorem 1, we get

$$\begin{aligned} t(n,k,2) &= \sum_{h=0}^{l-1} \tau_{k-1\downarrow}(kh+i) \\ &= \sum_{i=1}^{k-1}\sum_{h=0}^{l-1} \varphi_j(kh+i). \end{aligned}$$

It is straightforward to verify that the divisors of $kh+i$ that are multiples of j are of the following form

$$\chi_k(i,j)\left(\frac{kjd_h}{\gcd(k,j)} + i + kW_k(i,j)\right).$$

Hence,

$$\begin{aligned} t(n,k,2) &= \sum_{j=1}^{k-1} \chi_k(i,j) \left\lfloor \frac{j+\gcd(k,j)\big(l-1-W_k(i,j)\big)}{j} \right\rfloor \\ &= \sum_{j=1}^{k-1} \chi_k(i,j) \left\lfloor \frac{j+\gcd(k,j)\big(\frac{n-i}{k}-1-W_k(i,j)\big)}{j} \right\rfloor \\ &= \sum_{j=1}^{k-1} \chi_k(i,j) \left\lfloor 1+\frac{\gcd(k,j)}{kj}\big(n-i-k-kW_k(i,j)\big) \right\rfloor. \end{aligned}$$

□

Example 7. $k = 6$

1. For $i = 0$, $n = 6l$, we get

$$\begin{aligned} t(6l, 6, 2) &= \sum_{j=1}^{5} \left\lfloor \frac{n-6}{6j} \gcd(6, j) \right\rfloor \\ &= \frac{n-6}{2} + \left\lfloor \frac{n-6}{12} \right\rfloor + \left\lfloor \frac{n-6}{30} \right\rfloor . \end{aligned}$$

2. For $i = 1$, $n = 6l + 1$ we get

$$\begin{aligned} t(6l+1, 6, 2) &= \sum_{j=1}^{5} \chi_6(1, j) \left\lfloor 1 + \frac{\gcd(6, j)}{6j} \big(n - 7 - 6W_6(1, j)\big) \right\rfloor \\ &= \left\lfloor 1 + \frac{n-7}{6} \right\rfloor + \left\lfloor 1 \quad + \quad \frac{n-7-24}{30} \right\rfloor \\ &= \frac{n-1}{6} + \left\lfloor \frac{n-1}{30} \right\rfloor . \end{aligned}$$

3. For $i = 2$, $n = 6l + 2$ we get

$$\begin{aligned} t(6l+2, 6, 2) = & \sum_{j=1}^{5} \chi_6(2, j) \left\lfloor 1 + \frac{\gcd(6, j)}{6j} \big(n - 8 - 6W_6(2, j)\big) \right\rfloor \\ = & \left\lfloor 1 + \frac{n-8}{6} \right\rfloor + \left\lfloor 1 + \frac{2(n-8)}{12} \right\rfloor + \left\lfloor 1 + \frac{2(n-8-6)}{24} \right\rfloor + \left\lfloor 1 + \frac{n-8-18}{30} \right\rfloor \\ = & \frac{n-2}{3} + \left\lfloor \frac{n-2}{12} \right\rfloor + \left\lfloor \frac{n+4}{30} \right\rfloor . \end{aligned}$$

4. For $i = 3$, $n = 6l + 3$ we get

$$\begin{aligned} t(6l+3, 6, 2) &= \sum_{j=1}^{5} \chi_6(3, j) \left\lfloor 1 + \frac{\gcd(6, j)}{6j} \big(n - 9 - 6W_6(3, j)\big) \right\rfloor \\ &= \left\lfloor 1 + \frac{n-9}{6} \right\rfloor + \left\lfloor 1 + \frac{3(n-9)}{18} \right\rfloor + \left\lfloor 1 + \frac{n-9-12}{30} \right\rfloor \\ &= \frac{n-3}{3} + \left\lfloor \frac{n+9}{30} \right\rfloor . \end{aligned}$$

5. For $i = 4$, $n = 6l + 4$ we get

$$\begin{aligned} t(6l+4,6,2) &= \sum_{j=1}^{5} \chi_6(4,j) \left\lfloor 1 + \frac{\gcd(6,j)}{6j}\big(n - 10 - 6W_6(4,j)\big) \right\rfloor \\ &= \left\lfloor 1 + \frac{n-10}{6} \right\rfloor + \left\lfloor 1 + \frac{2(n-10)}{12} \right\rfloor + \left\lfloor 1 + \frac{2(n-10)}{24} \right\rfloor + \left\lfloor 1 + \frac{n-10-6}{30} \right\rfloor \\ &= \frac{n-4}{3} + \left\lfloor \frac{n+2}{12} \right\rfloor + \left\lfloor \frac{n+14}{30} \right\rfloor . \end{aligned}$$

6. For $i = 5$, $n = 6l + 5$ we get

$$\begin{aligned} t(6l+5,6,2) &= \sum_{j=1}^{5} \chi_6(5,j) \left\lfloor 1 + \frac{\gcd(6,j)}{6j}\big(n - 11 - 6W_6(5,j)\big) \right\rfloor \\ &= \left\lfloor 1 + \frac{n-11}{6} \right\rfloor + \left\lfloor 1 + \frac{n-11}{30} \right\rfloor \\ &= \frac{n-5}{6} + \left\lfloor \frac{n+19}{30} \right\rfloor . \end{aligned}$$

Using Theorem 6, we obtain the following table for $n \leq 20$.

$n\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$t(n,2)$
3	1																		1
4	1	1																	2
5	2	2	1																5
6	2	1	2	1															6
7	3	3	2	2	1														11
8	3	3	2	2	2	1													13
9	4	3	2	3	2	2	1												17
10	4	4	5	1	3	2	2	1											22
11	5	5	3	4	2	3	2	2	1										27
12	5	4	4	3	3	2	3	2	2	1									29
13	6	6	4	5	2	4	2	3	2	2	1								37
14	6	6	7	5	5	1	4	2	3	2	2	1							44
15	7	6	4	3	4	4	2	4	2	3	2	2	1						44
16	7	7	7	5	6	4	3	2	4	2	3	2	2	1					55
17	8	8	5	7	3	5	3	4	2	4	2	3	2	2	1				59
18	8	7	9	6	7	4	5	2	4	2	4	2	3	2	2	1			68
19	9	9	6	7	3	7	3	4	3	4	2	4	2	3	2	2	1		71
20	9	9	9	5	7	5	8	3	3	3	4	2	4	2	3	2	2	1	81

Table 1: $t(n,k,2)$, $2 \leq k \leq 19$, $3 \leq n \leq 20$.

Also, Theorem 6 allows us to obtain $t(n,2)$ for large values of n, the following table is introduced to illustrate a few.

n	100	500	1000	1500	2000	2500	3000	3500	4000
$t(n,2)$	1135	11103	28340	54652	70128	91440	136790	144687	169953

Table 2: Some values of $t(n,2)$.

4 Application

Let $\mathcal{P}_n$ be an n-side regular polygon. We say that an inscribed quadrilateral in $\mathcal{P}_n$ is proper if none of its sides belongs to $\mathcal{P}_n$.

Theorem 8. *Let $n \geq 9$ be an odd integer and let $\Diamond(n)$ be the number of inscribed, non-isometric and proper quadrilaterals in $\mathcal{P}_n$, using three equal chords. Then we have*

$$\Diamond(n) = \begin{cases} \dfrac{n-5}{4} + \left\lfloor \dfrac{n-5}{12} \right\rfloor, & \text{if } n \equiv 1 \pmod 4; \\[2ex] \dfrac{n-7}{4} + \left\lfloor \dfrac{n+1}{12} \right\rfloor, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. The chords belonging to an inscribed quadrilateral in $\mathcal{P}_n$, separate the number of vertices of $\mathcal{P}_n$ into four parts, which do not include the quadrilateral vertices. In other words, each such quadrilateral generates a partition of $n-4$ into four parts, using only two types of parts and vice versa. Then

$$\Diamond(n) = t(n-4,4,2),$$

and the result yields from Corollary 2. □

Figure 1, illustrates this idea in $\mathcal{P}_{19}$. The first quadrilateral is generated by the partition $15 = 1+1+1+12$, the second by $15 = 2+2+2+9$, the third by $15 = 3+3+3+6$ and the fourth by $15 = 3+4+4+4$.

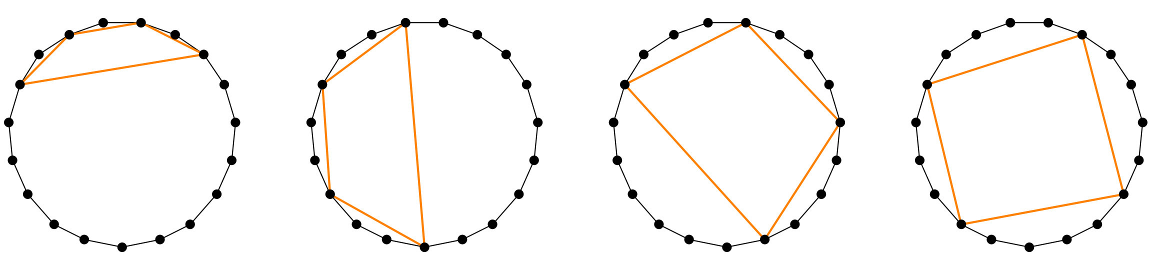

Figure 1: The non-isometric proper quadrilaterals inscribed in $\mathcal{P}_{19}$, using three equal chords.

5 Acknowledgement

The authors are indebted to the unknown referee for his valuable corrections and comments which have improved the quality of the paper. The authors are also grateful to Professors L. Benaissa and N. Hannoun for their help.

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2010 *Mathematics Subject Classification*: Primary 05A17; Secondary 11P83.
Keywords: integer partitions, partitions into parts of different sizes, partitions into parts of two sizes, number of divisors.

(Concerned with sequences A002133, A117955, and A117956.)

Received May 4 2010; revised version received October 3 2010; January 31 2011; February 28 2011. Published in *Journal of Integer Sequences*, March 25 2011.

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