Journal of Integer Sequences, Vol. 17 (2014),
Article 14.11.5

On the Diophantine Equation $x^4 + y^4 + z^4 + t^4 = w^2$

Alejandra Alvarado
Eastern Illinois University
Department of Mathematics and Computer Science
600 Lincoln Avenue
Charleston, IL 61920
USA
aalvarado2@eiu.edu

Jean-Joël Delorme
6 rue des Emeraudes
69006 Lyon
France
jean.joel.delorme@wanadoo.fr

Abstract

To our knowledge, only three parametric solutions to the equation $x^4+y^4+z^4+t^4 = w^2$ were previously known. In this paper, we study the equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$. We prove that it is possible to obtain infinitely many parametric solutions by finding points on an elliptic curve over a field $\mathbb{Q}(m)$ and we give several new parametric solutions.

1 Introduction

Jacobi and Madden [3] considered the equation

$$x^4 + y^4 + z^4 + t^4 = (x + y + z + t)^4. \tag{1}$$

They showed the existence of infinitely many integral solutions to (1). This is a special case of the equation

$$x^4 + y^4 + z^4 + t^4 = w^4, \tag{2}$$

for which Elkies [1] found an infinite family of integral solutions when $t = 0$. In this paper, we consider a special case of a similar equation

$$x^4 + y^4 + z^4 + t^4 = w^2. \tag{3}$$

2 Background

Consider the equation (3). We say that a solution is *trivial* if at least three of the numbers x, y, z, t, w are zero, for instance $(x, y, z, t, w) = (x, 0, 0, 0, x^2)$. If two and only two of the numbers x, y, z, t are zero, the equation has no nontrivial solution since Fermat proved that the equation $x^4 + y^4 = w^2$ has no solution in nonzero integers.

The first known parametric solution is nontrivial but very elementary:

$$(x, y, z, t, w) = (a^2, ab, b^2, ab, a^4 + b^4).$$

In the next solution, found by Fauquembergue [2], one of the numbers x, y, z, t, w is zero, for instance $z = 0$:

$$(x, y, z, t, w) = (ac, bc, 0, ab, a^4 + a^2b^2 + b^4)$$

where $a^2 + b^2 = c^2$. The following solution was also found by Fauquembergue [2], again assuming $a^2 + b^2 = c^2$:

$$\begin{aligned}(x, y, z, t, w) = \big(&2a^2bc^3, 2ab^2c^3, (a^2 - b^2)c^4, 2ab(a^4 + b^4),\\ &(a^6 + 2a^5b + 3a^4b^2 + 3a^2b^4 + 2ab^5 + b^6)(a^6 - 2a^5b + 3a^4b^2 + 3a^2b^4 - 2ab^5 + b^6)\big).\end{aligned}$$

These three parametric solutions yield nontrivial numerical solutions, unless $ab = 0$.

3 The Equation $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$

While investigating solutions to equation (3), the second author noticed some interesting properties. After some numerical results, he considered the following three cases.

- If $w = x^2 + y^2 + z^2 + t^2$ then $x^2y^2 + x^2z^2 + z^2t^2 + y^2z^2 + y^2t^2 + z^2t^2 = 0$; hence, there are only trivial solutions.
- If $w = x^2 + y^2 - z^2 - t^2$ then $x^2y^2 - x^2z^2 - y^2z^2 = t^2(x^2 + y^2 - z^2)$. This is an interesting but complicated case. We leave this for future work.

- If $w = x^2+y^2+z^2-t^2$ then $x^2y^2+x^2z^2+y^2z^2 = t^2(x^2+y^2+z^2)$. This case also looked interesting and will be discussed below, beginning with the following proposition. We believe our analysis of this case is new.

Proposition 1. *If $x^2+y^2+z^2 \neq 0$, then (x,y,z,t) satisfies*

$$x^4+y^4+z^4+t^4 = (x^2+y^2+z^2-t^2)^2 \tag{4}$$

if and only if

$$(x^2+y^2+z^2)(y^2z^2+z^2x^2+x^2y^2) = \square \tag{5}$$

and

$$t^2 = \frac{y^2z^2+z^2x^2+x^2y^2}{x^2+y^2+z^2}. \tag{6}$$

Proof. We have

$$x^4+y^4+z^4+t^4-(x^2+y^2+z^2-t^2)^2 = 2\big((x^2+y^2+z^2)t^2-(y^2z^2+z^2x^2+x^2y^2)\big).$$

□

Since (4) is homogeneous of degree four, from now on we will write solutions to this equation as $(x:y:z:t)$. The equation represents a surface in $\mathbb{P}^3$,

$$\mathcal{S}' = \left\{(x:y:z:t) \in \mathbb{P}^3 \mid x^4+y^4+z^4+t^4 = (x^2+y^2+z^2-t^2)^2\right\}.$$

Assuming $t \neq 0$, we can view the surface in affine form by corresponding $(x,y,z) \leftrightarrow (x:y:z:1)$. This gives a rather interesting looking surface in three dimensions; see Figure 1.

The main focus of this paper is to answer the following question.

Question 2. How many rational curves $m \mapsto (x(m), y(m), z(m))$ are on the surface $\mathcal{S}'$?

If $xyz \neq 0$, then (5) can be expressed as

$$(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) = \square.$$

This leads to the following lemma.

Lemma 3. *If $(x:y:z:t)$ is a solution to (4) such that $xyzt \neq 0$ then $\left(\frac{1}{x}:\frac{1}{y}:\frac{1}{z}:\frac{1}{t}\right)$ is also a solution to (4).*

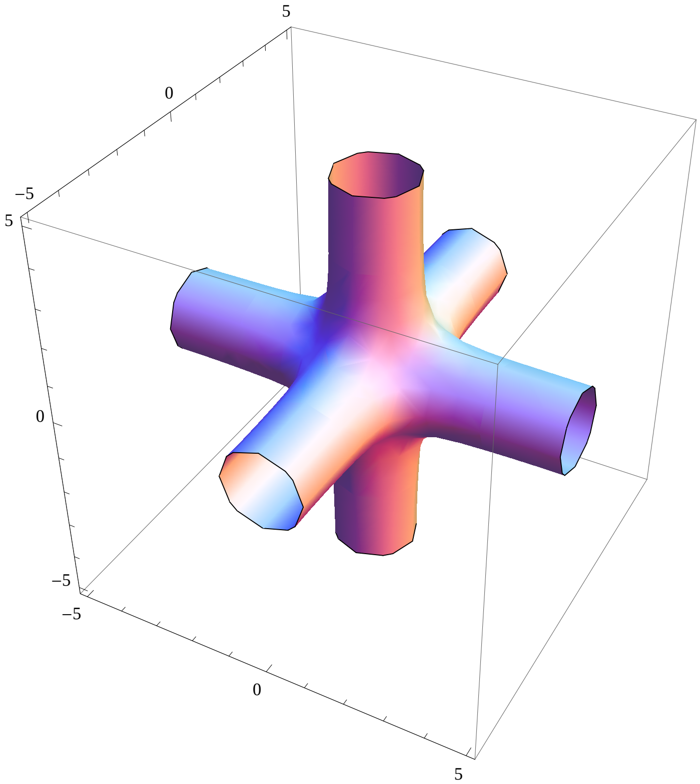

Figure 1: Plot of $\mathcal{S}'$ in affine space

Proof. Let $(X : Y : Z : T) = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right)$. Then

$$X^2 + Y^2 + Z^2 = \frac{1}{x^2y^2z^2}\left(y^2z^2 + z^2x^2 + x^2y^2\right) \text{ and}$$
$$Y^2Z^2 + Z^2X^2 + X^2Y^2 = \frac{1}{x^2y^2z^2}(x^2 + y^2 + z^2)$$

Hence
$$\frac{Y^2Z^2 + Z^2X^2 + X^2Y^2}{X^2 + Y^2 + Z^2} = \frac{x^2 + y^2 + z^2}{y^2z^2 + z^2x^2 + x^2y^2} = \frac{1}{t^2} = T^2$$

□

The following examples can be shown to be solutions to (4).

- The elementary solution $\mathcal{F}_0 = (a^2 : ab : b^2 : ab)$.
- The first solution of Fauquembergue $\mathcal{F}_1 = (ac : bc : 0 : ab)$ where $a^2 + b^2 = c^2$.

- The second solution of Fauquembergue $\mathcal{F}_2 = (2a^2bc^3 : 2ab^2c^3 : (a^2-b^2)c^4 : 2ab(a^4+b^4))$ where $a^2 + b^2 = c^2$.

An application of the previous lemma to the second solution of Fauquembergue, we deduce a new solution to (4).

Proposition 4. *If* $a^2 + b^2 = c^2$ *and*

$$\begin{aligned} x &= ac(a^2 - b^2)(a^4 + b^4) \\ y &= bc(a^2 - b^2)(a^4 + b^4) \\ z &= 2a^2b^2(a^4 + b^4) \\ t &= ab(a^2 + b^2)^2(a^2 - b^2) \end{aligned}$$

then $(x : y : z : t)$ *is a solution to (4).*

We will label this solution $\mathcal{D}_1$. For example, if $(a : b : c) = (4 : 3 : 5)$ then $(x : y : z : t) = (47180 : 35385 : 97056 : 52500)$ is a solution to (4).

4 An Elliptic Curve over $\mathbb{Q}(m)$

We begin this section by providing some background on elliptic surfaces, which can be defined as a one-parameter algebraic family of elliptic curves. See Silverman [5, Chapter 3].

Let C be a curve defined over a field k. Consider a rational map $C \to \mathbb{P}^1$. The collection of all such maps is denoted by $K = k(C)$. For example, if $C : a^2 + b^2 = c^2$ is defined over $\mathbb{Q}$, we have an isomorphism $C \to \mathbb{P}^1$ given by

$$(a : b : c) \mapsto m = \frac{a}{c - b} \iff (a : b : c) = (2m : m^2 - 1 : m^2 + 1).$$

Hence, $K = \mathbb{Q}(C) = \mathbb{Q}(m)$. This is the field we will consider.

Consider a family of curves

$$E_m : x_2^2 = x_1^3 + A(m)x_1 + B(m)$$

with rational functions $A(m), B(m) \in K$. If we rewrite our equation in homogeneous form, we form the elliptic surface

$$\mathcal{E} = \left\{((x_1 : x_2 : x_3), m) \in \mathbb{P}^2 \times C | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3\right\}.$$

We have a map $\pi : \mathcal{E} \to C$ defined by $((x_1 : x_2 : x_3), m) \mapsto m$. For $m \in \mathbb{Q}$ such that $4A(m)^3 + 27B(m)^2 \neq 0$, the *fiber*

$$\mathcal{E}_m = \pi^{-1}(m) = \left\{(x_1 : x_2 : 1) \in \mathbb{P}^2 | x_2^2x_3 = x_1^3 + A(m)x_1x_3^2 + B(m)x_3^3\right\}$$

is the elliptic curve E_m over $\mathbb{Q}$.

We say that the elliptic surface is *non-split* if the j-invariant

$$j : C \to \mathbb{P}^1 \text{ defined by } j(m) = 1728 \frac{4A(m)^3}{4A(m)^3 + 27B(m)^2}$$

is a non-constant function. A *parametrization of* $\mathcal{E}$ *by* C, or a *section to* π, is a map $\sigma : C \to \mathcal{E}$ such that the composition $\pi \circ \sigma : m \mapsto m$ is the identity map on C. There is always a trivial section on $\mathcal{E}$, namely the map $\sigma_0 : m \mapsto \mathcal{O}_m = ((0:1:0), m)$. In general, the collection $\mathcal{E}(C)$ of all sections is an abelian group, where we define

$$\left.\begin{array}{l} \sigma_1(m) = (P(m), m) \\ \\ \sigma_2(m) = (Q(m), m) \end{array}\right. \qquad \Longrightarrow \qquad (\sigma_1 \oplus \sigma_2)(m) = (P(m) \oplus Q(m), m).$$

We often abuse notation and write $E : x_2^2 = x_1^3 + Ax_1 + B$ as an elliptic curve over the function field $K = k(C)$. In fact, we have an isomorphism between the group of points of E over K and the group of sections of $\mathcal{E}$ over $\mathbb{Q}$.

$$\begin{array}{ccc} E(K) & \xrightarrow{\sim} & \mathcal{E}(C) \\ P(m) = (x_1(m) : x_2(m) : x_3(m)) & \mapsto & [\sigma : m \mapsto ((x_1(m) : x_2(m) : x_3(m)), m)] \end{array}$$

It will be helpful to consider $\mathcal{E}$ as a two-dimensional surface, where each section maps to a one-dimensional curve. The result [5, Theorem 6.1, Chapter 3] asserts that $E(K) \simeq \mathcal{E}(C)$ is a finitely generated abelian group whenever $\mathcal{E}$ is a non-split surface.

Let $m_0 \in \mathbb{Q}$ such that $E_0 = \pi^{-1}(m_0)$ is an elliptic curve over $k = \mathbb{Q}$. Silverman's "specialization theorem" [5, Theorem 11.4, Chapter 3] asserts that the map $\mathcal{E}(C) \to E_0(k)$ which sends a section $\sigma : m \mapsto (P(m), m)$ to the point $P_0 = P(m_0)$ is injective for all but finitely many $m_0 \in k$. In particular, if P_0 is a point of finite (infinite) order in $E_0(k)$ for some $m_0 \in k$, then $P(m)$ must be a point of finite (infinite) order in $E(K)$ as a function of m.

Let us return to the condition

$$(x^2 + y^2 + z^2)(y^2z^2 + z^2x^2 + x^2y^2) = \square.$$

If $a^2 + b^2 = c^2$ nonzero then we can express $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$. For the sake of space, we will leave our work below in terms of a, b, c. If we impose the condition $(x, y) = (a, b)$, we obtain

$$(c^2 + z^2)(c^2z^2 + a^2b^2) = \square.$$

Dividing by c^2, we can consider the following equation,

$$h^2 = z^4 + \frac{a^4 + 3a^2b^2 + b^4}{c^2} z^2 + a^2b^2 \tag{7}$$

From the preceding examples $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions to (8),

$$\begin{aligned}
(z,h)_{\mathcal{F}_0} &= \left(\frac{b^2}{a}, \frac{b(a^4+a^2b^2+b^4)}{a^2c}\right);\\
(z,h)_{\mathcal{F}_1} &= (0, ab);\\
(z,h)_{\mathcal{F}_2} &= \left(\frac{(a^2-b^2)c}{2\ ab}, \frac{(a^4+b^4)c^2}{4\ a^2b^2}\right);\\
(z,h)_{\mathcal{D}_1} &= \left(\frac{2\ a^2b^2}{(a^2-b^2)c}, \frac{(a^4+b^4)ab}{(a^2-b^2)^2}\right).
\end{aligned}$$

Comment 5. If we impose the condition $(x, y) = (a, b) = (2m, m^2 - 1)$, we obtain

$$\left((m^2+1)^2+z^2\right)\left((m^2+1)^2z^2+((2m)(m^2-1))^2\right) = \square.$$

Dividing by $(m^2 + 1)^2$, we can consider the following equation,

$$h^2 = z^4 + \frac{m^8+8m^6-2m^4+8m^2+1}{(m^2+1)^2}z^2 + \left((2m)(m^2-1)\right)^2 \tag{8}$$

From the preceding examples, $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{D}_1$, we know four solutions (z, h) to (8), imposing the condition $a^2 + b^2 = c^2$ and $(x, y) = (a, b)$. In terms of the parameter m we obtain

$$(z,h)_{\mathcal{F}_0} = \left(\frac{(m^2-1)^2}{2m}, \frac{(m^2-1)(m^4-2m^3+2m^2+2m+1)(m^4+2m^3+2m^2-2m+1)}{4m^2(m^2+1)}\right);$$

$$(z,h)_{\mathcal{F}_1} = (0, 4m(m^2-1));$$

$$(z,h)_{\mathcal{F}_2} = \left(\frac{-(m^2+1)(m^2+2m-1)(m^2-2m-1)}{4m(m^2-1)}, \frac{(m^2+1)^2(m^8-4m^6+22m^4-4m^2+1)}{16m^2(m^2-1)^2}\right);$$

$$(z,h)_{\mathcal{D}_1} = \left(\frac{8m^2(m^2-1)^2}{-(m^2+1)(m^2+2m-1)(m^2-2m-1)}, \frac{2m(m^2-1)(m^8-4m^6+22m^4-4m^2+1)}{((m^2+2m-1)(m^2-2m-1))^2}\right).$$

We will show that, in fact, there are infinitely many parametric solutions to equation (4) by showing there are infinitely many parametric solutions to (8).

Theorem 6. *Parametric solutions of the equation*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$$

may be obtained by finding points on an elliptic curve over the field $\mathbb{Q}(m)$.

Proof. By Proposition 1 and assuming $a^2 + b^2 = c^2$ nonzero, consider equation (8). Let $A = \dfrac{a^4 + 3a^2b^2 + b^4}{4(a^2 + b^2)}$ and $B = \dfrac{a^2b^2}{4}$, so that (8) can be expressed as

$$h^2 = z^4 + 4Az^2 + 4B \tag{9}$$

If we have a rational solution to (9) then we get a rational solution to

$$v^2 = u^3 + \alpha u^2 + \beta u \tag{10}$$

where $\alpha = -2A$ and $\beta = A^2 - B$, by

$$u = \frac{1}{2}(z^2 + 2A - h) \quad \text{and} \quad v = \frac{1}{2}z(z^2 + 2A - h).$$

Conversely, assuming $(u, v) \neq (0, 0)$, a rational solution to (10) leads to a rational solution to (9) by

$$z = \frac{v}{u} \quad \text{and} \quad h = \frac{v^2}{u^2} + 2A - 2u.$$

The discriminant of (10) is

$$(\alpha^2 - 4\beta)\beta^2 = 4B(A^2 - B)^2 = \frac{a^2b^2(a^4 + a^2b^2 + b^4)^4}{256(a^2 + b^2)^4}$$

which is nonzero since at least one of a, b are nonzero. Hence (10) defines an elliptic curve over the field $\mathbb{Q}(m)$ and every point (u, v) on this elliptic curve yields a point (z, h) satisfying (8), which implies a solution (x, y, z, t) to (4). □

We will show how to obtain new solutions, by adding points on the elliptic curve. We write E for the elliptic curve (10) over $\mathbb{Q}(m)$, $+$ for the addition of points on the curve (10), and P_S will denote a point on E yielding a solution S to (4). The addition of points on an elliptic curve is described in Silverman [6]. Note that instead of writing $P + P$ we will write $2P$.

Example 7. The solutions $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{D}_1$ to (4) are provided by the following points on (10):

$$P_{\mathcal{F}_0} = \left(\frac{(c-b)\ (a^2 - ab + b^2)\ (a^2 + ab + b^2)}{4\ (c+b)\ (a^2 + b^2)}\ ,\ \frac{b^2\ (c-b)\ (a^2 - ab + b^2)\ (a^2 + ab + b^2)}{4\ a\ (c+b)\ (a^2 + b^2)} \right)$$

$$P_{\mathcal{F}_1} = \left(\frac{(a^2 - ab + b^2)^2}{4\ (a^2 + b^2)}\ ,\ 0 \right)$$

$$P_{\mathcal{F}_2} = \left(\frac{a^2b^2}{4\ (a^2 + b^2)}\ ,\ \frac{ab\ (a^2 - b^2)}{8c} \right)$$

$$P_{\mathcal{D}_1} = \left(\frac{(a-b)^2\ (a^2 - ab + b^2)^2}{4\ (a+b)^2\ (a^2 + b^2)}\ ,\ \frac{a^2b^2\ (a-b)\ (a^2 + ab + b^2)^2}{2c\ (a+b)^3\ (a^2 + b^2)} \right)$$

Let us remind the reader of the Lutz-Nagell theorem, which will be used in the proof of Theorem 8 along with the "specialization theorem".

Theorem 8. *Let E be given by $y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Z}$. Let $P = (x, y) \in E(\mathbb{Q})$. Suppose P has finite order. Then $x, y \in \mathbb{Z}$. If $y \neq 0$ then y^2 divides $4A^3 + 27B^2$.*

Theorem 9. *There exists infinitely many points on (10).*

Proof. Let $N = \dfrac{a^2 - ab + b^2}{2c}$ and $L = \dfrac{a^2 + ab + b^2}{2c}$. Then (10) can be expressed as

$$E: \quad v^2 = u\left(u - N^2\right)\left(u - L^2\right) \tag{11}$$

It can be shown that the rank of this elliptic curve over $\mathbb{Q}(m)$ is at least one. To show the rank is at least one, specialize at say, $(a, b) = (3, 4)$. Then the point $P = (36/25, 21/10)$ is on the curve

$$E_1: \quad v^2 = u^3 - \frac{769}{50}u^2 + \frac{231361}{10000}u.$$

In order to use the Lutz-Nagell theorem, we need to express E_1 in Weierstrass form with integral coefficients:

$$E_1': \quad y^2 = x^3 - 1538x^2 + 231361x.$$

The point P on E_1 corresponds to $P' = (144, 2100)$ on E_1', and thus

$$2P' = (70980625/28224, 389867877575/4741632).$$

Since $2P'$ is not an integral point, P' is not a point of finite order on E_1', so by the "Specialization Theorem" the rank of (11) is of positive rank, these are the same ideas as in Ulas [7]. Also note, calculations found at least 17 distinct points on E. The maximum number of torsion points is 16, so the rank must be at least one. □

Remark 10. On E, the torsion points are $(0, 0)$, $(N^2, 0)$, $(L^2, 0)$, and $\mathcal{O}$, the point at infinity.

Theorem 11. *Every solution to the equation*

$$x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2 \qquad (4)$$

such that $(x, y) = (a, b)$, a and b nonzero, proceeds from exactly two points on E different from $(0, 0)$. If one of them is $P = (u, v)$, with $u \neq 0$, then the other one is $P' = (u', v') = \lambda(u, v)$, with $\lambda = \dfrac{\beta}{u^2}$.

Proof. Although E is defined over a function field, Figure 2 provides some intuition for our proof. Let $P = (u, v)$ be a point on E, with $u \neq 0$, which yields a solution to (4). If there exists a point $P' = (u', v')$ on E different from $(0, 0)$ yielding the same solution to (4) as

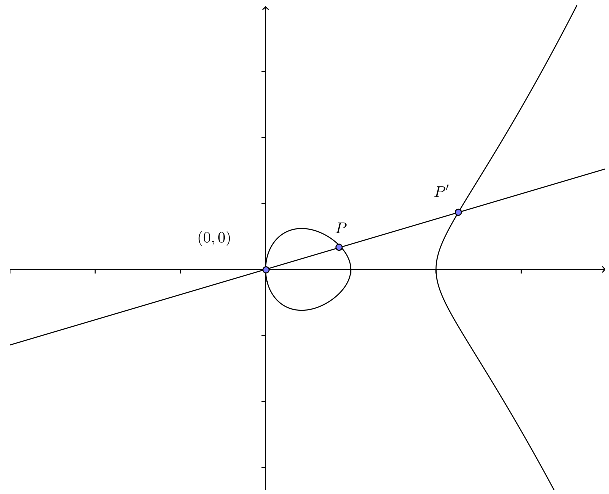

Figure 2: $E : v^2 = u^3 + \alpha u^2 + \beta u$

(u, v), then $z' = z$ so $u' \neq 0$ and $\dfrac{v'}{u'} = \dfrac{v}{u}$. Thus there exists a rational λ such that $u' = \lambda u$ and $v' = \lambda v$ and

$$0 = -v'^2 + u'^3 + \alpha u'^2 + \beta u' = -\lambda^2 v^2 + \lambda^3 u^3 + \alpha \lambda^2 u^2 + \beta \lambda u.$$

By substituting for v^2, we have

$$0 = -\lambda^2 (u^3 + \alpha u^2 + \beta u) + \lambda^3 u^3 + \alpha^2 \lambda^2 u^2 + \beta \lambda u = \lambda\ (\lambda - 1)\ u\ (\lambda u^2 - \beta).$$

Since $u' \neq 0$, then $\lambda \neq 0$. Assuming $(u', v') \neq (u, v)$ implies $\lambda \neq 1$. Thus $\lambda = \dfrac{\beta}{u^2}$.

To show (u', v') is on E, notice

$$u'^3 + \alpha u'^2 + \beta u' = \frac{\beta^3}{u^3} + \alpha\ \frac{\beta^2}{u^2} + \frac{\beta^2}{u} = \frac{\beta^2}{u^4}\left(\beta u + \alpha u^2 + u^3\right) = \frac{\beta^2}{u^4} v^2 = \left(\frac{\beta}{u^2} v\right)^2 = v'^2.$$

Both of these points yield the same solution to (4) since $x = a, y = b$ and $z = z'$. This defines a solution $(x : y : z : t)$, except possibly for the sign of t. □

Remark 12. If $P_{\mathcal{F}_2} = (u, v)$, we find that $2P_{\mathcal{F}_0} = (u', v')$, both yielding the same solution $\mathcal{F}_2$.

From the previous two theorems we deduce the following corollary.

Corollary 13. *There exists infinitely many parametric solutions to* $x^4 + y^4 + z^4 + t^4 = (x^2 + y^2 + z^2 - t^2)^2$.

5 Obtaining new parametric solutions

Before obtaining new parametric solutions, we can interpret Lemma 3 in terms of points in $E(K)$ where $K = \mathbb{Q}(m)$ and E as described earlier.

Proposition 14. *Let* $(u, v) \in E(K)$ *such that* $(u, v) \notin \{\mathcal{O}, (0, 0), (N^2, 0), (L^2, 0)\}$, *and let* $(u', v'), (u'', v'') \in E(K)$ *such that:*

$$(u, v) + (u', v') = (N^2, 0) \text{ and } (u, v) + (u'', v'') = (L^2, 0).$$

If (u, v) *yields a solution* $(x : y : z : t)$ *to (4) such that* $xyzt \neq 0$, *then* (u', v') *and* (u'', v'') *both yield* $\left(\frac{1}{y} : \frac{1}{x} : \frac{1}{z} : \frac{1}{t}\right)$, *except perhaps the signs of* $\frac{1}{z}$ *and* $\frac{1}{t}$.

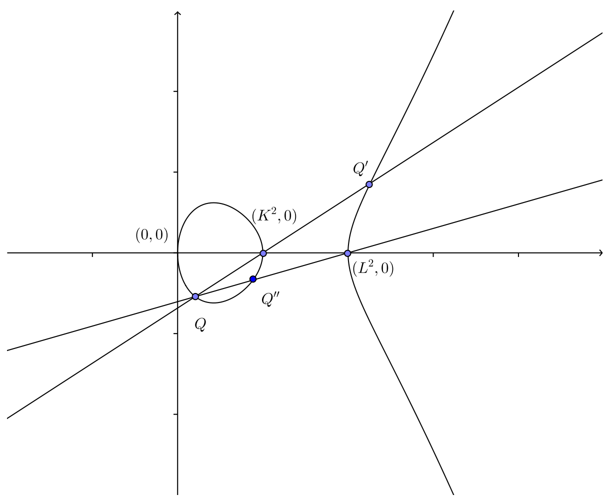

Figure 3: $E : v^2 = u^3 + \alpha u^2 + \beta u$

Proof. From the group law on equation (10), we deduce

$$(u', v') = \left(\frac{N^2(u - L^2)}{u - N^2}, \frac{N^2(N^2 - L^2)v}{(u - N^2)^2}\right), \quad (u'', v'') = \left(\frac{L^2(u - N^2)}{u - L^2}, \frac{L^2(L^2 - N^2)v}{(u - L^2)^2}\right).$$

If (u', v') yields $(x' : y' : z' : t')$ then $z' = \frac{v'}{u'} = \frac{(N^2 - L^2)v}{(u - N^2)(u - L^2)}$. From the relations $N^2 - L^2 = -ab$ and $v^2 = u(u - N^2)(u - L^2)$, we conclude

$$z' = -ab\frac{u}{v} = -\frac{ab}{z}.$$

Thus if (u, v) yields $(x : y : z : t) = (a : b : z : t)$, then (u', v') yields

$$(x' : y' : z' : t') = \left(a : b : \frac{ab}{z} : \frac{ab}{t}\right) = \left(\frac{1}{b} : \frac{1}{a} : \frac{1}{z} : \frac{1}{t}\right) = \left(\frac{1}{y} : \frac{1}{x} : -\frac{1}{z} : \frac{1}{t}\right),$$

where the signs of z' and t' may be positive or negative. The proof for z'' is similar. □

Next let $P_{\mathcal{D}_2} = P_{\mathcal{F}_0} + P_{\mathcal{D}_1}$. If $a^2 + b^2 = c^2$, we find $P_{\mathcal{D}_2} = (u, v)$ with

$$u = \frac{(c+a)(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)^2}{4(c-a)(a^2+b^2)(2a^3 + a^2c - b^2c)^2}$$
$$v = \frac{a^2b(a^4 + a^2b^2 + b^4)(2a^3 - a^2c + b^2c)(2b^3 - a^2c + b^2c)(2b^3 + a^2c - b^2c)}{4(c-a)^2(a^2+b^2)(2a^3 + a^2c - b^2c)^3}$$

By the same methods used in the proof of Theorem 6, we deduce the following solution $\mathcal{D}_2$:

Proposition 15. *If $a^2 + b^2 = c^2$ and if*

$$\begin{aligned}
x &= ab(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)\\
y &= b^2(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)\\
z &= a^2(2\ b^3 - a^2c + b^2c)(2\ b^3 + a^2c - b^2c)(2\ ab^2 + a^2c + b^2c)(-2\ ab^2 + a^2c + b^2c)\\
t &= ab(2\ a^3 - a^2c + b^2c)(2\ a^3 + a^2c - b^2c)(2\ a^2b + a^2c + b^2c)(-2\ a^2b + a^2c + b^2c)
\end{aligned}$$

then $\mathcal{D}_2 = (x : y : z : t)$ is a solution to (4).

Example 16. Since $(a : b : c) = (2m : m^2 - 1 : m^2 + 1)$, if $m = 2$, then

$$\begin{aligned}
x &= 1\ 899\ 301\ 428\\
y &= 1\ 424\ 476\ 071\\
z &= 282\ 491\ 696\\
t &= 1\ 165\ 848\ 372
\end{aligned}$$

Next let $P_{\mathcal{D}_3} = 2P_{\mathcal{D}_1}$. We find that $P_{\mathcal{D}_3} = (u, v)$ with:

$$u = \frac{(a^2+b^2)(a^4+b^4)^2}{16a^2b^2(a-b)^2(a+b)^2}, \quad v = \frac{PQRS(a^4+b^4)}{64a^3b^3c(a-b)^3(a+b)^3}$$

where

$$\begin{aligned}
P &= -a^3 + a^2b + ab^2 + b^3, & Q &= a^3 - a^2b + ab^2 + b^3\\
R &= a^3 + a^2b - ab^2 + b^3, & S &= a^3 + a^2b + ab^2 - b^3.
\end{aligned}$$

From this we deduce $\mathcal{D}_3$:

Proposition 17. *If* $a^2+b^2=c^2$ *and if*

$$\begin{aligned} x &= a^2bc(a^2-b^2)(a^2+b^2)(a^4+b^4)G \\ y &= ab^2c(a^2-b^2)(a^2+b^2)(a^4+b^4)G \\ z &= PQRSG \\ t &= 4ab(a^2-b^2)(a^2+b^2)^2(a^4+b^4)\ H \end{aligned}$$

with P, Q, R, S *as above, and*

$$\begin{aligned} G &= a^{12}+6a^{10}b^2-a^8b^4+4a^6b^6-a^4b^8+6a^2b^{10}+b^{12} \\ H &= a^{12}-2a^{10}b^2+7a^8b^4+4a^6b^6+7a^4b^8-2a^2b^{10}+b^{12}, \end{aligned}$$

then $\mathcal{D}_2=(x:y:z:t)$ *is a solution to (4).*

Now put $P_{\mathcal{D}_4}=P_{\mathcal{D}_3}+P_{\mathcal{F}_1}$. We find $P_{\mathcal{D}_4}=(u,v)$, with

$$u=\frac{(a^2-ab+b^2)^2P^2S^2}{4(a^2+b^2)^2Q^2R^2}, \quad v=\frac{a^2b^2c(a^2-b^2)(a^2-ab+b^2)^2(a^4+b^4)PS}{Q^3R^3}$$

From this we deduce the following solution $\mathcal{D}_4$:

Proposition 18. *If* $a^2+b^2=c^2$ *and if*

$$\begin{aligned} x &= acPQRSH \\ y &= bcPQRSH \\ z &= 4a^2b^2(a^2-b^2)(a^2+b^2)^2(a^4+b^4)H \\ t &= abPQRSG \end{aligned}$$

with the same P, Q, R, S, G, H *as in Proposition 17, then* $\mathcal{D}_4=(x:y:z:t)$ *is a solution to (4).*

The degree of this solution is 26 in $(a:b:c)$, and hence 52 in the homogeneous coordinates $(m:n)$ if we express $(a:b:c)=(2mn:m^2-n^2:m^2+n^2)$ for nonzero integers m,n.

Remark 19. The values z for $P_{\mathcal{D}_3}$ and z' for $P_{\mathcal{D}_4}$ satisfy $zz'=ab$, so except perhaps exchanging (x',y') and (y',x'), $\mathcal{D}_4$ is deduced from $\mathcal{D}_3$ by replacing (x,y,z) by $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$.

In summary, parametric solutions $(x:y:z:t)$ to (4) with their degree are shown in Table 1.

solution	$\mathcal{F}_0$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$
degree	4	4	12	16	28	48	52

Table 1: Degree of parametric solutions

6 Acknowledgment

The first author is thankful to Professor Edray Goins who helped visual this interesting problem. Both authors are thankful to Professor Andrew Bremner for introducing us. We are also grateful to the unknown referee for their careful reading.

References

[1] Noam D. Elkies, On $A^4 + B^4 + C^4 = D^4$, *Math. Comp.* **51** (1988), 825–835.

[2] Elie Fauquembergue, Question 266, *L'Intermédiaire des Mathématiciens* **1** (1894) 148.

[3] Lee W. Jacobi and Daniel J. Madden, , On $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$, *Amer. Math. Monthly* **115** (2008), 220–236.

[4] Titus Piezas III, *A Collection of Algebraic Identities*, https://sites.google.com/site/tpiezas/Home.

[5] Joseph H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer, 1994.

[6] Joseph H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

[7] Maciej Ulas, Rational points in arithmetic progressions on $y^2 = x^n + k$, *Canad. Math. Bull.* **55** (2012) 193–207.

2010 *Mathematics Subject Classification*: Primary 11D25; Secondary 11G05.
Keywords: Diophantine equation, elliptic curve, elliptic surface.

Received August 11 2014; revised version received October 5 2014. Published in *Journal of Integer Sequences*, November 7 2014.

Return to Journal of Integer Sequences home page.