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Reciprocals of the Gcd-Sum Functions

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Abstract

In this paper we study the reciprocals of the gcd-sum function and some related functions and improve some results of Tóth. The harmonic mean of the gcd function is also studied.

1 Introduction

Recently, L. Tóth published a paper [3] about the gcd-sum function $P(n)$, called also Pillai's function [2] defined by

$$P(n) = \sum_{k=1}^{n} \gcd(k, n).$$

In this well-written paper, Tóth not only listed many classical results about this function, its analogues and generalizations, but also proved several new results. The aim of this short note is to improve some results in Tóth's paper.

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Let $H(n)$ denote the harmonic mean of $\gcd(1,n), \gcd(2,n), \ldots, \gcd(n,n)$, namely,

$$H(n) := n\left(\sum_{j=1}^{n} \frac{1}{\gcd(j,n)}\right)^{-1} = \frac{n^2}{\sum_{d|n} d\varphi(d)}, \tag{1}$$

where $\varphi(n)$ denotes Euler's function. Theorem 4 of Tóth [3] states that

$$\sum_{n\leq x} \frac{H(n)}{n} = C_1 \log x + C_2 + O(x^{-1+\varepsilon}), \tag{2}$$

where C_1 and C_2 are computable constants, $\varepsilon > 0$ is a small positive constant. As a corollary, he deduced that

$$\sum_{n\leq x} H(n) = C_1 x + O(x^{\varepsilon}). \tag{3}$$

In this note we shall prove first the following

Theorem 1. *We have the asymptotic formula*

$$\sum_{n\leq x} H(n) = C_1 x + C_3 \log x + O((\log x)^{2/3}), \tag{4}$$

where C_1 and C_3 are constants.

As a corollary of Theorem 1, we have

Corollary 2.

$$\sum_{n\leq x} \frac{H(n)}{n} = C_1 \log x + C_2 + O(x^{-1}(\log x)^{2/3}). \tag{5}$$

Remark 3. Tóth [3] studied the average order of $\sum_{n\leq x} H(n)/n$ first and then deduced an asymptotic formula of $\sum_{n\leq x} H(n)$ through it. However in this note we study $\sum_{n\leq x} H(n)$ first, then deduce an asymptotic formula for $\sum_{n\leq x} H(n)/n$.

Now we study the reciprocals of $P(n)$ and some other related functions. We first recall the definitions of some functions.

An integer d is called a unitary divisor of n if $d|n$ and $(d, n/d) = 1$, notation $d\|n$. The unitary analogue of the function P is defined by

$$P^*(n) := \sum_{j=1}^{n} (j,n)_*,$$

where $(j,n)_* = \max\{d \in \mathbb{N} : d|j, d\|n\}$, which was first introduced in Tóth [4]. This function P^* (A145388) is multiplicative and for every prime power $p^\alpha (\alpha \geq 1)$ we have $P^*(p^\alpha) = 2p^\alpha - 1$.

Let $n > 1$ be an integer of canonical form $n = \prod_{j=1}^{r} p_j^{a_j}$. An integer d is called an exponential divisor of n if $d = \prod_{j=1}^{r} p_j^{b_j}$ such that $b_j | a_j (1 \leq j \leq r)$, notation $d|_e n$. By convention $1|_e 1$. The kernel of n is denoted by $\kappa(n) = \prod_{j=1}^{r} p_j$.

Two positive integers $m > 1, n > 1$ have common exponential divisors iff they have the canonical forms $n = \prod_{j=1}^{r} p_j^{a_j}$ and $m = \prod_{j=1}^{r} p_j^{b_j}$ with $a_j \geq 1, b_j \geq 1 (1 \leq j \leq r)$. The greatest common exponential divisor of n and m is $(n, m)_e = \prod_{j=1}^{r} p_j^{(a_j, b_j)}$. By convention define $(1, 1)_e = 1$ and note that $(1, m)_e$ does not exist for $m > 1$. Now define the function $P^{(e)}(n)$ by the relation

$$P^{(e)}(n) = \sum_{\substack{1 \leq j \leq n \\ \kappa(j) = \kappa(n)}} (j, n)_e.$$

The function $P^{(e)}$(see Tóth [5]) is multiplicative and for any prime power p^α $(\alpha \geq 1)$ we have

$$P^{(e)}(p^\alpha) = \sum_{1 \leq j \leq \alpha} p^{(j,\alpha)} = \sum_{d \mid \alpha} p^d \varphi(\alpha/d). \tag{6}$$

Now we recall another analogue of P. Let $n > 1$ be an integer and we say an integer j is regular $(\mathrm{mod}\ n)$ if there exists an integer x such that $j^2 x \equiv j \pmod{n}$. Tóth [6] introduced the function (A176345)

$$\tilde{P}(n) := \sum_{j \in \mathrm{Reg}_n} \gcd(j, n),$$

where $\mathrm{Reg}_n = \{1 \leq j \leq n : j \text{ is regular} (\mathrm{mod}\ n)\}$. Tóth[6] showed that $\tilde{P}$ is multiplicative and

$$\tilde{P}(n) = n \prod_{p \mid n} (2 - 1/p). \tag{7}$$

The above mentioned functions have been extensively studied and many papers about them have been published. See Tóth [3] and references therein.

Theorem 6 of Tóth [3] states that

$$\sum_{n \leq x} \frac{1}{P(n)} = K (\log x)^{1/2} + O((\log x)^{-1/2}), \tag{8}$$

$$\sum_{n \leq x} \frac{1}{P^*(n)} = K^* (\log x)^{1/2} + O((\log x)^{-1/2}), \tag{9}$$

$$\sum_{n \leq x} \frac{1}{\tilde{P}(n)} = \tilde{K} (\log x)^{1/2} + O((\log x)^{-1/2}), \tag{10}$$

$$\sum_{n \leq x} \frac{1}{P^{(e)}(n)} = K^{(e)} \log x + O(1), \tag{11}$$

where $K, K^*, \tilde{K}, K^{(e)}$ are computable constants.

In this note we shall prove the following result.

Theorem 4. *Suppose $N \geq 1$ is a fixed integer, then*

$$\sum_{n\leq x}\frac{1}{P(n)} = \sum_{j=0}^{N} K_j(\log x)^{1/2-j} + O((\log x)^{-1/2-N}), \tag{12}$$

$$\sum_{n\leq x}\frac{1}{P^*(n)} = \sum_{j=0}^{N} K_j^*(\log x)^{1/2-j} + O((\log x)^{-1/2-N}), \tag{13}$$

$$\sum_{n\leq x}\frac{1}{\tilde{P}(n)} = \sum_{j=0}^{N} \tilde{K}_j(\log x)^{1/2-j} + O((\log x)^{-1/2-N}), \tag{14}$$

$$\sum_{n\leq x}\frac{1}{P^{(e)}(n)} = K_0^{(e)}\log x + K_1^{(e)} + O(x^{-1}\log^2 x), \tag{15}$$

where $K_j, K_j^*, \tilde{K}_j, K_j^{(e)} (j \geq 0)$ are computable constants.

Notation. Throughout this paper, ε denotes a sufficiently small positive constant. $\zeta(s)$ denotes the Riemann zeta-function. For any real number t, $[t]$ denotes the greatest integer not exceeding t, $\{t\} = t - [t]$, and $\psi(t) = \{t\} - 1/2$. For any complex number z, $\sigma_z(n) = \sum_{d|n} d^z$ and $d_z(n)$ denotes the generalized divisor function, $\mu(n)$ denotes the Möbius function.

2 Proof of Theorem 1

Define for $\Re s > 1$ that

$$D_H(s) := \sum_{n=1}^{\infty} H(n)n^{-s}. \tag{16}$$

Since $H(n)$ is multiplicative, by Euler's product we get

$$D_H(s) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} H(p^\alpha)p^{-\alpha s}\right). \tag{17}$$

We evaluate $H(p^\alpha)$ first for any prime p and $\alpha \geq 1$. By (1) we have

$$\begin{aligned} H(p^\alpha) &= p^{2\alpha} \times (1 + \sum_{j=1}^{\alpha} p^{2j-1}(p-1))^{-1} \\ &= p^{2\alpha} \times (1 + \frac{p-1}{p}\sum_{j=1}^{\alpha} p^{2j})^{-1} \\ &= p^{2\alpha} \times (1 + \frac{p^{2\alpha+2} - p^2}{p(p+1)})^{-1} \\ &= p^{2\alpha} \times \left(\frac{p^{2\alpha+1}+1}{p+1}\right)^{-1} \\ &= \frac{p^{2\alpha}(p+1)}{p^{2\alpha+1}+1} = \frac{1+p^{-1}}{1+p^{-2\alpha-1}}. \end{aligned} \tag{18}$$

Hence we have

$$
\begin{aligned}
&1+\sum_{\alpha=1}^{\infty} H(p^{\alpha})p^{-\alpha s} \qquad (19)\\
&=1+\sum_{\alpha=1}^{\infty}\frac{1+p^{-1}}{1+p^{-2\alpha-1}}p^{-\alpha s}\\
&=1+(1+p^{-1})\sum_{\alpha=1}^{\infty}p^{-\alpha s}\sum_{m=0}^{\infty}\left(-p^{-2\alpha-1}\right)^{m}\\
&=1+(1+p^{-1})\sum_{m=0}^{\infty}(-1)^{m}p^{-m}\sum_{\alpha=1}^{\infty}p^{-\alpha s-2\alpha m}\\
&=1+(1+p^{-1})\sum_{m=0}^{\infty}(-1)^{m}p^{-m}\frac{p^{-s-2m}}{1-p^{-s-2m}}\\
&=1+(1+p^{-1})\frac{p^{-s}}{1-p^{-s}}+(1+p^{-1})\sum_{m=1}^{\infty}(-1)^{m}p^{-m}\frac{p^{-s-2m}}{1-p^{-s-2m}}\\
&=\frac{1+p^{-1-s}}{1-p^{-s}}+(1+p^{-1})\sum_{m=1}^{\infty}(-1)^{m}p^{-m}\frac{p^{-s-2m}}{1-p^{-s-2m}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-p^{-s})(1-p^{-s-1})(1+\sum_{\alpha=1}^{\infty}H(p^{\alpha})p^{-\alpha s}) \qquad (20)\\
&=1-p^{-2s-2}+(1-p^{-s})(1-p^{-s-1})(1+p^{-1})\sum_{m=1}^{\infty}(-1)^{m}p^{-m}\frac{p^{-s-2m}}{1-p^{-s-2m}}\\
&=1+O(p^{-2\sigma-2}+(1+p^{-\sigma})(1+p^{-\sigma-1})p^{-\sigma-3})\\
&=1+O(p^{-2\sigma-2}+p^{-\sigma-3}+p^{-2\sigma-3}+p^{-3\sigma-4}),
\end{aligned}
$$

where $\sigma=\Re s$.

From (17), (20) and noting

$$\zeta(s)=\prod_{p}\left(1-p^{-s}\right)^{-1}\ \ (\Re s>1) \qquad (21)$$

we get

$$D_H(s)=\zeta(s)\zeta(s+1)G(s)\ (\Re s>1), \qquad (22)$$

where

$$G(s)=\prod_{p}(1-p^{-s})(1-p^{-s-1})\left(1+\sum_{\alpha=1}^{\infty}H(p^{\alpha})p^{-\alpha s}\right) \qquad (23)$$

such that if we expand $G(s)$ into a Dirichlet series

$$G(s)=\sum_{n=1}^{\infty}g(n)n^{-s}, \qquad (24)$$

then this Dirichlet series is absolutely convergent for $\Re s > -1/2$.

From (22) and (24) we have

$$H(n) = \sum_{d|n} \sigma_{-1}(d) g(n/d). \tag{25}$$

For $\sigma_{-1}(n)$ we have

$$\begin{aligned}
\sum_{n\le x} \sigma_{-1}(n) \quad &= \sum_{nm\le x} \frac{1}{n} = \sum_{n\le x} \frac{1}{n} \sum_{m\le x/n} 1 \\
&= \sum_{n\le x} \frac{1}{n} \left[\frac{x}{n}\right] = \sum_{n\le x} \frac{1}{n} \left(\frac{x}{n} - \frac{1}{2} - \psi(\frac{x}{n})\right) \\
&= x \sum_{n\le x} n^{-2} - \frac{1}{2} \sum_{n\le x} n^{-1} - \sum_{n\le x} \frac{1}{n} \psi(\frac{x}{n}) \\
&= \frac{\pi^2 x}{6} - \frac{\log x}{2} + O((\log x)^{2/3}),
\end{aligned} \tag{26}$$

where in the last step we used the well-known bound(see Walfisz [7])

$$\sum_{n\le x} \frac{1}{n} \psi(\frac{x}{n}) \ll (\log x)^{2/3}. \tag{27}$$

From (25)-(27) we have

$$\begin{aligned}
\sum_{n\le x} H(n) \quad &= \sum_{m\le x} g(m) \sum_{n\le x/m} \sigma_{-1}(n) \\
&= \frac{\pi^2 x}{6} \sum_{m\le x} \frac{g(m)}{m} - \frac{\log x}{2} \sum_{m\le x} g(m) + \frac{1}{2} \sum_{m\le x} g(m) \log m \\
&\quad +O\left(\log^{2/3} x \sum_{m\le x} |g(m)|\right).
\end{aligned} \tag{28}$$

Recall that the Dirichlet series $\sum_{n=1}^{\infty} g(n) n^{-s}$ is absolutely convergent for $\Re s > -1/2$. Hence for any $U > 1$ we have

$$\sum_{U<n\le 2U} |g(n)| \ll U^{-1/2+\varepsilon}.$$

It follows that the infinite series $\sum_{m\ge 1} g(m) m^{-1}, \sum_{m\ge 1} |g(m)|, \sum_{m\ge 1} g(m) \log m$ are all convergent and that

$$\begin{aligned}
&\sum_{m\le x} \frac{g(m)}{m} = \sum_{m=1}^{\infty} \frac{g(m)}{m} + O(x^{-3/2+\varepsilon}), \\
&\sum_{m\le x} g(m) = \sum_{m=1}^{\infty} g(m) + O(x^{-1/2+\varepsilon}) \\
&\sum_{m\le x} g(m) \log m \ll 1, \ \sum_{m\le x} |g(m)| \ll 1.
\end{aligned} \tag{29}$$

Now Theorem 1 follows from (28) and (29).

3 Proof of Theorem 4

In this section we shall prove Theorem 4. We only prove (12) and (15) since the proofs of (12), (13) and (14) are the same.

3.1 The generalized divisor problem

Suppose $k \geq 2$ ($k \in \mathbb{N}$) is a fixed integer. The divisor function $d_k(n)$ denotes the number of ways n can be written as a product of k natural-number factors. It is an important problem in the analytic number theory to study the mean value of the divisor function $d_k(n)$. It is well-known that $d_k(n)$ are the coefficients of the Dirichlet series

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s} \ (\Re s > 1).$$

Now suppose z is a fixed complex number and let $d_z(n)$ denote the coefficients of the Dirichlet series

$$\zeta^z(s) := \sum_{n=1}^{\infty} d_z(n) n^{-s} \ (\Re s > 1), \tag{30}$$

where $\zeta^z(s) = e^{z \log \zeta(s)}$ such that for $\log s$ we take the main value $\log 1 = 0$. The function $d_z(n)$ is called the generalized divisor function.

Suppose $A > 0$ is an arbitrary but fixed real number and $N \geq 1$ is an arbitrary but fixed integer. Then uniformly for $|z| \leq A$ we have

$$\sum_{n \leq x} d_z(n) = x \sum_{j=1}^{N} c_j(z)(\log x)^{z-j} + O(x(\log x)^{\Re z - N - 1}), \tag{31}$$

where the functions $c_1(z), \cdots, c_N(z)$ are regular in the region $|z| \leq A$.

The above result is Theorem 14.9 of Ivić [1].

3.2 Proof of (12)

Define $P_1(n) = n/P(n)$. Then by Euler's product we have for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} P_1(n) n^{-s} = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} P_1(p^\alpha) p^{-\alpha s}\right). \tag{32}$$

We evaluate $P_1(p^\alpha)(\alpha \geq 1)$ first. The formula (14) of [3] reads

$$P(p^\alpha) = (\alpha+1)p^\alpha - \alpha p^{\alpha-1},$$

which implies that

$$P_1(p^\alpha) = \frac{p^\alpha}{(\alpha+1)p^\alpha - \alpha p^{\alpha-1}} = \frac{1}{\alpha+1} + O(p^{-1}). \tag{33}$$

Inserting (33) into (32) and recalling (21) we get that

$$\sum_{n=1}^{\infty} P_1(n)n^{-s} = \zeta^{1/2}(s)G_1(s),\ \Re s > 1, \tag{34}$$

where

$$G_1(s) = \prod_p (1-p^{-s})^{1/2} \left(1 + \sum_{\alpha=1}^{\infty} P_1(p^\alpha)p^{-\alpha s}\right) \tag{35}$$

such that if we expand $G_1(s)$ into a Dirichlet series

$$G_1(s) = \sum_{n=1}^{\infty} g_1(n)n^{-s}, \tag{36}$$

then this Dirichlet series is absolutely convergent for $\Re s > 1/2$. For the function g_1 we have the trivial estimate

$$\sum_{m\le y} |g_1(m)| \le y^{1/2+\varepsilon}. \tag{37}$$

From (37) we get by partial summation that

$$\sum_{m\le y} |g_1(m)|m^{-1} \ll 1,\ \ \sum_{m>y} |g_1(m)|m^{-1} \ll y^{-1/2+\varepsilon} \tag{38}$$

and for any fixed constant C that

$$\sum_{m\le y} \frac{g_1(m)\log^C m}{m} = \sum_{m=1}^{\infty} \frac{g_1(m)\log^C m}{m} + O(y^{-1/2+\varepsilon}). \tag{39}$$

Let e_C denote the value of the infinite series in (39).

Suppose β is a real number which is not a non-negative integer. By Taylor's expansion we have

$$(1-u)^\beta = \sum_{\ell=0}^{N} d_\ell^{(\beta)} u^\ell + O(|u|^{N+1}),\ |u| \le 1/2, \tag{40}$$

where $d_\ell^{(\beta)} = (-1)^l \beta(\beta-1)\cdots(\beta-\ell+1)/\ell!$.

By the hyperbolic approach, (31) with $A = z = 1/2$, (38) and (39) with $y = \sqrt{x}$ and (40) we get for any fixed $N \geq 1$ that

$$\begin{aligned}
\sum_{n\leq x} P_1(n) \quad &= \sum_{nm\leq x} d_{1/2}(n) g_1(m) \qquad (41)\\
&= \sum_{m\leq\sqrt{x}} g_1(m) \sum_{n\leq x/m} d_{1/2}(n) + \sum_{n\leq\sqrt{x}} d_{1/2}(n) \sum_{\sqrt{x}<m\leq x/n} g_1(m)\\
&= \sum_{m\leq\sqrt{x}} g_1(m) \sum_{n\leq x/m} d_{1/2}(n) + O(x^{3/4+\varepsilon})\\
&= \sum_{m\leq\sqrt{x}} g_1(m) \left(\frac{x}{m}\sum_{j=1}^{N} c_j(1/2)(\log\frac{x}{m})^{1/2-j} + O(\frac{x}{m}(\log x)^{-1/2-N})\right)\\
&\quad +O(x^{3/4+\varepsilon})\\
&= x\sum_{j=1}^{N} c_j(1/2)(\log x)^{1/2-j} \sum_{m\leq\sqrt{x}} \frac{g_1(m)}{m}\left(1-\frac{\log m}{\log x}\right)^{1/2-j}\\
&\quad +O(x(\log x)^{-1/2-N})\\
&= x\sum_{j=1}^{N} c_j(1/2)(\log x)^{1/2-j} \sum_{\ell=0}^{N} d_\ell^{(1/2-j)} \sum_{m\leq\sqrt{x}} \frac{g_1(m)}{m}\frac{\log^\ell m}{\log^\ell x}\\
&\quad +O\left(x\sum_{j=1}^{N}(\log x)^{1/2-j}\sum_{m\leq\sqrt{x}}\frac{|g_1(m)|}{m}\frac{\log^{N+1} m}{\log^{N+1} x}\right)\\
&\quad +O(x(\log x)^{-1/2-N})\\
&= x\sum_{j=1}^{N} c_j(1/2)(\log x)^{1/2-j} \sum_{\ell=0}^{N} d_\ell^{(1/2-j)} e_\ell \log^{-\ell} x\\
&\quad +O(x(\log x)^{-1/2-N})\\
&= x\sum_{j=1}^{N} \mathcal{K}_j(\log x)^{1/2-j} + O(x(\log x)^{-1/2-N}),
\end{aligned}$$

where

$$\mathcal{K}_j = \sum_{\substack{j=j_1+\ell\\ j_1\geq 1,\ell\geq 0}} c_{j_1}(1/2) d_\ell^{(1/2-j_1)} e_\ell = \sum_{\ell=0}^{j-1} c_{j-\ell}(1/2) d_\ell^{(1/2-j+\ell)} e_\ell \quad (1\leq j\leq N).$$

From (41) we get (12) immediately by partial summation and some easy calculations.

3.3 Proof of (15)

Now we prove (15). Define $P_1^{(e)}(n) = n/P^{(e)}(n)$. By Euler's product we have for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} P_1^{(e)}(n)n^{-s} = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s}\right). \tag{42}$$

Suppose p is a prime. From (6) it is easy to see that

$$P^{(e)}(p) = p, P^{(e)}(p^2) = p^2 + p, P^{(e)}(p^3) = p^3 + 2p, P^{(e)}(p^\alpha) = p^\alpha + O(p^{\alpha/2})\ (\alpha \ge 4).$$

Hence

$$\begin{aligned} &P_1^{(e)}(p) = 1, P_1^{(e)}(p^2) = \frac{1}{1+p^{-1}}, P_1^{(e)}(p^3) = \frac{1}{1+2p^{-2}}, \\ &P_1^{(e)}(p^\alpha) = \frac{p^\alpha}{p^\alpha + O(p^{\alpha/2})} = \frac{1}{1+O(p^{-\alpha/2})} = 1 + O(p^{-\alpha/2})\ (\alpha \ge 4). \end{aligned} \tag{43}$$

From (43) we get

$$\begin{aligned} &1 + \sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s} \\ &= 1 + \sum_{\alpha=1}^{\infty} p^{-\alpha s} + \sum_{\alpha=1}^{\infty}(P_1^{(e)}(p^\alpha) - 1)p^{-\alpha s} \\ &= 1 + \sum_{\alpha=1}^{\infty} p^{-\alpha s} + \sum_{\alpha=2}^{\infty}(P_1^{(e)}(p^\alpha) - 1)p^{-\alpha s} \\ &= \frac{1}{1-p^{-s}} + (\frac{1}{1+p^{-1}} - 1)p^{-2s} + (\frac{1}{1+2p^{-2}} - 1)p^{-3s} + \sum_{\alpha=4}^{\infty}(P_1^{(e)}(p^\alpha) - 1)p^{-\alpha s} \\ &= \frac{1}{1-p^{-s}} + (\frac{1}{1+p^{-1}} - 1)p^{-2s} + O\left(p^{-3\sigma-2} + \sum_{\alpha=4}^{\infty} p^{-\alpha/2-\alpha\sigma}\right) \\ &= \frac{1}{1-p^{-s}} + (\frac{1}{1+p^{-1}} - 1)p^{-2s} + O\left(p^{-3\sigma-2} + p^{-4\sigma-2}\right) \\ &= \frac{1}{1-p^{-s}} - p^{-1-2s} + O\left(p^{-2\sigma-2} + p^{-3\sigma-2} + p^{-4\sigma-2}\right) \\ &= \frac{1}{1-p^{-s}} - p^{-1-2s} + O\left(p^{-2\sigma-2} + p^{-4\sigma-2}\right). \end{aligned} \tag{44}$$

Hence we get

$$\begin{aligned} &(1-p^{-s})(1 + \sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s}) \\ &= 1 - p^{-1-2s}(1-p^{-s}) + O\left(p^{-2\sigma-2}(1+p^{-2\sigma})(1+p^{-\sigma})\right) \\ &= 1 - p^{-1-2s} + p^{-1-3s} + O\left(p^{-2\sigma-2}(1+p^{-3\sigma})\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} &(1-p^{-s})(1+p^{-1-2s})(1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s}) \\ &= 1-p^{-2-4s}+p^{-1-3s}+p^{-2-5s}+O\left(p^{-2\sigma-2}(1+p^{-3\sigma})(1+p^{-1-2\sigma})\right) \\ &= 1+p^{-1-3s}+O\left(p^{-2\sigma-2}(1+p^{-3\sigma})(1+p^{-1-2\sigma})\right) \end{aligned} \tag{46}$$

and

$$\begin{aligned} &(1-p^{-s})(1+p^{-1-2s})(1-p^{-1-3s})(1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s}) \\ &= 1-p^{-2-6s}+O\left(p^{-2\sigma-2}(1+p^{-3\sigma})(1+p^{-1-2\sigma})(1+p^{-1-3\sigma})\right) \\ &= 1+O\left(p^{-2-6\sigma}+p^{-2\sigma-2}(1+p^{-3\sigma})(1+p^{-1-2\sigma})(1+p^{-1-3\sigma})\right). \end{aligned} \tag{47}$$

We write for $\sigma > 1$

$$\begin{aligned} &1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s} \\ &= \frac{1}{1-p^{-s}} \times \frac{1}{1+p^{-1-2s}} \times \frac{1}{1-p^{-1-3s}} \\ &\times \left((1-p^{-s}) \times (1+p^{-1-2s}) \times (1-p^{-1-3s}) \times (1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s})\right) \\ &= \frac{1}{1-p^{-s}} \times \frac{1-p^{-1-2s}}{1-p^{-2-4s}} \times \frac{1}{1-p^{-1-3s}} \\ &\times \left((1-p^{-s}) \times (1+p^{-1-2s}) \times (1-p^{-1-3s}) \times (1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s})\right). \end{aligned} \tag{48}$$

From (48) we may write for $\sigma > 1$ that

$$\sum_{n=1}^{\infty} P_1^{(e)}(n)n^{-s} = \frac{\zeta(s)\zeta(3s+1)}{\zeta(2s+1)} G_{P_1^{(e)}}(s), \tag{49}$$

where

$$G_{P_1^{(e)}}(s) = \zeta(4s+2)\prod_p \left((1-p^{-s})(1+p^{-1-2s})(1-p^{-1-3s})(1+\sum_{\alpha=1}^{\infty} P_1^{(e)}(p^\alpha)p^{-\alpha s})\right). \tag{50}$$

From (47) we see that if we write the function $G_{P_1^{(e)}}(s)$ into a Dirichlet series

$$G_{P_1^{(e)}}(s) = \sum_{n=1}^{\infty} g_{P_1^{(e)}}(n)n^{-s}, \tag{51}$$

then this Derichlet series is absolutely for $\sigma > -1/6$. This fact implies that

$$\sum_{n\le x}|g_{P_1^{(e)}}(n)| \ll 1,\ \sum_{n\le x}\frac{g_{P_1^{(e)}}(n)}{n} = \sum_{n=1}^{\infty}\frac{g_{P_1^{(e)}}(n)}{n} + O(x^{-7/6+\varepsilon}). \tag{52}$$

From (49), (51) and (52) we get

$$\begin{aligned}
\sum_{n\le x} P_1^{(e)}(n) &= \sum_{n_1 n_2^2 n_3^3 n_4\le x}\frac{\mu(n_2)}{n_2 n_3} g_{P_1^{(e)}}(n_4) \qquad (53)\\
&= \sum_{n_4\le x} g_{P_1^{(e)}}(n_4)\sum_{n_2^2 n_3^3\le\frac{x}{n_4}}\frac{\mu(n_2)}{n_2 n_3}\sum_{n_1\le\frac{x}{n_2^2 n_3^3 n_4}} 1\\
&= \sum_{n_4\le x} g_{P_1^{(e)}}(n_4)\sum_{n_2^2 n_3^3\le\frac{x}{n_4}}\frac{\mu(n_2)}{n_2 n_3}\left(\frac{x}{n_2^2 n_3^3 n_4} + O(1)\right)\\
&= x\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4}\sum_{n_2^2 n_3^3\le\frac{x}{n_4}}\frac{\mu(n_2)}{n_2^3 n_3^4} + O(\sum_{n_4\le x}|g_{P_1^{(e)}}(n_4)|\times\log^2 x)\\
&= x\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4}\sum_{n_2\le\sqrt{\frac{x}{n_4}}}\frac{\mu(n_2)}{n_2^3}\sum_{n_3\le(\frac{x}{n_4 n_2^2})^{1/3}}\frac{1}{n_3^4} + O(\log^2 x)\\
&= x\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4}\sum_{n_2\le\sqrt{\frac{x}{n_4}}}\frac{\mu(n_2)}{n_2^3}\left(\zeta(4) + O(\frac{n_4 n_2^2}{x})\right) + O(\log^2 x)\\
&= x\zeta(4)\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4}\sum_{n_2\le\sqrt{\frac{x}{n_4}}}\frac{\mu(n_2)}{n_2^3} + O(\log^2 x)\\
&= x\zeta(4)\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4}\left(\frac{1}{\zeta(3)} + O(\frac{n_4}{x})\right) + O(\log^2 x)\\
&= x\frac{\zeta(4)}{\zeta(3)}\sum_{n_4\le x}\frac{g_{P_1^{(e)}}(n_4)}{n_4} + O(\log^2 x)\\
&= x\frac{\zeta(4)}{\zeta(3)}\sum_{n_4=1}^{\infty}\frac{g_{P_1^{(e)}}(n_4)}{n_4} + O(\log^2 x),
\end{aligned}$$

where we used the following easy estimates($y \ge 2$)

$$\sum_{n\le y} n^{-4} = \zeta(4) + O(y^{-3}), \tag{54}$$

$$\sum_{n\le y}\mu(n)n^{-3} = \frac{1}{\zeta(3)} + O(y^{-2}), \tag{55}$$

$$\sum_{n\le y} n^{-1} \ll \log y. \tag{56}$$

Now (15) follows from (53) by partial summation.

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