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Ramanujan and Labos Primes, Their Generalizations, and Classifications of Primes

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Abstract

We study the parallel properties of the Ramanujan primes and a symmetric counterpart, the Labos primes. Further, we study all primes with these properties (generalized Ramanujan and Labos primes) and construct two kinds of sieves for them. Finally, we give a further natural generalization of these constructions and pose some conjectures and open problems.

1 Introduction

The very well-known Bertrand postulate (1845) states that, for every $x > 1$, there exists a prime in the interval $(x, 2x)$. This postulate quickly became a theorem when, in 1851, it was unexpectedly proved by Chebyshev (for a very elegant version of his proof, see Theorem 9.2 in [8]). In 1919, Ramanujan ([6]; [7, pp. 208–209]) gave a quite new and very simple proof of Bertrand's postulate. Moreover, in his proof of a *generalization* of Bertrand's, the following sequence of primes appeared ([11], A104272)

$$2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, \ldots \tag{1}$$

Definition 1. For $n \geq 1$, the *n-th Ramanujan prime* is the largest prime (R_n) for which $\pi(R_n) - \pi(R_n/2) = n$.

Let us show that, equivalently, R_n is *the smallest positive integer $g(n)$ with the property that, if $x \geq g(n)$, then $\pi(x) - \pi(x/2) \geq n$.*

Proof. Indeed, evidently, $g(n)$ is prime. Note that, if for $x > R_n$, we have $\pi(x) - \pi(x/2) \le n - 1$, then, evidently, there exists a prime $q > x$ for which $\pi(q) - \pi(q/2) = n$. This contradicts the maximality of R_n. Thus $g(n) \le R_n$. On the other hand, if $g(n) < R_n$, then for all $x \in [g(n), R_n]$ we have $\pi(x) - \pi(x/2) \ge n$. Let $p \ge g(n)$ be the nearest prime less than R_n. Then, for $k \ge 0$, $\pi(p) - \pi(p/2) = n + k$. Since $\pi(R_n) - \pi(R_n/2) = n$, there exists exactly $k+1$ primes $\nu_1 < \cdots < \nu_{k+1}$ in interval $[\frac{p+1}{2}, \frac{q-1}{2}]$. For $x = 2\nu_{k+1} \in [p+1, q-1]$, we have

$$\pi(x) - \pi(x/2) = \pi(2\nu_{k+1}) - \pi(\nu_{k+1}) \le \pi(p) - \pi(p/2) - k - 1 = n - 1.$$

The contradiction shows that $g(n) = R_n$. □

In [12], Sondow obtained some estimates for R_n and, in particular, proved that $R_n > p_{2n}$ for every $n > 1$. Laishram [3] proved that $R_n < p_{3n}$ (a short proof of this result follows from the more general Theorem 30 in this paper, see Remark 29). Further, Sondow proved that $R_n \sim p_{2n}$ as $n \to \infty$. From this, denoting the counting function of the Ramanujan primes by π_R, we have $R_{\pi_R(x)} \sim 2\pi_R(x) \ln \pi_R(x)$. Since $R_{\pi_R(x)} \le x < R_{\pi_{R(x)}+1}$, we have $x \sim p_{2\pi_R(x)} \sim 2\pi_R(x) \ln \pi_R(x)$ as $x \to \infty$, and may conclude that

$$\pi_R(x) \sim \frac{x}{2\ln x} \sim \frac{\pi(x)}{2}. \tag{2}$$

Below we prove several other properties of the Ramanujan primes. *Everywhere below p_n denotes the n-th prime.* An important role is played by the following property.

Theorem 2. *If p is an odd Ramanujan prime such that $p_m < p/2 < p_{m+1}$, then the interval $(p,\ 2p_{m+1})$ contains a prime.*

In 2003, Labos introduced the following sequence of primes (cf. [11], sequence A080359). We call them *Labos primes*, denoting the n-th Labos prime by L_n.

Definition 3. For $n \ge 1$, the *n-th Labos prime* is the smallest positive integer (L_n) for which $\pi(L_n) - \pi(L_n/2) = n$.

The first Labos primes are (see sequence A080359 in [11]):

$$2, 3, 13, 19, 31, 43, 53, 61, 71, 73, 101, 103, 109, 113, 139, 157, 173, \ldots \tag{3}$$

Note that, since (see [11])

$$\pi(R_n) - \pi(R_n/2) = n, \tag{4}$$

by Definition 3 we have

$$L_n \le R_n. \tag{5}$$

Note also that, obviously, $L_n \sim p_{2n}$ *as* $n \to \infty$.

For the Labos primes we prove a symmetric statement to Theorem 2.

Theorem 4. *If p is an odd Labos prime, such that $p_m < p/2 < p_{m+1}$, then the interval $(2p_m,\ p)$ contains a prime.*

It is clear that Theorems 2-4 are connected with some left-right symmetry in the distribution of primes. Unfortunately, we do not lave a *precise* left-right symmetry since the inequalities of type $R_1 \le L_1 \le R_2 \le L_2 \le \cdots$ are broken from the very outset. In Theorem 17 we show that this deficiency could be removed by the consideration some additional primes with the close properties. Based on this theorem, we give a natural simple classification of the primes.

2 Proof of Theorems 2,4

We start with four conditions on an odd prime p.

Condition 5. Let $p = p_n$, with $n > 1$. All integers $(p+1)/2, (p+3)/2, \ldots, (p_{n+1}-1)/2$ are composite numbers.

Condition 6. Let $p \ge 5$ and $p_m < p/2 < p_{m+1}$. The interval $(p,\ 2p_{m+1})$ contains a prime.

Condition 7. Let $p = p_n$ with $n \ge 3$. All integers $(p-1)/2, (p-3)/2, \ldots, (p_{n-1}+1)/2$ are composite numbers.

Condition 8. Let $p_m < p/2 < p_{m+1}$. The interval $(2p_m,\ p)$ contains a prime.

Lemma 9. *Conditions 5 and 6 are equivalent.*

Proof. If Condition 5 is valid, and $p_m < p/2 < p_{m+1}$, then $p_{m+1} > (p_{n+1}-1)/2$, i.e., $p_{m+1} \ge (p_{n+1}+1)/2$. Thus $2p_{m+1} > p_{n+1} > p_n = p$, and Condition 6 is valid. Conversely, let Condition 6 be satisfied and $p = p_n$. Then from the condition $p_{m+1} > p/2 > p_m$ we have $2p_m < p_n < p_{n+1} < 2p_{m+1}$, or $p_m < p_n/2 < p_{n+1}/2 < p_{m+1}$. Since the interval (p_m, p_{m+1}) contains no primes, the interval $(p_n/2, p_{n+1}/2) \subset (p_m, p_{m+1})$ also contains no primes, and Condition 5 follows. □

Analogously, we obtain the equivalence of the second pair of conditions.

Lemma 10. *Conditions 7 and 8 are equivalent.*

Now we are able to prove Theorems 2-4.

Proof. In view of Lemma 9, to prove of Theorem 1 it is sufficient to prove that, for Ramanujan primes, Condition 5 is satisfied. If Condition 5 is not satisfied, then suppose that $p_m = R_n < p_{m+1}$ and k is the least positive integer such that $q = (p_m + k)/2$ is a prime not exceeding $(p_{m+1}-1)/2$. Thus

$$R_n = p_m < 2q < p_{m+1} - 1. \tag{6}$$

From Definition 1 it follows that $R_n - 1$ is the maximum integer for which the equality

$$\pi(R_n - 1) - \pi((R_n - 1)/2) = n - 1 \tag{7}$$

holds. However, according to (6), $\pi(2q) = \pi(R_n - 1) + 1$, and in view of the minimality of the prime q, in the interval $((R_n-1)/2, q)$ there is no prime. Thus $\pi(q) = \pi((R_n-1)/2) + 1$ and

$$\pi(2q) - \pi(q) = \pi(R_n - 1) - \pi((R_n - 1)/2) = n - 1.$$

Since, by (6), $2q > R_n$, this contradicts the property of maximality of R_n in (7). Thus Theorem 2 follows. Theorem 4 is proved quite analogously, using Lemma 10. □

3 Pseudo-Ramanujan primes, over-Ramanujan primes and their Pseudo-Labos and over-Labos analogues

Definition 11. Non-Ramanujan primes satisfying Condition 6 (or, equivalently, Condition 5) are *pseudo-Ramanujan primes.*

Denote the sequence of pseudo-Ramanujan primes by $\{R_n^*\}$ (see sequence (1.7)). The first pseudo-Ramanujan primes are (see sequence A164288 in [11]):

$$109, 137, 191, 197, 283, 521, 617, 683, 907, 991, 1033, 1117, 1319, \ldots$$

Definition 12. An *over-Ramanujan prime* is a prime satisfying Condition 6 (or, equivalently, Condition 5).

Denote the sequence of over-Ramanujan primes by $\{R'_n\}$ (see sequence A164368 in [11]). Note that all Ramanujan primes greater than 2 are also over-Ramanujan primes. It is easy to see that $R'_1 = 11$. Furthermore, let us prove the following simple criterion.

Proposition 13. *$p_n \geq 5$ is an over-Ramanujan prime if and only if $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$.*

Proof. 1) Let $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$. From this it follows that, if $p_k < p_n/2 < p_{k+1}$, then there are no primes between $p_n/2$ and $p_{n+1}/2$. Thus $p_{n+1}/2 < p_{k+1}$ as well. Therefore, we have $2p_k < p_n < p_{n+1} < 2p_{k+1}$, i.e., p_n is an over-Ramanujan prime. Conversely, if p_n is an over-Ramanujan prime, then $2p_k < p_n < p_{n+1} < 2p_{k+1}$, and $\pi(\frac{p_n}{2}) = \pi(\frac{p_{n+1}}{2})$. □

Definition 14. Non-Labos primes satisfying Condition 8 (or, equivalently, Condition 7) are *pseudo-Labos primes.*

Denote the sequence of pseudo-Labos primes by $\{L_n^*\}$. The first pseudo-Labos primes are (A164294 in [11]):

$$131, 151, 229, 233, 311, 571, 643, 727, 941, 1013, 1051, 1153, 1373, \ldots$$

Definition 15. An *over-Labos prime* is a prime satisfying Condition 8 (or, equivalently, Condition 7).

Denote the sequence of over-Labos primes by $\{L'_n\}$ (see sequence A194598 in [11]). Note that all Labos primes grater than 3 are also over-Labos primes. It is easy to verify that $L'_1 = 13$. Analogously to Proposition 13 we obtain the following criterion for over-Labos primes.

Proposition 16. *$p_n \geq 5$ is an over-Labos prime if and only if $\pi(\frac{p_{n-1}}{2}) = \pi(\frac{p_n}{2})$.*

Theorem 17. *Consider ALL primes $\{R'_n\}$ and $\{L'_n\}$ for which Theorems 2-4 are correspondingly true, then*

$$R'_1 \leq L'_1 \leq R'_2 \leq L'_2 \leq \cdots \tag{8}$$

Proof. Note that intervals of the form $(2p_m,\ 2p_{m+1})$ containing not more than one prime, contain neither over-Ramanujan nor over-Labos primes. If an interval contains only two primes, then the first prime is an over-Ramanujan prime (R'), while the second one is an over-Labos prime (L'), and we see that $R' < L'$; on the other hand, if it contains k primes, then beginning with the second one and up to the $(k-1)$-st we have primes which are simultaneously over-Ramanujan and over-Labos primes. Thus, taking into account that the last prime is only an over-Labos prime we have for this interval

$$R'_1 < L'_1 = R'_2 < L'_2 = R'_3 < \cdots < L'_{k-1} = R'_{k-1} < L'_k.$$

The following interval containing at least two primes begins with an over-Ramanujan prime and the process repeats. □

4 On difference $R_n - L_n$

Consider positive records of the difference $R_n - L_n$. They are (A182366 in [11])

$$8, 10, 24, 36, 60, 64, 84, 114, 124, 144, 202, 226, 228, \ldots \tag{9}$$

at

$$n = 2, 4, 10, 14, 43, 95, 145, 167, 287, 415, 560, 635, 982, \ldots\ .$$

However, we do not know a proof that $\limsup_{n\to\infty}(R_n - L_n) = \infty$. We prove a weaker statement.

Proposition 18. $\limsup_{n\to\infty}(R_{n+1} - L_n) = \infty$.

Proof. If there exists a constant C, such that $R_{n+1} - L_n \le C$, then for every $n \ge 1$, there exist not more than C intervals $(2, 4), (3, 6), \ldots$ containing exactly n primes. Therefore, we have $\le C$ of the first intervals containing one prime, $\le 2C$ of the first intervals containing one or two primes,..., $\le nC$ of the first intervals containing not more than n primes. Hence interval $((n+1)C, 2(n+1)C$ contains more than n primes. However, for large n it is impossible, since the number of primes in the interval $(1, N)$ for $N = 2(n+1)C$ is equivalent to $N/\ln N = o(n)$. □

Now consider the following problem. Let us call a prime p *compatible* with another prime q, if the intervals $(p/2, q/2)$ and $(p, q]$, if $q > p$, (or intervals $(q/2, p/2)$ and $(q, p\,]$, if $q < p$) contain the same number of primes. It is clear that, if p compatible with q, then q compatible with p. If p is compatible with no other prime, we call it a *peculiar* prime. It is required to describe the peculiar primes. We give a solution of this problem in the following form.

Proposition 19. *A prime p is peculiar if and only if it is simultaneously Ramanujan and Labos prime.*

Proof. Let $p = L_n = R_n$ (a case $L_{n-1} = R_n$, evidently, is impossible). Then, by the Definitions 1-3, p is the smallest and the largest prime for which $\pi(p) - \pi(p/2) = n$. Thus the difference $\pi(x) - \pi(x/2) = n$, where x is prime, occurs only once. However, if there exists

$q \neq p$ for which $\pi(q) - \pi(q/2) = \pi(p) - \pi(p/2) = n$, then the difference $\pi(x) - \pi(x/2) = n$ with a prime x occurs at least twice. The contradiction shows that p is a peculiar prime. Conversely, if a prime p is peculiar, then, for any $q \neq p$, we have $\pi(p/2) - \pi(q/2) \neq \pi(p) - \pi(q)$ and, consequently, $\pi(q) - \pi(q/2) \neq \pi(p) - \pi(p/2)$. Thus the difference $\pi(x) - \pi(x/2) = \pi(p) - \pi(p/2)$, where x is prime, appears only once. This means that $L_n = R_n = p$, where $n = \pi(p) - \pi(p/2)$. □

Thus the peculiar primes are (A164554 in [11])

$$2, 71, 101, 181, 239, 241, 269, 349, 373, 409, 419, 433, 439, 491, \ldots \qquad (10)$$

Corollary 20. *If there exist infinitely many peculiar primes, then* $\liminf_{n\to\infty}(R_n - L_n) = 0$.

5 Prime gaps

Note that, as it follows from Lemma 9 and Theorem 2, if we consider a run of consecutive Ramanujan primes $p = R_l, \ldots, q = R_k$, then the interval $[\frac{1}{2}(p+1),\ \frac{1}{2}(q+1)]$ is free from primes. However, this note is far from a complete characterization of the prime gaps. For example, we have a run $\{2521, 2531\}$ of consecutive Ramanujan primes which gives a "prime gap" $[\frac{2521+1}{2},\ \frac{2531+1}{2}] = [1261, 1266]$. However, the real prime gap is much larger: (1259,1277). A better result can be obtained using over-Ramanujan primes. Indeed, the considered property of Ramanujan primes is valid for all over-Ramanujan primes, while runs of consecutive over-Ramanujan primes, generally speaking, are longer. For example, instead of the run $\{2521, 2531\}$ of Ramanujan primes, we have the run $\{2521, 2531, 2539, 2543, 2549\}$ of over-Ramanujan primes. This gives the interval $[\frac{2521+1}{2},\ \frac{2549+1}{2}] = [1261, 1275]$, which is free from primes and very close to the real gap. In general, since over-Ramanujan primes satisfy Condition 1, to every run of consecutive over-Ramanujan primes $p = R'_l, \ldots, q = R'_k$ there corresponds the interval $[\frac{1}{2}(p+1),\ \frac{1}{2}(q+1)]$ which contains no primes. Note that the prime q' following q gives an additional improvement of the lower estimate of size (l) of the considered prime gap. Indeed, we know that q' is necessarily an over-Labos prime. Since the over-Labos primes satisfy Condition 3, all numbers $\frac{q'-1}{2}, \frac{q'-3}{2}, \ldots, \frac{q+1}{2}$ are composite. Hence $l \geq \frac{q'-p}{2}$. For example, consider the run $\{227, 229, 233, 239, 241\}$ of over-Ramanujan primes (all of which are Ramanujan). The following prime is $q' = 251$. Thus, for the gap containing $(227+1)/2 = 114$ we have $l \geq \frac{251-227}{2} = 12$ (the exact value of l here is 14).

6 The first sieve for the selection of the over-Ramanujan primes from all primes

Recall that Bertrand's sequence $\{b(n)\}$ is defined by $b(1) = 2$, and, for $n \geq 2$, $b(n)$ is the largest prime less than $2b(n-1)$ (see A006992 in [11]):

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503, 5003, \ldots \qquad (11)$$

Put

$$B_0 = \{b^{(0)}(n)\} = \{b(n)\}. \qquad (12)$$

Further we build sequences $B_1 = \{b^{(1)}(n)\}, B_2 = \{b^{(2)}(n)\}, \ldots$ according to the following inductive rule: if the sequences $B_0, \ldots, B_{k-1}$ have been defined already, let us consider the minimal prime $p^{(k)} \notin \bigcup_{i=1}^{k-1} B_i$. Then the sequence $\{b^{(k)}(n)\}$ is defined by $b^{(k)}(1) = p^{(k)}$, and, for $n \geq 2$, $b^{(k)}(n)$ is the largest prime less than $2b^{(k)}(n-1)$. Consequently:

$$B_1 = \{11, 19, 37, 73, \ldots\}, \tag{13}$$

$$B_2 = \{17, 31, 61, 113, \ldots\}, \tag{14}$$

$$B_3 = \{29, 53, 103, 199, \ldots\}, \tag{15}$$

etc., such that, putting $p^{(1)} = 11$, we obtain the sequence

$$\{p^{(k)}\}_{k\geq 1} = \{11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 109, 127, \ldots\}. \tag{16}$$

Sequence (16) coincides with sequence (1) of Ramanujan primes from the second term up to the 12-th term, but the 13-th term of this sequence is 109 which is the first term of the pseudo-Ramanujan primes.

Theorem 21. *For* $n \geq 1:$

$$p^{(n)} = R'_n. \tag{17}$$

Proof. The least omitted prime in (11) is $p^{(1)} = 11 = R'_1$; the least omitted prime in the union of (12) and (13) is $p^{(2)} = 17 = R'_2$. We use induction. Suppose we have already defined the primes

$$p^{(1)} = 11, p^{(2)}, \ldots, p^{(n-1)} = R'_{n-1}.$$

Let q be the least prime which is omitted in the union $\bigcup_{i=1}^{n-1} B_i$, such that $q/2$ is in the interval (p_m, p_{m+1}). According to our algorithm, q which is dropped should not be the largest prime in the interval $(p_{m+1}, 2p_{m+1})$. Then there are primes in the interval $(q, 2p_{m+1})$; let r be one of them. We have $2p_m < q < r < 2p_{m+1}$. This means that q, in view of its minimality among the dropped primes which are more than $R'_{n-1} = p^{(n-1)}$, is the least over-Ramanujan prime larger than R'_{n-1} and the least prime of the form $p^{(k)}$ larger than $p^{(n-1)}$. Therefore, $q = p^{(n)} = R'_n$. □

Analogously, using sequence $\{c_n\}$ defined by $c(1) = 2$, and, for $n \geq 2$, $c(n)$ is the smallest prime more than $2c(n-1)$ (see A055496 in [11]), one can construct a sieve for over-Labos primes.

7 The second sieve for the selection of the over-Ramanujan primes from all primes

Theorem 17 on the precise symmetry between distributions of the sequences $\{R'_n\}$ and $\{L'_n\}$ allows us to construct the second sieve for over-Ramanujan primes.

Consider consecutive intervals of the form $(2p_n,\ 2p_{n+1}),\ n = 1, 2, \ldots$ Remove all the intervals containing less than two primes. For every remaining interval, we write the primes (in increasing order) except for the last one. Then all remaining primes are over-Ramanujan. Indeed, by the definition, an over-Ramanujan prime cannot be the last prime in every such interval. Let us demonstrate this sieve. For the primes 2,3,5,7,11,... consider the intervals

$$(4, 6), (6, 10), (10, 14), (14, 22), (22, 26), (26, 34), (34, 38), \ldots . \tag{18}$$

Remove those intervals containing less than two primes. We have the following sequence of intervals:

$$(10, 14), (14, 22), (26, 34,), (38, 46), (46, 58), (58, 62), (62, 74), \ldots . \tag{19}$$

Now we write all primes from these intervals, excluding the *last* primes. Then we obtain sequence (16). Analogously we obtain the second sieve for over-Labos primes. This sequence can be obtained in a parallel way, since, by definition, an over-Labos prime cannot be the first prime in any interval of the considered form. Therefore, here, for every remaining interval, we write the primes (in increasing order) except of the *first* one. Then all remaining primes are over-Labos (cf. sequence A164333 in [11]):

$$13, 19, 31, 43, 53, 61, 71, 73, 101, 103, 109, 113, 131, 139, 151, 157, \ldots . \tag{20}$$

8 A classification of primes

In connection with the considered construction, let us consider the following classification of primes.

1) The first two primes 2,3 form a separate set of primes. 2) If $p \geq 11$ is an over-Ramanujan but not an over-Labos prime, then, in connection with the second sieve, we call p a *right prime* (cf. A166307 in [11]):

$$11, 17, 29, 41, 47, 59, 67, 97, 107, 127, 137, 149, 167, 179, 197, 227, \ldots .$$

3) If $p \geq 5$ is an over-Labos but not an over-Ramanujan prime, then we call p a *left prime* The first terms of this sequence are (cf. A182365 in [11]):

$$13, 19, 31, 43, 53, 61, 73, 103, 113, 131, 139, 157, 173, 193, 199, 251, \ldots .$$

4) If p is simultaneously an over-Ramanujan and an over-Labos prime, then we call it a *central prime* (sequence A166252 in [11]):

$$71, 101, 109, 151, 181, 191, 229, 233, 239, 241, 269, 283, 311, 349, \ldots .$$

Note that, by Proposition 19, every peculiar prime more than 2 is central. Conversely is not true. The non-peculiar central primes are (cf. A182451 in [11]):

$$109, 151, 191, 229, 233, 283, 311, 571, 643, 683, 727, 941, 991, 1033, \ldots$$

5) Finally, the rest of the primes are naturally called *isolated primes* (sequence A166251 in [11]):

$$5, 7, 23, 37, 79, 83, 89, 163, 211, 223, 257, 277, 317, 331, 337, 359, \ldots .$$

Note that from the second sieve the following results follow.

Proposition 22. *Let l_n, r_n denote the n-th left prime and the n-th right prime, respectively. Then $l_n \sim r_n$ as $n \to \infty$.*

Proof. The prime number theorem yields $p_{n+1} - p_n = o(p_n)$. Since, by the construction, l_n and r_n belong to the same interval of the form $(2p_{m(n)},\ 2p_{m(n)+1})$, we have $r_n - l_n < 2(p_{m(n)+1} - p_{m(n)}) = o(l_n)$, and the statement follows. □

Proposition 23. *The sequence of lengths of the runs of the consecutive isolated primes is unbounded if and only if there exist arbitrary long sequences of consecutive primes $p_k, p_{k+1}, \ldots, p_m$, such that every interval $(\frac{p_i}{2}, \frac{p_{i+1}}{2})$, $i = k, k+1, \ldots, m-1$, contains a prime.*

Proof. Since the isolated primes more than 3 are the only primes which are neither over-Labos nor over-Ramanujan, then the statement follows from the condition and Propositions 13,16. □

9 On density of over-Ramanujan and over-Labos primes

Let $\pi_{R'}(x)$ be the counting function of over-Ramanujan numbers not exceeding x. It follows from (2), Theorem 2 and Definition 12 that, if $\lim_{n\to\infty} \pi_{R'}(x)/\pi(x)$ exists, then it is more than or equal to $\frac{1}{2}$. Berend [2] beautifully shown that this fact follows from Definition 12 only.

Proposition 24. ([2])

$$\liminf_{n\to\infty} \pi_{R'}(x)/\pi(x) \geq \frac{1}{2}.$$

Proof. In the range from 7 up to n there are $\pi(n) - 3$ primes. Put

$$h = h(n) = \pi(n/2) - 2.$$

Then $p_{h+2} \leq n/2$ and interval $(p_{h+2}, n/2]$ is free from primes. Furthermore, consider intervals

$$(2p_2, 2p_3), (2p_3, 2p_4), \ldots, (2p_{h+1}, 2p_{h+2}).$$

Our $\pi(n) - 3$ primes are somehow distributed in these h intervals. Suppose $k = k(n)$ of these intervals contain at least one prime and $h - k$ contain no primes. Then, for exactly k primes, there is no primes between them and the next $2p_j$, and for the other $\pi(n) - 3 - k$ there is. Therefore, since $k(n) \leq h(n) \leq \pi(n/2) - 2$, then, for $\varepsilon > 0$ and $n > n_\varepsilon$, we have:

$$\frac{\pi(n) - k(n)}{\pi(n)} \geq \frac{\pi(n) - \pi(n/2)}{\pi(n)} \geq \frac{1}{2} - \varepsilon.$$

□

Unfortunately, the analysis of the sieves obtained in section 7 seems much more difficult than the analysis of the sieve of Eratosthenes for primes. Nevertheless, some very simple probabilistic arguments lead to a very plausible conjecture about the density of over-Ramanujan and over-Labos primes. First of all, let us show that events R' : "a prime is over-Ramanujan" and L' : "a prime is over-Labos" are independent.

Proof. Indeed, denoting events r : "a prime is right", l : "a prime is left" and Is : "a prime is isolated", we have

$$\mathrm{P}[R'|L'] = 1 - \mathrm{P}[l] - \mathrm{P}[Is];\ \mathrm{P}[R'|\overline{L'}] = 1 - \mathrm{P}[r] - \mathrm{P}[Is]. \tag{21}$$

Hence, in view of $\mathrm{P}[l] = \mathrm{P}[r]$ (cf. Proposition 5), we have

$$\mathrm{P}[R'|L'] = \mathrm{P}[R'|\overline{L'}]. \tag{22}$$

□

Conjecture 25.

$$\pi_{R'}(x) \sim (1 - e^{-1})\pi(x) = 0.63212\cdots\pi(x). \tag{23}$$

The following proof is heuristic.

Proof. Consider asymptotically $\frac{\pi(n)}{2}$ intervals of the form $(2p_m,\ 2p_{m+1})$ covering all $\pi(n)$ primes. It is well known ([5]) that, for large n, an interval between two random consecutive primes on the average has length $\ln p_n$. Thus a random interval of the considered form has length $2\ln p_n$ and, according to the Cramér model, the number of primes in such a random interval has the binomial $(2\ln p_n,\ \frac{1}{\ln p_n})$ distribution which, for large $\ln p_n$, has a good approximation by a Poisson distribution with parameter $\lambda = 2$. Thus we accept that a random interval contains k primes with probability $\mathrm{P}[X = k] = \frac{2^k}{k!}e^{-2},\ k = 0, 1, 2\ldots$. Since $\mathrm{P}[X = 0] = e^{-2}$, then the number of intervals containing at least one prime is about $\frac{\pi(n)}{2}(1-e^{-2})$. This number corresponds to our condition, since we consider *only* such intervals. Furthermore, since $\mathrm{P}[X = 1] = 2e^{-2}$, then the probability that such an interval contains an only prime is $2e^{-2}/(1-e^{-2})$ and, consequently, we have about $\frac{\pi(n)}{2}(1-e^{-2})\cdot\frac{2e^{-2}}{(1-e^{-2})} = \pi(n)e^{-2}$ intervals containing, by our terminology, isolated primes and this number coincides with the number of isolated primes. This means that the probability that a prime is isolated is e^{-2}. On the other hand, this probability equals $\mathrm{P}[\overline{R'}\cdot\overline{L'}] = (\mathrm{P}[\overline{R'}])^2$. Therefore, $\mathrm{P}[\overline{R'}] = e^{-1}$ and $\mathrm{P}[R'] = 1 - e^{-1}$ which justifies the conjecture. □

Greg Martin [4] did the corresponding calculations for the first million primes p, and found that approximately 61.2% of them have a prime in the interval $(p, 2p_{n+1})$. Since in this case $\ln p_n$ is small (less than 17), an error of about 2% is quite acceptable.

Remark 26. It could be done also the following simple explanation of appearance of the constant $1-1/e$ in Conjecture 25. Consider a random prime p. Let it be in interval $(2p_n, 2p_{n+1})$. We accept that p could be to the left or to the right from the midpoint $p_n + p_{n+1}$ with the same frequency. Thus the mathematical expectation of the distance between p and $2p_{n+1}$ is the difference $p_{n+1} - p_n$ which in average is $\ln n$. Accepting for large n that the frequency of appearance a prime approximately is $\frac{1}{\ln n}$, we see that it is natural to accept that the frequency of the appearance of a prime to the right from p in the considered interval is close to $1 - (1 - \frac{1}{\ln n})^{\ln n}$ which for large n is close to $1 - e^{-1}$. Thus the frequency that a random p is over-Ramanujan is $1 - e^{-1}$.

Note that, if Conjecture 25 is true, then, using (2), for the counting function $\pi_{R^*}(x)$ of pseudo-Ramanujan primes we have

$$\pi_{R^*}(x) \sim (\frac{1}{2} - \frac{1}{e})\pi(x) = 0.13212...\pi(x), \tag{24}$$

so that the proportion of Ramanujan primes among all over-Ramanujan primes is approximately 0.79099. Using Theorem 17, we note that, if Conjecture 25 is true, then, for the counting function $\pi_{L'}(x)$ of over-Labos primes we have

$$\pi_{L'}(x) \sim \pi_{R'}(x) \sim (1 - e^{-1})\pi(x). \tag{25}$$

Therefore, if Conjecture 25 is true, then, for the counting functions $\pi_l(x),\ \pi_r(x), \pi_c(x)$ and $\pi_{is}(x)$ of the left, right, central and isolated primes, respectively, of our classes of primes, we have

$$\pi_l(x) \sim \pi_r(x) \sim (1 - e^{-1})e^{-1}\pi(x) = 0.2325 \cdots \pi(x), \tag{26}$$

$$\pi_c(x) \sim (1 - e^{-1})^2\pi(x) = 0.3995 \cdots \pi(x), \tag{27}$$

$$\pi_{is}(x) \sim e^{-2} = 0.1353 \cdots \pi(x), \tag{28}$$

so that $\pi_r(x) + \pi_l(x) + \pi_c(x) + \pi_{is}(x) = \pi(x)$.

10 A generalization

Let us consider a natural generalization of Ramanujan primes.

Definition 27. For a real $v > 1$, a *v-Ramanujan prime* is the largest prime $(R_v(n))$ for which $\pi(R_v(n)) - \pi(R_v(n/v)) = n$.

As in case $v = 2$, equivalently $R_v(n)$ is the *smallest integer with the property: if $x \geq R_v(n)$, then $\pi(x) - \pi(x/v) \geq n$.* Note that, evidently,

$$R_v(n) \sim p_{((v/(v-1))n)} \tag{29}$$

as $\to \infty$. Let $\pi_R^{(v)}(x)$ be the counting function of v-Ramanujan primes. Then (cf. (2))

$$\pi_R^{(v)}(x) \sim (1 - 1/v)\pi(x). \tag{30}$$

Put

$$\kappa(v) = \begin{cases} 0, & \text{if } v \text{ is not the ratio between two primes;} \\ r, & \text{if } v = \frac{r}{q} \text{ where } r \text{ and } q \text{ are primes.} \end{cases} \tag{31}$$

The following theorem is proved in the same way as Theorem 1.

Theorem 28. *Let $v > 1$ be a given real number. If $p > \max(2v,\ \kappa(v))$ is v-Ramanujan prime such that $p_m < p/v < p_{m+1}$, then the interval $(p,\ \lceil vp_{m+1}\rceil + \varepsilon)$ contains a prime.*

Remark 29. The condition $p > \max(2v,\ \kappa(v))$ allows us to avoid the cases $p = 2v$ and $p = vq$ with a prime q, when the condition $p_n < p/v < p_{n+1}$ is impossible.

Let us find an upper bound on the n-th v-Ramanujan prime.

Theorem 30. *If* $n \geq \frac{1}{k}\max(6k,\ e^v,\ v^{(0.79677\frac{k-1}{k}v-1)^{-1}})$, *then, for* $v \geq 1.25507\frac{k}{k-1}$, *we have*

$$R_v(n) \leq p_{kn}. \tag{32}$$

Proof. It is sufficient to show that $\pi(\frac{p_{kn}}{v}) \leq (k-1)n$. Indeed, then we have $\pi(p_{kn}) - \pi(\frac{p_{kn}}{v}) \geq kn - (k-1)n = n$. We use the following known results ([1], [9]-[10]):

$$p_n < n\ln n + n\ln\ln n,\ \ n \geq 6; \tag{33}$$

$$p_n > n\ln n; \tag{34}$$

$$\pi(x) < 1.25506\frac{x}{\ln x},\ \ x > 1. \tag{35}$$

Note that $\frac{p_{kn}}{v} > \frac{kn}{v} > \frac{kn}{e^v}$. Hence, by the condition, $\frac{p_{kn}}{v} > 1$. By (33)-(35),

$$\pi(\frac{p_{kn}}{v}) < 1.25506\frac{p_{kn}}{v\ln(\frac{p_{kn}}{v})} < 1.25506\frac{kn}{v}\cdot\frac{\ln(kn)+\ln(\ln(kn))}{\ln(\frac{kn\ln(kn)}{v})}$$

$$= 1.25506\frac{kn}{v}(1+\frac{\ln v}{\ln(\frac{kn\ln(kn)}{v})}).$$

Taking into account that, by the condition, $\ln(kn) > v$, we have

$$\pi(\frac{p_{kn}}{v}) < 1.25506\frac{kn}{v}(1+\frac{\ln v}{\ln(kn)}).$$

Finally, note that, by the condition, $\frac{\ln v}{\ln(kn)} \leq 0.7968\frac{k-1}{k}v - 1$. Therefore,

$$\pi(\frac{p_{kn}}{v}) < 1.25506\cdot 0.79677(k-1)n < (k-1)n.$$

□

Corollary 31.

$$R_3(n) < p_{2n},\ \ n \geq 1. \tag{36}$$

$$R_{1.8}(n) < p_{4n},\ \ n \geq 1. \tag{37}$$

Proof. By Theorem 30, for $v = 3,\ k = 2$, we get the required inequality for $n \geq 279$. Using a computer verification for $n < 279$, we obtain (36). In case $v = 1.8,\ k = 4$, we get the required inequality for $n \geq 2370$. Using a computer verification for $n < 2370$, we obtain (10.9). □

The first 1.8-Ramanujan primes are

$$2, 11, 17, 37, 43, 59, 61, 79, 97, 101, 103, 137, 163, 167, 191, 211, \dots . \tag{38}$$

Remark 32. In case $v = 2,\ k = 3$, by Theorem 6, we find that $R_n = R_2(n) < p_{3n}$, for $n \geq 22398$. Computer verification for $n < 22398$ leads to Laishram's result [3].

Definition 33. A prime $p > \max(2v,\ \kappa(v))$ is a *v-over-Ramanujan prime* if, as soon as $p_m < p/v < p_{m+1}$, the interval $(p,\ vp_{m+1})$ contains a prime.

Definition 34. A v-over-Ramanujan not v-Ramanujan prime is a v-pseudo-Ramanujan prime.

Now v-Labos primes, v-over-Labos primes and v-pseudo-Labos primes are introduced quite symmetrically (see Section 3). In particular, the following statements are valid.

Theorem 35. *Let $v > 1$ be a given real number. If $p > \max(2v,\ \kappa(v))$ is v-Labos prime, such that $p_m < p/v < p_{m+1}$, then the interval $(\lfloor vp_m \rfloor - \varepsilon,\ p)$ contains a prime.*

Theorem 36. *For the sequences $\{R'_v(n)\}$ and $\{L'_v(n)\}$ of v-over-Ramanujan and v-over-Labos primes, we have*

$$R'_v(1) \leq L'_v(1) \leq R'_v(2) \leq L'_v(2) \leq \cdots \tag{39}$$

A generalization of the first sieve for v-over-Ramanujan primes, $v \geq 2$, is based on the Bertrand-like sequence $\{b_v(n)\}$, defined by $b_v(1) = 2$, and, for $n \geq 2$, as the largest prime less than $\lceil vb_v(n-1) \rceil + \varepsilon$. A generalization of the second sieve for v-over-Ramanujan primes is based on the sequence of intervals

$$(\lfloor 2v \rfloor - \varepsilon, \lceil 3v \rceil + \varepsilon), (\lfloor 3v \rfloor - \varepsilon, \lceil 5v \rceil + \varepsilon), (\lfloor 5v \rfloor - \varepsilon, \lceil 7v \rceil + \varepsilon), \dots \tag{40}$$

with the removing intervals containing less than two primes (cf. (19)). For every remaining interval, we write the primes (in increasing order) except for the last one. Then all remaining primes are v-over-Ramanujan. For example, if $v = 3$, we obtain the following sequence of 3-over-Ramanujan primes (sequence A164952 in [11]):

$$2, 3, 11, 17, 23, 29, 41, 43, 59, 61, 71, 73, 79, 97, 101, 103, 107, \dots . \tag{41}$$

Furthermore, one can obtain a v-classification of primes, including v-left, v-right, v-central and v-isolated primes (see Section 8). In particular, if $l_v(n),\ r_v(n)$ denote the n-th v-left prime and the n-th v-right prime, respectively, then $l_n \sim r_n$ as $n \to \infty$. Consider now a natural generalization of Proposition 19 with the similar proof.

Proposition 37. *Let $\pi_{R'_v}(x)$ be the counting function of v-over-Ramanujan numbers not exceeding x. Then*

$$\liminf_{n \to \infty} \pi_{R'_v}(x)/\pi(x) \geq 1 - \frac{1}{v}.$$

A generalization of Conjecture 25 (with a similar heuristic proof) is the following.

Conjecture 38.

$$\pi_{R'_v}(x) \sim (1 - e^{-(v-1)})\pi(x). \tag{42}$$

Note that, if Conjecture 38 is true, then, using (30), for the counting function $\pi_{R^*_v}(x)$ of v-pseudo-Ramanujan primes we have

$$\pi_{R^*_v}(x) \sim (\frac{1}{v} - e^{-(v-1)})\pi(x), \tag{43}$$

so that the proportion of v-pseudo-Ramanujan primes among all v-over-Ramanujan primes is $(\frac{1}{v} - e^{-(v-1)})/(1 - e^{-(v-1)})$. This proportion tends to 1 as $v \to 1$, and decreases to 0 as $v \to \infty$. Using Theorem 36, we note that, if Conjecture 38 is true, then, for the counting function $\pi_{L'_v}(x)$ of v-over-Labos primes, we have

$$\pi_{L'_v}(x) \sim \pi_{R'_v}(x) \sim (1 - e^{-(v-1)})\pi(x). \tag{44}$$

Furthermore, if Conjecture 38 is true, then, for the counting functions $\pi_{l_v}(x), \pi_{r_v}(x), \pi_{c_v}(x)$ and $\pi_{is_v}(x)$ of the v-left, v-right, v-central and v-isolated primes, respectively, of the considered classes of primes, we have

$$\pi_{l_v}(x) \sim \pi_{r_v}(x) \sim (1 - e^{-(v-1)})e^{-(v-1)}\pi(x), \tag{45}$$

$$\pi_{c_v}(x) \sim (1 - e^{-(v-1)})^2\pi(x), \tag{46}$$

$$\pi_{is_v}(x) \sim e^{-2(v-1)}\pi(x), \tag{47}$$

so that $\pi_{r_v}(x) + \pi_{l_v}(x) + \pi_{c_v}(x) + \pi_{is_v}(x) = \pi(x)$.

11 Other open problems

Conjecture 39. (*cf. Proposition* 23). There exist arbitrary long sequences of consecutive primes $p_k, p_{k+1}, \ldots, p_m$, such that every interval $(\frac{p_i}{2}, \frac{p_{i+1}}{2}),\ i = k, k+1, \ldots, m-1$, contains a prime.

Conjecture 40. (*cf. Proposition* 18). $\limsup_{n\to\infty}(R_n - L_n) = \infty$.

Conjecture 41. (*cf. Proposition* 19). There exist infinitely many peculiar primes.

Problem 42. For $v > 1$, to estimate the smallest pseudo-v-Ramanujan prime, the smallest v-central prime and the smallest v-isolated prime.

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