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Mean Values of a Class of Arithmetical Functions

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Abstract

In this paper we consider a class of functions $\mathcal{U}$ of arithmetical functions which include $\tilde{P}(n)/n$, where $\tilde{P}(n) := n\prod_{p|n}(2-\frac{1}{p})$. For any given $U \in \mathcal{U}$, we obtain the asymptotic formula for $\sum_{n\le x} U(n)$, which improves a result of De Koninck and Kátai.

1 Introduction

In 1933, Pillai [10] introduced the function

$$P(n) = \sum_{k=1}^{n} \gcd(k, n),$$

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and proved that

$$P(n)=\sum_{d|n} d\varphi(n/d), \quad \text{and} \quad \sum_{d|n} P(d)=nd(n)=\sum_{d|n}\sigma(d)\varphi(n/d),$$

where φ is Euler's function, $d(n)$ and $\sigma(n)$ denote the number of divisors of n and the sum of the divisors of n respectively. Many authors investigated the properties of $P(n)$, see [2, 3, 4, 5, 6, 10, 13]; it is Sloane's sequence A018804. Chidambaraswamy and Sitaramachandrarao [6] showed that, given an arbitrary $\epsilon>0$,

$$\sum_{n\le x} P(n)=e_1x^2\log x+e_2x^2+O(x^{1+\theta+\epsilon}),$$

where e_1, e_2 are computable constants and $0<\theta<1/2$ is some exponent contained in

$$\sum_{n\le x} d(n)=x\log x+(2\gamma-1)x+O(x^{\theta+\epsilon}). \tag{1}$$

The asymptotic formula (1) is the well-known Dirichlet divisor problem. The latest value of θ is $\theta=131/416$ proved by Huxley [8].

Tóth [12] first defined the gcd-sum function over regular integers modulo n by the relation

$$\tilde{P}(n)=\sum_{k\in \mathrm{Reg}_n} \gcd(k,n), \tag{2}$$

where $\mathrm{Reg}_n=\{k: 1\le k\le n \text{ and } k \ \text{ is regular } (\mathrm{mod}\ n)\}$, and proved that $\tilde{P}(n)$ is multiplicative and for every $n\ge 1$,

$$\tilde{P}(n)=n\prod_{p|n}(2-\frac{1}{p}). \tag{3}$$

It is sequence A176345 in Sloane's Encyclopedia. He also obtained the following asymptotic formula

$$\sum_{n\le x}\tilde{P}(n)=\frac{x^2}{2\zeta(2)}(K_1\log x+K_2)+O(x^{3/2}\delta(x)), \tag{4}$$

where K_1 and K_2 are certain constants and $\delta(x)$ is given by

$$\delta(x)=\exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}).$$

Zhang and Zhai [15] showed that the estimate of $\sum_{n\le x}\tilde{P}(n)$ is closely related to the square-free divisor problem and improved the error term of (4) under RH.

De Koninck and Kátai [7] introduced two wide classes of arithmetical functions $\mathcal{R}$ and $\mathcal{U}$, the first of which includes the function $P(n)/n$, and the second of which includes $\tilde{P}(n)/n$. More precisely, the class $\mathcal{R}$ is made of the following functions R. Firstly let $\gamma(n)$ denote the kernel of $n\ge 2$, that is $\gamma(n)=\prod_{p|n} p$ (with $\gamma(1)=1$). Then, given an arbitrary positive constant c, an arbitrary real number $\alpha > 0$ and a multiplicative function $\kappa(n)$ satisfying $| \kappa(n) | \leq \frac{c}{\gamma(n)^\alpha}$ for all $n \geq 2$, let $R \in \mathcal{R}$ be defined by

$$R(n) = R_{\kappa,c,\alpha}(n) := d(n) \sum_{d \| n} \kappa(d) = d(n) \prod_{p^a \| n} (1 + \kappa(p^a)). \tag{5}$$

It is easily seen that if we let $\kappa(p^a) = -\frac{a/(a+1)}{p}$, then the corresponding function $R(n)$ is precisely $P(n)/n$.

De Koninck and Kátai [7] showed that

$$T(x) := \sum_{n \leq x} R(n) = A_0 x \log x + B_0 x + O(x^{\beta+\epsilon}), \tag{6}$$

with

$$\beta = \begin{cases} \theta, & \text{if } \alpha \geq 1 - \theta; \\ 1 - \alpha, & \text{if } \alpha < 1 - \theta; \end{cases}$$

where θ is the exponent in (1), A_0, B_0 are certain constants.

As for the class of functions $\mathcal{U}$, it is made of the functions

$$U(n) = U_{h,c,\alpha}(n) := 2^{\omega(n)} \sum_{d | n} h(d),$$

where $\omega(n)$ stands for the number of distinct prime factors of n, and h is a multiplicative function satisfying $|h(n)| \leq \frac{c}{\gamma(n)^\alpha}$ for all $n \geq 2$. It is easily seen that by taking $h(p) = -\frac{1}{2p}$ and $h(p^a) = 0$, for $a \geq 2$, we obtain the particular case $U(n) = \tilde{P}(n)/n$. De Koninck and Kátai [7] proved that

$$S(x) := \sum_{n \leq x} U(n) = t_1 x \log x + t_2 x + O(\frac{x}{\log x}), \tag{7}$$

where t_1, t_2 are certain constants.

In this paper, we shall prove the following

Theorem 1. *Suppose* $0 \leq \alpha < 1$. *Then we have*

$$S(x) = t_1 x \log x + t_2 x + O(x^{1-\alpha+\epsilon} + x^{1/2+\epsilon}). \tag{8}$$

Remark 2. (i) From our proof we see that the evaluation of $S(x)$ is closely related to the distribution of the zeros of the Riemann zeta function. The exponent $1/2$ can be reduced to $4/11$ if RH is true.

(ii) The exponent $1-\alpha$ in the error term of Theorem 1 is best possible when α is small. For example, if we take $h(n) = n^{-\alpha}$ with $0 < \alpha < 1/2$, then our proof with slight modifications yields

$$\sum_{n \leq x} U(n) = t_1 x \log x + t_2 x + t_3 x^{1-\alpha} \log x + t_4 x^{1-\alpha} + O(x^{1/2+\epsilon}).$$

We are also interested in the short interval case. In this case, the restrictions on α and RH can be removed. Actually, we have the following Theorem 3.

Theorem 3. *Suppose* (1) *holds for* $1/4 < \theta < 1/3$. *Then for* $x^{\theta+2\epsilon} \le y \le x$, *we have*

$$\sum_{x<n\le x+y} U(n) = H(x+y) - H(x) + O(yx^{-\frac{\epsilon}{2}} + x^{\theta+\epsilon}), \tag{9}$$

where $H(x) = t_1 x \log x + t_2 x$.

2 Preliminary Lemmas

Lemma 4. *Let* s *be a complex number with* $\Re s > 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{U(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} G(s),$$

where $G(s)$ *can be written as a Dirichlet series* $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$, *which is absolutely convergent for* $\Re s > 1-\alpha$. *Moreover* $g(n)$ *satisfies* $|g(n)| \ll n^{-\alpha+\epsilon}$.

Proof. For $\Re s > 1$, by Euler product representation we have

$$F(s) := \sum_{n=1}^{\infty} \frac{U(n)}{n^s} = \prod_p \left(1 + \sum_{\beta=1}^{\infty} \frac{U(p^\beta)}{p^{\beta s}}\right),$$

where $U(p^\beta) = 2(1 + h(p) + \cdots + h(p^\beta)), \beta \ge 1$. Thus

$$\begin{aligned} 1 + \sum_{\beta=1}^{\infty} \frac{U(p^\beta)}{p^{\beta s}} &= 1 + \sum_{\beta=1}^{\infty} \frac{2}{p^{\beta s}} + 2\sum_{\beta=1}^{\infty} p^{-\beta s} \sum_{j=1}^{\beta} h(p^j) \\ &= \frac{1-p^{-2s}}{(1-p^{-s})^2} + 2\sum_{\beta=1}^{\infty} p^{-\beta s} \sum_{j=1}^{\beta} h(p^j) \\ &= \frac{1-p^{-2s}}{(1-p^{-s})^2} \times \left(1 + \frac{2(1-p^{-s})^2}{1-p^{-2s}} \sum_{\beta=1}^{\infty} p^{-\beta s} \sum_{j=1}^{\beta} h(p^j)\right), \end{aligned}$$

hence we get

$$\sum_{n=1}^{\infty} \frac{U(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} G(s),$$

where

$$G(s) = \prod_p \left(1 + \frac{2(1-p^{-s})^2}{1-p^{-2s}} \sum_{\beta=1}^{\infty} p^{-\beta s} \sum_{j=1}^{\beta} h(p^j)\right).$$

From the above formula, it is easy to see that $G(s)$ can be expanded to a Dirichlet series $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$, which is absolutely convergent for $\Re s > 1-\alpha$, if we notice that $|h(p)| \le \frac{c}{p^\alpha}$. Therefore $|g(n)| \ll n^{-\alpha+\epsilon}$. □

Lemma 5. *Let*

$$\sum_{n=1}^{\infty}\frac{d^{(2)}(n)}{n^s}=\frac{\zeta^2(s)}{\zeta(2s)},\quad \Re s>1,$$

where $d^{(2)}(n)$ *denote the number of square-free divisors of* n*. Then for any real numbers* $x\geq 1$*, we have*

$$D^{(2)}(x):=\sum_{n\leq x}d^{(2)}(n)=c_1x\log x+c_2x+\Delta^{(2)}(x)$$

with $\Delta^{(2)}(x)=O(x^{1/2}\log x)$*, where*

$$c_1=\frac{1}{\zeta(2)},\quad c_2=\frac{2\gamma-1}{\zeta(2)}-\frac{2\zeta'(2)}{\zeta^2(2)}.$$

Moreover, if RH is true, then $\Delta^{(2)}(x)=O(x^{4/11+\epsilon})$.

Proof. The first result is due to Mertens [9] and the second one is due to Baker [1]. □

Lemma 6.

$$\sum_{n\leq x}|g(n)|\ll x^{1-\alpha+\epsilon}.$$

Proof. It follows from $|g(n)|\ll n^{-\alpha+\epsilon}$. □

Lemma 7. *Let* $k\geq 2$ *be a fixed integer ,* $1<y\leq x$ *be large real numbers and*

$$\mathcal{A}(x,y;k,\epsilon):=\sum_{\substack{x<nm^k\leq x+y\\ m>x^{\epsilon}}}1.$$

Then we have

$$\mathcal{A}(x,y;k,\epsilon)\ll yx^{-\epsilon}+x^{1/4}.$$

Proof. This is Lemma 3 of Zhai [14]. □

3 Proof of Theorem 1

Notice that

$$\frac{\zeta^2(s)}{\zeta(2s)}=\sum_{\ell=1}^{\infty}\frac{d^{(2)}(\ell)}{\ell^s},\quad G(s)=\sum_{m=1}^{\infty}\frac{g(m)}{m^s}. \tag{10}$$

By the Dirichlet convolution, we have

$$\sum_{n\leq x}U(n)=\sum_{m\ell\leq x}g(m)d^{(2)}(\ell)=\sum_{m\leq x}g(m)\sum_{\ell\leq x/m}d^{(2)}(l),$$

and Lemma 5 applied to the inner sum gives

$$\sum_{n\leq x}U(n)=\sum_{m\leq x}g(m)\left\{\frac{c_1x}{m}\log(\frac{x}{m})+\frac{c_2x}{m}+O\left((\frac{x}{m})^{1/2+\epsilon}\right)\right\}$$

$$= c_1 x \left\{ \left(\log x + \frac{c_2}{c_1} \right) \sum_{m \le x} \frac{g(m)}{m} - \sum_{m \le x} \frac{g(m)\log m}{m} \right\} + O\left(x^{1/2+\epsilon} \sum_{m \le x} \frac{|g(m)|}{m^{1/2+\epsilon}} \right)$$

$$= c_1 x \left\{ \left(\log x + \frac{c_2}{c_1} \right) \sum_{m=1}^{\infty} \frac{g(m)}{m} - \sum_{m=1}^{\infty} \frac{g(m)\log m}{m} + O(x^{-\alpha+\epsilon}) \right\} + O\left(x^{1/2+\epsilon} \sum_{m \le x} \frac{|g(m)|}{m^{1/2+\epsilon}} \right),$$

if we notice by Lemma 6 that both of the infinite series $\sum_{m=1}^{\infty} \frac{g(m)}{m}$, $\sum_{m=1}^{\infty} \frac{g(m)\log m}{m}$ are absolutely convergent, and

$$\sum_{m > x} \frac{g(m)}{m} \ll x^{-\alpha+\epsilon}, \quad \sum_{m > x} \frac{g(m)\log m}{m} \ll x^{-\alpha+\epsilon}. \tag{11}$$

Then we have

$$\sum_{n \le x} U(n) = t_1 x \log x + t_2 x + O(x^{1-\alpha+\epsilon}) + O\left(x^{1/2+\epsilon} \sum_{m \le x} \frac{|g(m)|}{m^{1/2+\epsilon}} \right), \tag{12}$$

where

$$t_1 = \frac{1}{\zeta(2)} \sum_{m=1}^{\infty} \frac{g(m)}{m} = \frac{G(1)}{\zeta(2)},$$

$$t_2 = \frac{1}{\zeta(2)} \left\{ (2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}) \sum_{m=1}^{\infty} \frac{g(m)}{m} - \sum_{m=1}^{\infty} \frac{g(m)\log m}{m} \right\}$$

$$= \frac{1}{\zeta(2)} \left\{ (2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}) G(1) - G'(1) \right\}.$$

By Lemma 6, we have

$$\sum_{m \le x} \frac{|g(m)|}{m^{1/2+\epsilon}} \le \sum_{m \le x} \frac{1}{m^{1/2+\alpha+\epsilon}} \le \begin{cases} x^{\epsilon}, & \alpha \ge 1/2; \\ x^{1/2-\alpha+\epsilon}, & \alpha < 1/2, \end{cases}$$

Theorem 1 follows from the above estimates and Eq. (12).

4 Proof of Theorem 3

By Lemma 4, we have

$$U(n) = \sum_{n = n_1 n_2 n_3^2} d(n_1) g(n_2) \mu(n_3),$$

where $d(n)$ is the divisor function. Then

$$\sum_{x < n \le x+y} U(n) = \sum_{x < n_1 n_2 n_3^2 \le x+y} d(n_1) g(n_2) \mu(n_3) = \Sigma_1 + O\left(\Sigma_2 + \Sigma_3 \right), \tag{13}$$

where

$$\Sigma_1 = \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} g(n_2)\mu(n_3) \sum_{\frac{x}{n_2 n_3^2} < n_1 \le \frac{x+y}{n_2 n_3^2}} d(n_1),$$

$$\Sigma_2 = \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_2 > x^\epsilon}} d(n_1)|g(n_2)|,$$

$$\Sigma_3 = \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_3 > x^\epsilon}} d(n_1)|g(n_2)|.$$

Recalling (1), the inner sum in Σ_1 is

$$\frac{(x+y)}{n_2 n_3^2}\log\frac{(x+y)}{n_2 n_3^2} - \frac{x}{n_2 n_3^2}\log\frac{x}{n_2 n_3^2} + (2\gamma - 1)\frac{y}{n_2 n_3^2} + O\left(\frac{x^\theta}{n_2^\theta n_3^{2\theta}}\right)$$
$$= \frac{(x+y)\log(x+y) - x\log x}{n_2 n_3^2} - y\frac{\log(n_2 n_3^2)}{n_2 n_3^2} + (2\gamma - 1)\frac{y}{n_2 n_3^2} + O\left(\frac{x^\theta}{n_2^\theta n_3^{2\theta}}\right).$$

Inserting the above expression into Σ_1 and after some easy calculations, we get

$$\Sigma_1 = H(x+y) - H(x) + O\left(yx^{-\epsilon} + y^{-\alpha\epsilon+\epsilon^2} + x^{\theta+\epsilon}\right). \tag{14}$$

For Σ_2, we have

$$|g(n_2)| \ll n_2^{-\alpha+\epsilon} \ll x^{-\alpha\epsilon+\epsilon^2},$$

if we notice that $n_2 > x^\epsilon$, and hence

$$\Sigma_2 \ll x^{-\alpha\epsilon+\epsilon^2} \sum_{x < n_1 n_2 n_3^2 \le x+y} d(n_1) = x^{-\alpha\epsilon+\epsilon^2} \sum_{x < n \le x+y} d_*(n),$$

where

$$d_*(n) = \sum_{n = n_1 n_2 n_3^2} d(n_1) \ll n^{\epsilon^2}.$$

Therefore we have

$$\Sigma_2 \ll x^{-\alpha\epsilon+\epsilon^2} \sum_{x < n \le x+y} n^{\epsilon^2} \ll yx^{-\alpha\epsilon+\epsilon^2}. \tag{15}$$

Since $d(n) \ll n^{\epsilon^2}, \quad g(n_2) \ll 1$, by Lemma 7 we have

$$\Sigma_3 \ll x^{\epsilon^2} \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_3 > x^\epsilon}} 1 \ll x^{\epsilon^2} \sum_{\substack{x < n n_3^2 \le x+y \\ n_3 > x^\epsilon}} d(n)$$
$$\ll x^{2\epsilon^2} \sum_{\substack{x < n n_3^2 \le x+y \\ n_3 > x^\epsilon}} 1 = x^{2\epsilon^2}\mathcal{A}(x, y; 2, \epsilon)$$
$$\ll yx^{-\epsilon+2\epsilon^2} + x^{1/4+\epsilon^2}. \tag{16}$$

Then Theorem 3 follows from Eqs. (13)–(16).

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