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GCD Property of the Generalized Star of David in the Generalized Hosoya Triangle

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Abstract

The *generalized Hosoya triangle* is an arrangement of numbers in which each entry is a product of two generalized Fibonacci numbers. We prove the GCD property for the star of David of length two. We give necessary and sufficient conditions such that the star of David of length three satisfies the GCD property. We propose some open questions and a conjecture for the star of David of length bigger than or equal to four. We also study GCD properties and modularity properties of generalized Fibonacci numbers.

1 Introduction

The nth *generalized Fibonacci number* is a linear combination of the elements F_{n-2} and F_{n-1} with integer coefficients. That is, if we take two integers a and b, then the first and second generalized Fibonacci numbers are a and b, respectively, and the nth generalized Fibonacci number has the form $aF_{n-2} + bF_{n-1}$, where $n > 2$. If $a = b = 1$, then we obtain the nth regular Fibonacci number.

The *generalized Hosoya triangle* is a triangular arrangement of numbers that resembles the Pascal triangle. Each entry is the product of two generalized Fibonacci numbers instead of a binomial coefficient. In particular, the regular Hosoya triangle [5, 7] consists of a triangular array of numbers where each entry is the product of two Fibonacci numbers. Hosoya [5] published this triangle in 1976.

Several authors have studied greatest common divisor (GCD) properties of some geometric configurations in the Pascal triangle and the regular Hosoya triangle (see, for example, [1, 2, 3, 4, 6, 8]). We study GCD properties of a special configuration of points in the generalized Hosoya triangle called the *generalized star of David (GSD)*. We form a GSD by the vertices of two triangles from a regular hexagon in a generalized Hosoya triangle. We say that a star of David has length l if a side of the regular hexagon has length l.

Many authors have studied properties of the star of David of length two and/or its generalizations (see, for example [1, 2, 3, 4, 6, 8]). With the exception of the work done by Flórez and Junes [1] all these papers consider the star of David in the Pascal triangle.

Hoggatt and Hansell [4] proved that the product of all points in a star of David of length two in the Pascal triangle forms a perfect square. Hillman and Hogart [3] proved that the GCD of each triangle of the star of David of length two in the Pascal triangle gives the same number. These two properties are called the *product* and *GCD properties*, respectively, of the star of David. Flórez and Junes [1] proved that both properties are true for any star of David of length two in the regular Hosoya triangle.

We prove the GCD property of the star of David of length two in the generalized Hosoya triangle. We give necessary and sufficient conditions such that the star of David of length three in the generalized Hosoya triangle satisfies the GCD property. These conditions depend on L_2–divisibility, where L_2 is the second Lucas number. To prove these results, we introduce new properties of generalized Fibonacci numbers. In particular, we provide an analog to the identity $\gcd(F_n, F_m) = F_{\gcd(n,m)}$ for generalized Fibonacci numbers and show that if a generalized Fibonacci number G_n is divisible by F_w, then the GCD of G_n and G_{n+kw} is F_w for any non–zero integer k.

We propose a conjecture that gives necessary and sufficient conditions to determine whether the star of David of length $l \geq 4$ has the GCD property. These conditions depend on L_{l-1}–divisibility, where L_{l-1} is the $(l-1)$th Lucas number.

2 Preliminaries and Examples

In this section we give some examples and introduce notation and definitions that we are going to use throughout the paper. Some of them are well known, but we prefer to restate them here to avoid ambiguities.

2.1 The Generalized Hosoya Triangle

We denote by $\{G_n(a,b)\}_{n\in\mathbb{N}}$ the *generalized Fibonacci sequence* with integers a and b. That is,

$$G_1(a,b) = a, G_2(a,b) = b \text{ and } G_n(a,b) = G_{n-1}(a,b) + G_{n-2}(a,b) \text{ for all } n \in \mathbb{N} \setminus \{1,2\}.$$

It is clear that $G_n(1,1) = F_n$. When there is no ambiguity with a and b, we denote the nth term of the generalized Fibonacci sequence by G_n instead of $G_n(a,b)$. The first eight terms of the generalized Fibonacci sequence with integers a and b are

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, \text{ and } 8a+13b.$$

Notice that every element in this sequence is a linear combination of the integers a and b with Fibonacci coefficients. In general, we have that $G_n = a\,F_{n-2} + b\,F_{n-1}$ for all $n \in \mathbb{N}$ (see, for example, [7, Thm. 7.1, p. 109].)

The *generalized Hosoya sequence* $\{H_{a,b}(r,k)\}_{r\geq k\geq 1}$ is defined by the double recursion

$$H_{a,b}(r,k) = H_{a,b}(r-1,k) + H_{a,b}(r-2,k)$$

and

$$H_{a,b}(r,k) = H_{a,b}(r-1,k-1) + H_{a,b}(r-2,k-2)$$

where $r > 2$ and $1 \leq k \leq r$, with initial conditions

$$H_{a,b}(1,1) = a^2; \quad H_{a,b}(2,1) = ab; \quad H_{a,b}(2,2) = ab; \quad H_{a,b}(3,2) = b^2.$$

It is easy to see that if we let $a = b = 1$ in the generalized Hosoya sequence, then we obtain the regular Hosoya sequence $\{H(r,k)\}_{r\geq k\geq 1}$ as Koshy defines it [7, pp. 187–188]. It is known that

$$H(r,k) = F_k\,F_{r-k+1}$$

for all natural numbers r, k such that $k \leq r$, see [7, p. 188]. This and Proposition 1 show that our definition of $\{H_{a,b}(r,k)\}_{r\geq k\geq 1}$ is the "right" generalization for $\{H(r,k)\}_{r\geq k\geq 1}$.

Proposition 1. *If r and k are natural numbers such that $k \leq r$, then*

$$H_{a,b}(r,k) = G_k\, G_{r-k+1},$$

for all integers $a, b \in \mathbb{Z}$.

Proof. We prove first that $H_{a,b}(r,1) = a\, G_r$ for all $r \in \mathbb{N}$.

Since $H_{a,b}(r,1) = H_{a,b}(r-1,1) + H_{a,b}(r-2,1)$ for all $r \geq 3$, where $H_{a,b}(1,1) = a^2$ and $H_{a,b}(2,1) = ab$, a straightforward argument by strong induction on r shows that

$$H_{a,b}(r,1) = a\, G_r \text{ for all } r \in \mathbb{N}. \tag{a}$$

Using a similar argument, we can prove that

$$H_{a,b}(r,2) = b\, G_{r-1} \text{ for all } r \in \mathbb{N} \setminus \{1\}. \tag{b}$$

Now, for any $k \in \mathbb{N}$ let $P(k)$ be the statement: $H_{a,b}(r,k) = G_k\, G_{r-k+1}$ for all natural numbers r such that $r \geq k$. We want to prove by strong induction that $P(k)$ is true for all $k \in \mathbb{N}$.

Statements (a) and (b) show that $P(1)$ and $P(2)$ are true. This proves the basis step. We assume now that $P(1), \ldots, P(k)$ are true for a fixed natural number $k \geq 2$ and prove that $P(k+1)$ is true. Let r be a natural number such that $r \geq k+1$. Therefore, $r-1 \geq k$ and $r-2 \geq k-1$. Since $P(k-1)$ and $P(k)$ are true, we can write

$$H_{a,b}(r-2,k-1) = G_{k-1}\, G_{r-k} \quad \text{and} \quad H_{a,b}(r-1,k) = G_k\, G_{r-k}.$$

This and the recurrence $H_{a,b}(r,k+1) = H_{a,b}(r-1,k) + H_{a,b}(r-2,k-1)$ show that

$$H_{a,b}(r,k+1) = G_{k+1}\, G_{r-k}.$$

Thus, $P(k+1)$ is true. This proves the proposition. □

If there is no ambiguity with the integers a and b we write $H(r,k)$ instead of $H_{a,b}(r,k)$. The generalized Hosoya sequence gives rise to the *generalized Hosoya triangle* where the entry in position k, taken from left to right, of the rth row is equal to $H(r,k)$ (see Table 1).

$$
\begin{array}{ccccccccccc}
 & & & & & H(1,1) & & & & & \\
 & & & & H(2,1) & & H(2,2) & & & & \\
 & & & H(3,1) & & H(3,2) & & H(3,3) & & & \\
 & & H(4,1) & & H(4,2) & & H(4,3) & & H(4,4) & & \\
 & H(5,1) & & H(5,2) & & H(5,3) & & H(5,4) & & H(5,5) & \\
H(6,1) & & H(6,2) & & H(6,3) & & H(6,4) & & H(6,5) & & H(6,6)
\end{array}
$$

Table 1: Generalized Hosoya Triangle.

If P is a point in a generalized Hosoya triangle, then it is clear that there are two unique positive integers r and k such that $r \geq k$ with $P = H(r, k)$. We call the pair (r, k) the *rectangular coordinates* of the point P.

We now give a more convenient system of coordinates for points in the generalized Hosoya triangle. Proposition 1 shows that every entry of the generalized Hosoya triangle is the product of two generalized Fibonacci numbers. In particular, if we use Proposition 1 for all entries of Table 1, we obtain Table 2.

					$G_1\,G_1$					
				$G_1\,G_2$		$G_2\,G_1$				
			$G_1\,G_3$		$G_2\,G_2$		$G_3\,G_1$			
		$G_1\,G_4$		$G_2\,G_3$		$G_3\,G_2$		$G_4\,G_1$		
	$G_1\,G_5$		$G_2\,G_4$		$G_3\,G_3$		$G_4\,G_2$		$G_5\,G_1$	
$G_1\,G_6$		$G_2\,G_5$		$G_3\,G_4$		$G_4\,G_3$		$G_5\,G_2$		$G_6\,G_1$

Table 2: Generalized Hosoya Triangle.

Notice that any diagonal of Table 2 is the collection of all generalized Fibonacci numbers multiplied by a particular G_n. More precisely, an nth *diagonal* in the generalized Hosoya triangle is the collection of all generalized Fibonacci numbers multiplied by G_n. We distinguish between *slash diagonals* and *backslash diagonals*, with the obvious meaning. We write $\mathrm{S}(G_n)$ and $\mathrm{B}(G_m)$ to mean the slash diagonal and backslash diagonal, respectively (see Figure 1). We formally define these two diagonals as

$$\mathrm{S}(G_n) = \{H(n+i-1, n)\}_{i=1}^{\infty} = \{G_n\, G_i | i \in \mathbb{N}\},$$

and

$$\mathrm{B}(G_m) = \{H(m+i-1, i)\}_{i=1}^{\infty} = \{G_i\, G_m | i \in \mathbb{N}\}.$$

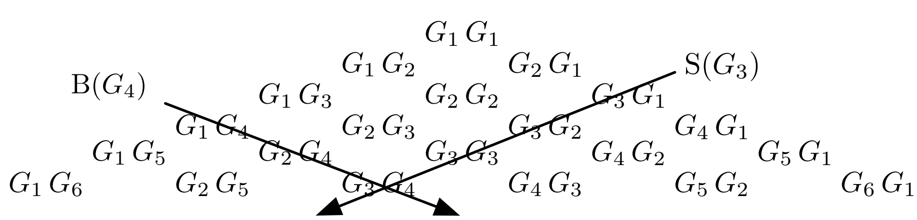

Figure 1: The slash diagonal $\mathrm{S}(G_3)$ and backslash diagonal $\mathrm{B}(G_4)$.

We can associate an ordered pair of natural numbers to every element of a generalized Hosoya triangle. If P is a point in a generalized Hosoya triangle, then there are two generalized Fibonacci numbers G_m and G_n such that $P \in \mathrm{B}(G_m) \cap \mathrm{S}(G_n)$ (Figure 1 depicts this fact for $m = 4$ and $n = 3$.) Thus, $P = G_m\, G_n$. Therefore, the point P corresponds to the pair (m, n). It is clear that this correspondence is a bijection between points of a generalized Hosoya triangle and ordered pairs of natural numbers. The pair (m, n) is called the *diagonal coordinate* of P. Thus, we can view Proposition 1 as a way to change from rectangular coordinates to diagonal coordinates and vice–versa.

Throughout this paper we use only diagonal coordinates to refer to any point in a generalized Hosoya triangle.

We now give some examples of generalized Hosoya triangles. We can construct different triangles by fixing values for the integers a and b. For the first example we fix $a = b = 1$ and obtain the following numerical sequence.

$$G_1 = 1, \quad G_2 = 1, \quad G_3 = 2, \quad G_4 = 3, \quad G_5 = 5, \quad G_6 = 8, \ldots$$

Substituting these values in Table 2 we obtain the regular Hosoya triangle (see [5, 7] and Table 3).

					1					
				1		1				
			2		1		2			
		3		2		2		3		
	5		3		4		3		5	
8		5		6		6		5		8

Table 3: Regular Hosoya Triangle.

We now fix $a = 7$ and $b = 2$ and obtain the following numerical sequence.

$$G_1 = 7, \quad G_2 = 2, \quad G_3 = 9, \quad G_4 = 11, \quad G_5 = 20, \quad G_6 = 31, \ldots$$

Substituting these values in Table 2 we obtain a Hosoya triangle with $a = 7$ and $b = 2$ (see Table 4 part (a)). Similarly, if we fix $a = 5$ and $b = 8$ we obtain another numerical sequence.

$$G_1 = 5, \quad G_2 = 8, \quad G_3 = 13, \quad G_4 = 21, \quad G_5 = 34, \quad G_6 = 55, \ldots$$

The Hosoya triangle generated by this new sequence is depicted in Table 4 part (b).

					49					
				14		14				
			63		4		63			
		77		18		18		77		
	140		22		81		22		140	
217		40		99		99		40		217

(a)

					25					
				40		40				
			65		64		65			
		105		104		104		105		
	170		168		169		168		170	
275		272		273		273		272		275

(b)

Table 4: Generalized Hosoya triangles.

2.2 Generalized Star of David

Let E be a regular hexagon formed with points lying in a generalized Hosoya triangle. We say that E has length l if a side of the hexagon contains l points from the generalized Hosoya triangle. We denote the corner points of E by a_1, a_2, a_3 and b_1, b_2, b_3. The *star of David* of length l is a configuration formed by the six corner points of E. That is, the star of David is a configuration of six points in the Hosoya triangle formed by two triangles with vertices a_1, a_2, a_3 and b_1, b_2, b_3 of E. Figure 2 part (a) represents a star of David of length two. The continuous lines in Figure 2 parts (b) and (c) show stars of David of length three and four, respectively.

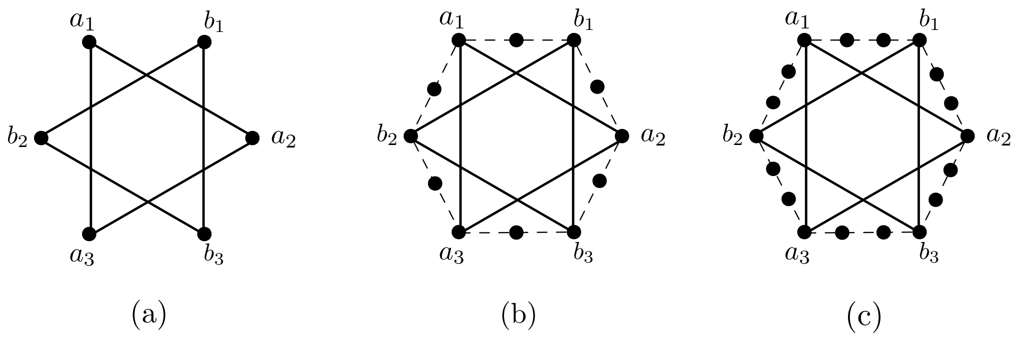

Figure 2: Stars of David.

We first consider some examples of the generalized star of David of length two as in Figure 2 part (a). We can obtain a complete characterization of its vertices a_1, a_2, a_3 and b_1, b_2, b_3, by knowing the location of one. For instance, if (m, n) are the diagonal coordinates of a_2, then we have

$$\begin{array}{llll} a_1 = G_{m+1}\, G_{n-2}, & a_2 = G_m\, G_n & \text{and} & a_3 = G_{m+2}\, G_{n-1}, \\ b_1 = G_m\, G_{n-1}, & b_2 = G_{m+2}\, G_{n-2} & \text{and} & b_3 = G_{m+1}\, G_n. \end{array}$$

We need to give values to a, b, m and n to consider some particular examples. For instance, if we take $a = b = 1$ (the regular Hosoya triangle) and $m = 5$, $n = 4$, then we obtain

$$a_1 = 8, \quad a_2 = 15, \quad a_3 = 26, \quad b_1 = 10, \quad b_2 = 13, \quad b_3 = 24.$$

Similarly, if we take $a = 7$, $b = 2$ (generalized Hosoya triangle in Table 4 part (a)), $m = 3$ and $n = 3$, then we obtain

$$a_1 = 77, \quad a_2 = 81, \quad a_3 = 40, \quad b_1 = 18, \quad b_2 = 140, \quad b_3 = 99.$$

If we now take $a = 5$, $b = 8$ (generalized Hosoya triangle in Table 4 part (b)), $m = 2$ and $n = 4$, then we have

$$a_1 = 104, \quad a_2 = 168, \quad a_3 = 273, \quad b_1 = 104, \quad b_2 = 168, \quad b_3 = 273.$$

Notice that in all three cases $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = 1$. Flórez and Junes proved in [1, Thm. 3.2] that this is always true for any star of David of length two in the regular Hosoya triangle. We prove a more general result. Namely, we show that $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = (\gcd(a, b))^2$ for any star of David of length two in any generalized Hosoya triangle (see Theorem 9).

We now consider some examples of the generalized star of David of length three as in Figure 2 part (b). We can obtain a complete characterization of its vertices a_1, a_2, a_3 and b_1, b_2, b_3 by knowing the location of one. For instance, if (m, n) are the diagonal coordinates of a_2, then we have

$$\begin{array}{llll} a_1 = G_{m+2}\, G_{n-4}, & a_2 = G_m\, G_n & \text{and} & a_3 = G_{m+4}\, G_{n-2}, \\ b_1 = G_m\, G_{n-2}, & b_2 = G_{m+4}\, G_{n-4} & \text{and} & b_3 = G_{m+2}\, G_n. \end{array}$$

We also need to give values to a, b, m and n to consider some particular examples. If we take $a = b = 1$ (the regular Hosoya triangle) and $m = 3$, $n = 5$, then we get

$$a_1 = 5, \quad a_2 = 10, \quad a_3 = 26, \quad b_1 = 4, \quad b_2 = 13, \quad b_3 = 25.$$

Therefore, $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = 1$. However, if we take $m = 8$ and $n = 16$, then

$$a_1 = 7920, \quad a_2 = 20727, \quad a_3 = 54288, \quad b_1 = 7917, \quad b_2 = 20736, \quad b_3 = 54285,$$

$\gcd(a_1, a_2, a_3) = 9$ and $\gcd(b_1, b_2, b_3) = 3$. In this case $\gcd(a_1, a_2, a_3) \neq \gcd(b_1, b_2, b_3)$. We give in this paper a complete characterization of all stars of David of length three in the generalized Hosoya triangle for which $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3)$ (see Theorem 12).

We can construct similar examples for the star of David of length four. In general, if (m, n) are the diagonal coordinates of a_2, then the vertices of the star of David of length l in the generalized Hosoya triangle are given by

$$\begin{array}{llll} a_1 = G_{m+(l-1)}\, G_{n-2(l-1)}, & a_2 = G_m\, G_n & \text{and} & a_3 = G_{m+2(l-1)}\, G_{n-(l-1)}, \\ b_1 = G_m\, G_{n-(l-1)}, & b_2 = G_{m+2(l-1)}\, G_{n-2(l-1)} & \text{and} & b_3 = G_{m+(l-1)}\, G_n, \end{array}$$

for positive integers m and n such that $n > 2(l - 1)$. We require that $n > 2(l - 1)$ so that a_1, a_3, b_1 and b_2 correspond to "real" points in the generalized Hosoya triangle.

We finish this subsection with a definition. We say that a star of David of length l has the *GCD property* if $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3)$.

2.3 Auxiliary Results

The *multiplication property of the GCD* says that if a and b are co–prime, then $\gcd(ab, c) = \gcd(a, c)\gcd(b, c)$. We use this property to prove Lemma 8, which plays a special role in the proof of the main results, Theorems 9 and 12. In addition, we also use the following proposition.

Proposition 2 ([1, Proposition 2.2])**.** *If* $\gcd(a, c) = \gcd(b, d) = 1$*, then* $\gcd(ab, cd) = \gcd(a, d)\gcd(b, c)$*, where* $a, b, c, d \in \mathbb{Z}$*.*

3 GCD Properties of Generalized Fibonacci Numbers

In this section we study GCD properties of generalized Fibonacci numbers as well as some of its modularity properties. In particular, we provide an analog to the identity $\gcd(F_n, F_m) = F_{\gcd(n,m)}$ for generalized Fibonacci numbers. That is, $\gcd(G_n, G_m)$ is always a divisor of $\gcd(a, b)\ F_{|m-n|}$ with equality when $|m - n| \in \{1, 2\}$. We also prove that if a generalized Fibonacci number G_n is divisible by F_w, then the GCD of G_n and G_{n+kw} is F_w for any non–zero integer k.

We use these results to prove the GCD property for the star of David of length two and three in the generalized Hosoya triangle. We write $a \mid b$ to mean that a divides b.

Lemma 3. *If* $\gcd(a, b) = d$, $a' = a/d$ *and* $b' = b/d$, *then* $G_n(a, b) = dG_n(a', b')$ *for* $n \in \mathbb{N}$.

Proof. We know that $G_n(a, b) = aF_{n-2} + bF_{n-1}$ for all $n \in \mathbb{N}$. Thus,

$$G_n(a, b) = da'F_{n-2} + db'F_{n-1} = d(a'F_{n-2} + b'F_{n-1}) = dG_n(a', b').$$

This proves the Lemma. □

Recall that we use G_m for $G_m(a, b)$ for any integer m if there is no ambiguity. Similarly, we use G'_m for $G_m(a', b')$ if $d = \gcd(a, b)$, $a' = a/d$ and $b' = b/d$.

Lemma 4. *If* a, b, n *and* w *are integers, then*

$$G_{n+w}(a, b) = (aF_{w-1} + bF_w)F_{n-2} + (aF_w + bF_{w+1})F_{n-1} = G_n\left(G_{w+1}, G_{w+2}\right).$$

Proof. It is well known that $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$, see [7, Ex. 47, p. 89]. Thus,

$$\begin{aligned} G_{n+w}(a, b) &= aF_{w+n-2} + bF_{n-1+w} \\ &= a(F_{w-1}F_{n-2} + F_wF_{n-1}) + b(F_{n-2}F_w + F_{n-1}F_{w+1}) \\ &= (aF_{w-1} + bF_w)F_{n-2} + (aF_w + bF_{w+1})F_{n-1} \\ &= G_{w+1}F_{n-2} + G_{w+2}F_{n-1} \\ &= G_n\left(G_{w+1}, G_{w+2}\right). \end{aligned}$$

This proves the lemma. □

Theorem 5. *Let* $d = \gcd(a, b)$. *If* n *and* w *are natural numbers, then* $\gcd(G_n, G_{n+w}) \mid d\, F_w$. *Moreover, if* $w = 1, 2$, *then* $\gcd(G_n, G_{n+w}) = d$.

Proof. The Lemma 3 implies that $\gcd(G_n, G_{n+w}) = d \gcd(G'_n, G'_{n+w})$. Thus, it is enough to prove the theorem for $\gcd(G'_n, G'_{n+w})$.

We first prove that $\gcd(G'_n, G'_{n+1}) = 1$. We use induction on n. If $n = 1$, then

$$\gcd(G'_1, G'_2) = \gcd(a', b') = 1.$$

This proves the basis step. We now suppose that $\gcd(G'_k, G'_{k+1}) = 1$ for a fixed natural number k and prove $\gcd(G'_{k+1}, G'_{k+2}) = 1$.

Let r be the natural number such that $\gcd(G'_{k+1}, G'_{k+2}) = r$. Thus,

$$r \mid G'_{k+1} \quad \text{and} \quad r \mid G'_{k+2}. \tag{1}$$

Therefore, $r \mid (G'_{k+2} - G'_{k+1})$. Thus, $r \mid G'_k$. This and (1) imply that $r \mid \gcd(G'_k, G'_{k+1})$. Therefore, the inductive hypothesis implies that $r = 1$. This proves that

$$\gcd(G'_n, G'_{n+1}) = 1 \quad \text{for every } n \in \mathbb{N}. \tag{2}$$

We now prove that $\gcd(G'_n, G'_{n+w})$ divides F_w. Let d' be a divisor of $\gcd(G'_n, G'_{n+w})$. Therefore,

$$d' \mid G'_n \quad \text{and} \quad d' \mid G'_{n+w}. \tag{3}$$

So, d' divides any linear combination of G'_n and G'_{n+w}. In particular,

$$d' \mid (G'_{n+w} - F_{w-1}G'_n). \tag{4}$$

We prove that $d' \mid F_w G'_{n+1}$. Since $G'_k = G_k(a', b')$ for any positive integer k, from Lemma 4 we have

$$G'_{n+w} = (a'F_{w-1} + b'F_w)F_{n-2} + (a'F_w + b'F_{w+1})F_{n-1}.$$

Therefore,

$$G'_{n+w} - F_{w-1}G'_n = [(a'F_{w-1} + b'F_w)F_{n-2} + (a'F_w + b'F_{w+1})F_{n-1}] - F_{w-1}(a'F_{n-2} + b'F_{n-1}).$$

That is,

$$\begin{aligned}
G'_{n+w} - F_{w-1}G'_n &= b'F_wF_{n-2} + (a'F_w + b'(F_{w+1} - F_{w-1}))F_{n-1} \\
&= b'F_wF_{n-2} + (a'F_w + b'F_w)F_{n-1} \\
&= F_w(a'F_{n-1} + b'(F_{n-2} + F_{n-1})) \\
&= F_w(a'F_{n-1} + b'F_n) \\
&= F_wG'_{n+1}.
\end{aligned}$$

This and (4) imply that $d' \mid F_wG'_{n+1}$. From (2) and (3) we have that $\gcd(d', G'_{n+1}) = 1$. This and $d' \mid F_wG'_{n+1}$ imply that $d' \mid F_w$. We have shown that any divisor of $\gcd(G'_n, G'_{n+w})$ is a divisor of F_w. This proves that $\gcd(G'_n, G'_{n+w})$ divides F_w.

We now prove the second claim. Equality (2) proves the second claim for $w = 1$. If $w = 2$, then $\gcd(G'_n, G'_{n+2})$ divides F_2. Therefore, $\gcd(G'_n, G'_{n+2}) = 1 = \gcd(a', b')$. This proves the theorem. □

Corollary 6. *Let m, n, s and t be positive integers. If $|m - n| \in \{1, 2\}$ and $|s - t| \in \{1, 2\}$, then*

$$\gcd(G_m\, G_s, G_n\, G_t) = \gcd(G_m, G_t)\, \gcd(G_s, G_n).$$

Proof. Let $d = \gcd(a, b)$, $a' = a/d$ and $b' = b/d$. Since $|m - n| \in \{1, 2\}$ and $|s - t| \in \{1, 2\}$, Theorem 5 shows that $\gcd(G'_m, G'_n) = 1$ and $\gcd(G'_s, G'_t) = 1$. This and Proposition 2 imply that

$$\gcd(G'_m G'_s, G'_n G'_t) = \gcd(G'_m, G'_t) \gcd(G'_s, G'_n).$$

Multiplying both sides of this equality by d^2 we obtain

$$d^2 \gcd(G'_m G'_s, G'_n G'_t) = d \gcd(G'_m, G'_t) d \gcd(G'_s, G'_n). \tag{5}$$

Using Lemma 3 one can easily see that

$$\gcd(G_m, G_t) = d \gcd(G'_m, G'_t),$$

$$\gcd(G_s, G_n) = d \gcd(G'_s, G'_n), \text{ and}$$

$$\gcd(G_m G_s, G_n G_t) = d^2 \gcd(G'_m G'_s, G'_n G'_t).$$

Substituting these three equalities in (5) we obtain that

$$\gcd(G_m G_s, G_n G_t) = \gcd(G_m, G_t) \gcd(G_s, G_n).$$

This proves the corollary. □

Theorem 7. *Let $a, b \in \mathbb{Z}$ and $n, w \in \mathbb{N}$ such that $\gcd(a, b) = 1$. Then*

$$G_n \equiv 0 \pmod{F_w} \text{ if and only if } \gcd(G_n, G_{n-w}) = \gcd(G_n, G_{n+w}) = F_w.$$

Proof. The proof of necessity is straightforward.

We prove sufficiency. We suppose that $G_n \equiv 0 \pmod{F_w}$ and prove that $\gcd(G_n, G_{n-w}) = \gcd(G_n, G_{n+w}) = F_w$. Since $G_n \equiv 0 \pmod{F_w}$, there is a $k \in \mathbb{Z}$ such that $G_n = kF_w$. From Lemma 4 we have that for every $m \in \mathbb{Z}$

$$\begin{aligned} G_{m+w} &= (aF_{w-1} + bF_w)F_{m-2} + (aF_w + bF_{w+1})F_{m-1} \\ &= (aF_{w-1} + bF_w)F_{m-2} + (aF_w + b(F_{w-1} + F_w))F_{m-1} \\ &= (aF_{m-2} + bF_{m-1})F_{w-1} + F_w(bF_{m-2} + aF_{m-1} + bF_{m-1}). \end{aligned}$$

Thus,

$$G_{m+w} = G_m F_{w-1} + F_w(bF_{m-2} + aF_{m-1} + bF_{m-1}). \tag{6}$$

Taking n instead of m in (6) and using that $G_n = kF_w$ we obtain

$$\begin{aligned} G_{n+w} &= kF_w F_{w-1} + F_w(bF_{n-2} + aF_{n-1} + bF_{n-1}) \\ &= F_w(kF_{w-1} + (bF_{n-2} + aF_{n-1} + bF_{n-1})). \end{aligned}$$

Therefore, $G_{n+w} \equiv 0 \pmod{F_w}$. Thus, $F_w \mid \gcd(G_{n+w}, G_n)$. This and Theorem 5 imply that $F_w = \gcd(G_{n+w}, G_n)$. Now, taking $n - w$ instead of m in (6) we obtain

$$G_n = G_{n-w} F_{w-1} + F_w(bF_{n-w-2} + aF_{n-w-1} + bF_{n-w-1}).$$

Since $G_n = kF_w$, we have that

$$\begin{aligned} G_{n-w}F_{w-1} &= kF_w - F_w(bF_{n-w-2} + aF_{n-w-1} + bF_{n-w-1}) \\ &= F_w(k - (bF_{n-w-2} + aF_{n-w-1} + bF_{n-w-1})). \end{aligned}$$

This and $\gcd(F_w, F_{w-1}) = 1$ imply that $G_{n-w} \equiv 0 \pmod{F_w}$. So, $F_w | \gcd(G_{n-w}, G_n)$. This and Theorem 5 imply that $F_w = \gcd(G_{n-w}, G_n)$, proving sufficiency. □

Lemma 8. *Let $\alpha, \beta, \delta, \gamma, \rho, \phi \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ such that $\gcd(a, b) = 1$ and let $D = \gcd(G_\alpha G_\beta, G_\delta G_\gamma, G_\rho G_\phi)$.*

1. *If $\alpha = \delta + 1, \rho = \delta + 2,\ \beta = \gamma - 2,$ and $\phi = \gamma - 1$, then $D = 1$.*
2. *If $\delta = \alpha + 1,\ \rho = \alpha + 2,\ \beta = \gamma - 1,$ and $\phi = \gamma - 2$, then $D = 1$.*
3. *If $\alpha = \delta + 2,\ \rho = \delta + 4,\ \beta = \gamma - 4,$ and $\phi = \gamma - 2$, then $D = \gcd(G_\beta, G_\rho, G_\delta G_\gamma)$.*
4. *If $\delta = \alpha + 2,\ \rho = \alpha + 4,\ \beta = \gamma - 2,$ and $\phi = \gamma - 4$, then $D = \gcd(G_\alpha, G_\gamma, G_\rho G_\phi)$.*

Proof. It is known that

$$\begin{aligned} \gcd(G_\alpha G_\beta, G_\delta G_\gamma, G_\rho G_\phi) &= \gcd(\gcd(G_\alpha G_\beta, G_\delta G_\gamma), G_\rho G_\phi) \qquad (7) \\ &= \gcd(\gcd(G_\alpha G_\beta, G_\rho G_\phi), G_\delta G_\gamma). \qquad (8) \end{aligned}$$

We prove part (1). The proof of part (2) is similar and we omit it. Since $|\alpha - \delta| = 1$ and $|\beta - \gamma| = 2$, Corollary 6 and (7) imply that

$$D = \gcd(\gcd(G_\alpha, G_\gamma) \gcd(G_\delta, G_\beta), G_\rho G_\phi). \qquad (9)$$

From Theorem 5, $|\alpha - \rho| = 1$ and $|\beta - \phi| = 1$, we have $\gcd(G_\rho, G_\alpha) = \gcd(G_\phi, G_\beta) = 1$. Thus,

$$\gcd(\gcd(G_\gamma, G_\alpha), G_\rho) = \gcd(\gcd(G_\delta, G_\beta), G_\phi) = 1.$$

This, Proposition 2 and (9) imply that

$$D = \gcd(\gcd(G_\alpha, G_\gamma), G_\phi)\ \gcd(\gcd(G_\delta, G_\beta), G_\rho).$$

Using Theorem 5, $|\gamma - \phi| = 1$ and $|\delta - \rho| = 2$, we have that $D = 1$. This proves part (1).

We now prove part (3). The proof of part (4) is similar and we omit it. Theorem 5, $|\alpha - \delta| = 2$ and $|\gamma - \phi| = 2$ imply that $\gcd(G_\alpha, G_\delta) = 1$ and $\gcd(G_\gamma, G_\phi) = 1$. Therefore, $\gcd(\gcd(G_\alpha, G_\phi), G_\delta) = 1$ and $\gcd(\gcd(G_\alpha, G_\phi), G_\gamma) = 1$. Thus,

$$\gcd(\gcd(G_\alpha, G_\phi), G_\delta G_\gamma) = 1. \qquad (10)$$

Since $|\alpha - \rho| = 2$ and $|\beta - \phi| = 2$, Corollary 6 and (8) imply that

$$D = \gcd(\gcd(G_\alpha, G_\phi) \gcd(G_\beta, G_\rho), G_\delta G_\gamma). \qquad (11)$$

Theorem 5, $|\alpha - \rho| = 2$ and $|\beta - \phi| = 2$ imply that $\gcd(G_\alpha, G_\rho) = \gcd(G_\beta, G_\phi) = 1$. Therefore, $\gcd(\gcd(G_\alpha, G_\phi), \gcd(G_\beta, G_\rho)) = 1$. This and the multiplication property of the GCD imply that

$$\gcd\left(\gcd(G_\alpha, G_\phi)\gcd(G_\beta, G_\rho), G_\delta G_\gamma\right) = \gcd\left(\gcd(G_\alpha, G_\phi), G_\delta G_\gamma\right)\ \gcd\left(\gcd(G_\beta, G_\rho), G_\delta G_\gamma\right).$$

This, (10) and (11) prove that $D = \gcd\left(G_\beta, G_\rho, G_\delta G_\gamma\right)$. This proves the lemma. □

4 GCD Property for the Star of David of Length Two and Three

In this section we prove the GCD property for the star of David of length two and give necessary and sufficient conditions for the star of David of length three to have the GCD property. We also prove that 9 divides the coordinates of the vertices a_2 or b_2 in Figure 2 part (b) if and only if the GCD of each triangle gives distinct numbers.

We remind the reader that if $d = \gcd(a, b)$, $a' = a/d$ and $b' = b/d$, then we use G_m for $G_m(a, b)$ and G'_m for $G_m(a', b')$ for any integer m.

Theorem 9. *Let $d = \gcd(a, b)$. If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the star of David of length two, then $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = d^2$.*

Proof. Let $a' = a/d$, $b' = b/d$, and (m, n) be the diagonal coordinates of a_2. We prove first that $\gcd(a_1, a_2, a_3) = d^2$. It is known that $a_1 = G_{m+1}\, G_{n-2}$, $a_2 = G_m\, G_n$ and $a_3 = G_{m+2}\, G_{n-1}$. This and Lemma 3 imply that

$$\gcd(a_1, a_2, a_3) = d^2 \gcd\left(G'_{m+1}\, G'_{n-2}, G'_m\, G'_n, G'_{m+2}\, G'_{n-1}\right).$$

This and Lemma 8 part (1) show that $\gcd(a_1, a_2, a_3) = d^2$.

The proof of $\gcd(b_1, b_2, b_3) = d^2$ is analogous to the proof of $\gcd(a_1, a_2, a_3) = d^2$. Indeed, the proof follows from Lemma 3 and Lemma 8 part (2). This proves the theorem. □

Lemma 10. *Let $d = \gcd(a, b)$ and $a' = a/d$, $b' = b/d$. If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the star of David of length three where (m, n) are the diagonal coordinates of a_2, then*

1. $\gcd(a_1, a_2, a_3) = d^2 \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$.
2. $\gcd(b_1, b_2, b_3) = d^2 \gcd(G'_m, G'_n, G'_{n-4} G'_{m+4})$.

Proof. Since (m, n) are the diagonal coordinates of a_2,

$$\begin{array}{llll} a_1 = G_{m+2}\, G_{n-4}, & a_2 = G_m\, G_n & \text{and} & a_3 = G_{m+4}\, G_{n-2}, \\ b_1 = G_m\, G_{n-2}, & b_2 = G_{m+4}\, G_{n-4} & \text{and} & b_3 = G_{m+2}\, G_n. \end{array}$$

This and Lemma 3 imply that

$$\gcd(a_1, a_2, a_3) = d^2 \gcd\left(G'_{m+2}\, G'_{n-4}, G'_m\, G'_n, G'_{m+4}\, G'_{n-2}\right).$$

This and Lemma 8 part (3) show that $\gcd(a_1, a_2, a_3) = d^2 \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$. This proves part (1). The proof of part (2) is similar and we omit it. □

Theorem 11. *Let $d = \gcd(a, b)$ and $a' = a/d$, $b' = b/d$. If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the star of David of length three where (m, n) are the diagonal coordinates of a_2, then*

1. $G'_m \equiv 0 \pmod 9$ *and* $G'_n \equiv 0 \pmod 9$ *if and only if* $\gcd(a_1, a_2, a_3) < \gcd(b_1, b_2, b_3)$.
2. $G'_{m+4} \equiv 0 \pmod 9$ *and* $G'_{n-4} \equiv 0 \pmod 9$ *if and only if* $\gcd(a_1, a_2, a_3) > \gcd(b_1, b_2, b_3)$.

Proof. We prove part (1). For sufficiency, we assume that $G'_m \equiv 0 \pmod 9$ and $G'_n \equiv 0 \pmod 9$ and prove that $\gcd(a_1, a_2, a_3) = 3\, d^2$ and $\gcd(b_1, b_2, b_3) = 9\, d^2$.

Since $G'_m \equiv 0 \pmod 9$ and $G'_n \equiv 0 \pmod 9$, it is clear that $G'_m \equiv 0 \pmod 3$ and $G'_n \equiv 0 \pmod 3$. This and Theorem 7 imply that

$$\gcd(G'_m, G'_{m+4}) = 3 \quad \text{and} \quad \gcd(G'_n, G'_{n-4}) = 3. \tag{12}$$

Therefore, $3 \mid \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$. We now show that 3 is the only prime divisor of $\gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$. If p is a prime number and $p \mid \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$, then one can easily see that $p \mid \gcd(G'_{n-4}, G'_n)$ or $p \mid \gcd(G'_{m+4}, G'_4)$. This and (12) imply that $p = 3$. Thus, $\gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n) = 3^k$ for some integer $k \geq 1$. We show that $k = 1$. If $k > 1$, then $9 \mid G'_{n-4}$ and $9 \mid G'_{m+4}$. Since $G'_m \equiv 0 \pmod 9$ and $G'_n \equiv 0 \pmod 9$, we can conclude that $9 \mid \gcd(G'_m, G'_{m+4})$ and $9 \mid \gcd(G'_n, G'_{n-4})$. This contradicts (12). Thus,

$$\gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n) = 3. \tag{13}$$

We now prove that $\gcd(G'_m, G'_n, G'_{n-4} G'_{m+4}) = 9$. From $G'_m \equiv 0 \pmod 9$, $G'_n \equiv 0 \pmod 9$ and (12), we can write that $9 \mid \gcd(G'_m, G'_n, G'_{n-4} G'_{m+4})$. We show that 3 is the only prime divisor of $\gcd(G'_m, G'_n, G'_{n-4} G'_{m+4})$. If p is a prime and $p \mid \gcd(G'_m, G'_n, G'_{n-4} G'_{m+4})$, then one can easily see that $p \mid \gcd(G'_{n-4}, G'_n)$ or $p \mid \gcd(G'_{m+4}, G'_4)$. This and (12) imply that $p = 3$. Thus, $\gcd(G'_m, G'_n, G'_{n-4} G'_{m+4}) = 3^l$ for some integer $l \geq 2$. We show that $l = 2$. If $l > 2$, then $3^3 \mid G'_{n-4} G'_{m+4}$. Therefore, $9 \mid G'_{n-4}$ or $9 \mid G'_{m+4}$. Since $G'_m \equiv 0 \pmod 9$ and $G'_n \equiv 0 \pmod 9$, we can conclude that $9 \mid \gcd(G'_m, G'_{m+4})$ or $9 \mid \gcd(G'_n, G'_{n-4})$. This contradicts (12). Thus,

$$\gcd(G'_m, G'_n, G'_{n-4} G'_{m+4}) = 9.$$

This equality, Lemma 10 and (13) show that $\gcd(a_1, a_2, a_3) = 3\, d^2$ and $\gcd(b_1, b_2, b_3) = 9\, d^2$, proving sufficiency of part (1).

Conversely, we assume that $\gcd(a_1, a_2, a_3) < \gcd(b_1, b_2, b_3)$. This and Lemma 10 imply that

$$\gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n) < \gcd(G'_m, G'_n, G'_{n-4} G'_{m+4}). \tag{14}$$

Therefore, $\gcd(G'_m, G'_n, G'_{n-4}G'_{m+4}) > 1$. If p is a prime such that $p \mid \gcd(G'_m, G'_n, G'_{n-4}G'_{m+4})$, then it is easy to see that $p \mid \gcd(G'_m, G'_{m+4})$ or $p \mid \gcd(G'_n, G'_{n-4})$. This and Theorem 5 imply that $p = 3$. Thus,

$$\gcd(G'_m, G'_n, G'_{n-4}G'_{m+4}) = 3^t \text{ for some } t \geq 1. \tag{15}$$

Therefore, $3 \mid G'_m$ and $3 \mid G'_n$. This and Theorem 7 imply that $\gcd(G'_m, G'_{m+4}) = 3$ and $\gcd(G'_n, G'_{n-4}) = 3$. Thus, $3 \mid \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$. In particular, we have that $3 \leq \gcd(G'_{n-4}, G'_{m+4}, G'_m G'_n)$. This, (14) and (15) imply that

$$\gcd(G'_m, G'_n, G'_{n-4}G'_{m+4}) = 3^t \text{ for some } t \geq 2.$$

Thus, $G'_m \equiv 0 \pmod 9$ and $G'_n \equiv 0 \pmod 9$. This proves necessity of part (1).

The proof of part (2) is analogous to the proof of part (1). Indeed, in the proof of part (1) we need to interchange the roles of a_i and b_i for $i = 1, 2, 3$, instead of m use $m + 4$, instead of n use $n - 4$, and also use Theorem 7. This completes the proof. □

Theorem 12. *Let $d = \gcd(a, b)$ and $a' = a/d$, $b' = b/d$. If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the star of David of length three where (m, n) are the diagonal coordinates of a_2, then*

$$\gcd(G'_{m+i}, G'_{n-i}, L_2^2) \neq L_2^2 \text{ for } i = 0, 4 \text{ if and only if } \gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3).$$

Proof. It is a straightforward application of Theorem 11. □

5 Open Questions and a Conjecture

Theorem 9 shows that any star of David of length two has the GCD property. Theorem 12 proves that the star of David of length three has the GCD property provided that the integers a, b, m and n satisfy some L_2^2-divisibility conditions. It is natural to ask the following questions.

Open Question 1. What conditions should we impose on a, b, m and n so that the star of David of length $l \geq 2$ satisfies the GCD property?

Open Question 2. If a star of David of length $l \geq 2$ satisfies the GCD property, then what is the common value of $\gcd(a_1, a_2, a_3)$ and $\gcd(b_1, b_2, b_3)$?

Theorem 9 answers open questions 1 and 2 for $l = 2$. Theorem 12 answers open question 1 for $l = 3$. We propose the following conjecture to answer open question 1 for $l \geq 4$.

Conjecture 13. Let $d = \gcd(a, b)$ and $a' = a/d$, $b' = b/d$. If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the star of David of length l where (m, n) are the diagonal coordinates of a_2, then

$$\gcd(G'_{m+i}, G'_{n-i}, Q) \neq Q \text{ for } i = 0, \ldots, 2(l-1) \text{ if and only if } \gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3),$$

where Q depends only on l.

l	Q	Relationship between Q and Lucas number
3	3^2	L_2^2
4	4^2	L_3^2
5	7^2	L_4^2
6	11^2	L_5^2
7	18	L_6
8	29^2	L_7^2
9	47^2	L_8^2
10	76^2	L_9^2
11	123^2	L_{10}^2
12	119^2	L_{11}^2
13	322	L_{12}

Table 5: Values of Q for some l.

Table 5 gives empirical values of Q for some $l \geq 4$. Notice that the value of Q for $l = 3$ is given by Theorem 12. We were not able to find the values of Q for $l \geq 14$. We believe that the values of Q for $l = 19$ and $l = 25$ follow the same pattern as $l = 7$ and $l = 13$. These numerical results, for $l \geq 4$ were found using *Mathematica*®.

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