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On the Number of Fixed-Length Semiorders

Yangzhou Hu[1]
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139
United States
yangzhou@mit.edu
yangzhou13@gmail.com

Abstract

A *semiorder* is a partially ordered set P with two certain forbidden induced subposets. This paper establishes a bijection between n-element semiorders of length H and $(n+1)$-node ordered trees of height $H+1$. This bijection preserves not only the number of elements, but also much additional structure. Based on this correspondence, we calculate the generating functions and explicit formulas for the numbers of labeled and unlabeled n-element semiorders of length H. We also prove several concise recurrence relations and provide combinatorial proofs for special cases of the explicit formulas.

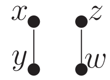
1 Introduction and Main Theorem

We will use partially ordered set (poset) notation and terminology from [5, Ch. 3]. A *semiorder* is a poset without the following induced subposets:

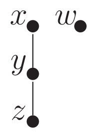
- $(\mathbf{2}+\mathbf{2})$: four distinct elements x, y, z, w, such that $x > y, z > w$, and other pairs are incomparable;

[1]The author's research was part of an undergraduate research project at M.I.T. under the supervision of Richard Stanley.

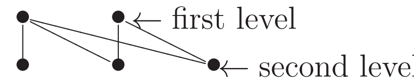
- $(\mathbf{3}+\mathbf{1})$: four distinct elements x, y, z, w, such that $x > y > z$, and other pairs are incomparable.



$(\mathbf{2}+\mathbf{2})$-structure



$(\mathbf{3}+\mathbf{1})$-structure

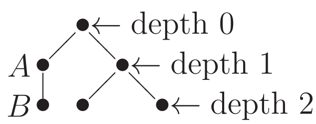


(a) A semiorder with 5 elements and length 1

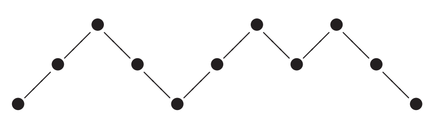
In other words, semiorders are $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{3}+\mathbf{1})$-free posets. Every semiorder can also be regarded as a partial ordering P of a subset of $\mathbb{R}$ defined by $x < y$ in P if $x < y - 1$ in $\mathbb{R}$. The *length* H of a semiorder is the length of a longest chain. Every semiorder R with n elements, up to isomorphism, can be uniquely represented as an integer vector $\rho(R) = (r_1, r_2, \ldots, r_n)$, where r_i represents the number of elements smaller than the i^{th} element, and $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$, $r_i \leq n-i$, for all $1 \leq i \leq n$. For instance, the above graph (a) presents a semiorder R with 5 elements, length 1, and vector $\rho(R) = (3, 2, 0, 0, 0)$. For further basic information on semiorders, see [4].

There is much interest in enumerating the number of posets with certain properties. For example, Bousquet-Mélou et al. enumerated the number of $(\mathbf{2}+\mathbf{2})$-free posets [1]. It is a classical result of Wine and Freund [6] that the number of nonisomorphic n-element semiorders is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, while Chandon, Lemaire, and Pouget [3] showed (in an equivalent form) that if $f(n)$ is the number of n-element *labeled* semiorders (or semiorders on an n-element set), then $\sum_{n\geq 0} f(n)\frac{x^n}{n!} = \sum_{n\geq 0} C_n(1-e^{-x})^n$. For a general principle implying this result, see Lemma 13. In this paper, we deal with semiorders of length at most H. That is, we enumerate the number of posets which are $(\mathbf{2}+\mathbf{2})$-free, $(\mathbf{3}+\mathbf{1})$-free, and of length at most H. We carry out the enumeration by establishing a bijection between semiorders and ordered trees of a fixed height.

An *ordered tree* is a rooted tree that has been embedded in the plane so that the relative order of subtrees at each node is part of its structure. The *height* H of an ordered tree is the number of edges in a chain of maximum length. The following graph (b) shows an ordered tree with 6 nodes and height 2.



(b) An ordered tree with 6 nodes and height 2



(c) A Dyck path with semilength 5 and height 2

A *Dyck path* of semilength n is a lattice path in the Euclidean plane from $(0,0)$ to $(2n, 0)$ whose steps are either $(1,1)$ or $(1,-1)$ and the path never goes below the x-axis. The height H of a Dyck path is the maximal y-coordinate among all points on the path. The above graph (c) shows a Dyck path with semilength 5 and height 2.

It is well-known that there is a one-to-one correspondence between (i) ordered trees with $n+1$ nodes and height H and (ii) Dyck paths with semilength n and height H. This paper establishes a bijection between n-element semiorders of length H and $(n+1)$-node ordered trees of height $H+1$. Thus these semiorders simultaneously correspond to Dyck paths of semilength n and height $H+1$.

Theorem 1 (Main Theorem)**.** *For $n \geq 1$ and $H \geq 0$, the number of nonisomorphic n-element unlabeled semiorders of length H is equal to the number of $(n+1)$-node ordered trees of height $H+1$, which is also equal to the number of Dyck paths with semilength n and height $H+1$.*

Section 2 gives a recurrence proof and a bijective proof for Theorem 1. Section 3 calculates the generating functions and explicit formulas for the number of unlabeled as well as labeled semiorders with fixed lengths. Section 4 proves some concise recurrence relations, and Section 5 shows explicit formulas for the number of semiorders of certain lengths H, and provides simple bijective proofs for these formulas.

2 Proof of Main Theorem 1

Before proving Theorem 1, we first define some terminology that is used later in the proof.

Definition 2. A node A in an ordered tree has *depth* i if the distance from node A to the root is i. In particular, the depth of the root is 0.

In this paper we regard the root as the uppermost node, and all other nodes are below the root. We say node B is attached to node A if node B has depth $i+1$ and node A has depth i, and these two nodes are adjacent. Refer to graph (b) as an example.

Definition 3. An element a of a semiorder is on the i^{th} *level* $(i \geq 1)$ if i is the largest integer for which there exist $i-1$ elements $a_1, a_2, \ldots, a_{i-1}$ satisfying $a_1 > a_2 > \cdots > a_{i-1} > a$. Refer to graph (a) as an example.

Proposition 4. *For a length H semiorder, and for $1 \leq i \leq H$, there is at least one element on the i^{th} level that is larger than all elements on the $(i+1)^{\text{th}}$ level.*

Proof. Suppose that there does not exist an element on the i^{th} level that is larger than all elements on the $(i+1)^{\text{th}}$ level. Since every element on the $(i+1)^{\text{th}}$ level must be smaller than at least one element on the i^{th} level, there must exist two elements a and c on the $(i+1)^{\text{th}}$ level which are smaller than two distinct elements b and d on the i^{th} level, respectively, and b is not larger than c, while d is not larger than a. Then $\{b > a,\ d > c\}$ forms a $(\mathbf{2}+\mathbf{2})$-structure, a contradiction. Therefore, at least one element on the i^{th} level is larger than all elements on the $(i+1)^{\text{th}}$ level. $\square$

Proposition 5. *For a length H semiorder, and for $1 \leq i < j \leq H+1$, $j-i \geq 2$, every element on the i^{th} level is larger than all elements on the j^{th} level.*

Proof. Assume to the contrary that there exist an element b on the i^{th} level and an element a on the j^{th} level such that b is not larger than a. By Definition 3, there exist $j-1$ elements $a_1, a_2, \ldots, a_{j-1}$, such that $a_1 > a_2 > \cdots > a_{j-1} > a$, and then a_{j-2} and a_{j-1} should be on the $(j-2)^{\text{th}}$ and $(j-1)^{\text{th}}$ level, respectively. Since element b is on the i^{th} level, and $j-i \geq 2$, element b cannot be smaller than any of a_{j-2}, a_{j-1}, or a. In addition, since b is not larger than a and $a_{j-2} > a_{j-1} > a$, b is not comparable with any of a_{j-2}, a_{j-1} or a. Hence, $\{a_{j-2} > a_{j-1} > a,\ b\}$ forms a $(\mathbf{3}+\mathbf{1})$-structure, a contradiction. Therefore every element on the i^{th} level should be larger than all elements on the j^{th} level, for $j-i \geq 2$. □

We are now ready to prove the Main Theorem 1. We give two proofs here: one considers the recurrence formulas of the two numbers in the theorem, and the other directly establishes a bijection between semiorders and ordered trees.

2.1 Recurrence proof

Let $t(n,h,k)$ denote the number of $(n+1)$-node ordered trees of height $h+1$, for which exactly k nodes have depth $h+1$, $1 \leq k \leq n$. Let $f(n,h,k)$ be the number of n-element semiorders of length h, and exactly k elements are on the last level. We show that $t(n,h,k)$ and $f(n,h,k)$ have the same initial value and recurrence formula in the following lemmas, and thus they are equal.

Lemma 6. *For $h \geq 1$, we have*

$$t(n,h,k) = \sum_{m=1}^{n-k} \binom{m+k-1}{m-1} \cdot t(n-k, h-1, m). \tag{1}$$

Proof. Say we have an $(n-k+1)$-node ordered tree of height h, and assume that exactly m nodes have depth h, $1 \leq m \leq n-k$. Consider adding k nodes to the tree to get a new tree with $n+1$ nodes and height $h+1$, and the newly added nodes are exactly the set of nodes of depth $h+1$. Thus we need to attach the k new nodes to the m nodes of depth h, and every new node is uniquely attached to one node. Let the m nodes be $A_1, A_2, \ldots, A_m$, and the number of new nodes attached to A_i be r_i, $1 \leq i \leq m$. Then we have $r_1 + r_2 + \cdots + r_m = k,\ \ r_i \geq 0,\ \ 1 \leq i \leq m$.

The number of integer solutions to the above equation is $\binom{m+k-1}{m-1}$. Therefore, we have $\binom{m+k-1}{m-1}$ ways to add the k nodes. Summing up all possible m's, we obtain

$$t(n,h,k) = \sum_{m=1}^{n-k} \binom{m+k-1}{m-1} \cdot t(n-k, h-1, m).$$

□

Lemma 7. *For $h \geq 1$, we have*

$$f(n,h,k) = \sum_{m=1}^{n-k} \binom{m+k-1}{m-1} \cdot f(n-k, h-1, m). \tag{2}$$

Proof. We say that an element of a semiorder is *good* if the element is on the last level of the semiorder. Say we have an $(n-k)$-element semiorder S of length $h-1$ and m good elements, $1 \leq m \leq n-k$. Consider adding k elements to S to get a new semiorder S' with n elements and length h, and the newly added elements are exactly the set of good elements of S'. Call the original m good elements $a_1, a_2, \ldots, a_m$, and the k new elements $b_1, b_2, \ldots, b_k$. Then in the semiorder S', we have that $a_1, a_2, \ldots, a_m$ are the only elements on the h^{th} level, and $b_1, b_2, \ldots, b_k$ are the only elements on the $(h+1)^{\text{th}}$ level.

If we remove all elements on the first $h-1$ levels of S', then we get a length one semiorder P with $m+k$ elements, and there are exactly m elements on the upper level and k elements on the lower level. On the other hand, given a semiorder S and a semiorder P as above, we can uniquely determine the semiorder S', because based on Proposition 5, the k elements on the $(h+1)^{\text{th}}$ level of S' must be smaller than all elements on the i^{th} level of S', for $1 \leq i \leq h-1$. Therefore, the semiorder P uniquely determines the way to add the k new elements.

Let P with $\rho(P) = (p_1, p_2, \ldots, p_{m+k})$ represent one such semiorder. Then we have

$$\begin{cases} k = p_1 \geq p_2 \geq \cdots \geq p_m \geq 0; \\ p_{m+1} = p_{m+2} = \cdots = p_{m+k} = 0. \end{cases} \tag{3}$$

Notice that $\{p_2, p_3, \ldots, p_m\}$ is an $(m-1)$-element multiset with elements from $\{0, 1, \ldots, k\}$, and thus we have $\binom{k+1+m-1-1}{m-1} = \binom{m+k-1}{m-1}$ such multisets. Therefore, there are $\binom{m+k-1}{m-1}$ possible semiorder P's. Summing up all possible m's, we have

$$f(n,h,k) = \sum_{m=1}^{n-k} \binom{m+k-1}{m-1} \cdot f(n-k, h-1, m).$$

□

Proof of the Main Theorem 1. For $h = 0$, the $(n+1)$-element ordered tree of height $h+1 = 1$ can only be the tree with n nodes adjacent to the root; meanwhile, the n-element semiorder of length 0 can only be the one with n elements and any two of the elements are incomparable. As a result, we have

$$t(n,0,k) = f(n,0,k) = \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{if } n \neq k. \end{cases}$$

For $h \geq 1$, by Lemma 6 and 7, $t(n,h,k)$ and $f(n,h,k)$ have the same recurrence formula. Therefore $t(n,h,k) = f(n,h,k)$ for every $n \geq 1$, $h \geq 0$, and $1 \leq k \leq n$. Summing on k completes the proof of Theorem 1. □

2.2 Bijective proof

Recall that an element of a semiorder is good if it is on the last level of the semiorder. Based on the idea in the recurrence proof, we can construct a one-to-one map from $(n+1)$-element ordered trees of height $H+1$ with k nodes of depth $H+1$ to n-element semiorders of length H with k good elements.

For an ordered tree with $n+1$ nodes and height $H+1$, let us assume that there are x_i nodes of depth i, $0 \le i \le H+1$. Since the root is the only node of depth 0, we have

$$\sum_{i=1}^{H+1} x_i = n. \tag{4}$$

Let s_j^i denote the number of nodes of depth i that are adjacent to the j^{th} node of depth $i-1$, $1 \le j \le x_{i-1}$, $1 \le i \le H+1$. Since every node of depth i should be adjacent to exactly one node of depth $i-1$, we must have

$$\sum_{j=1}^{x_{i-1}} s_j^i = x_i. \tag{5}$$

Let $u_j^i = \sum_{k=j}^{x_{i-1}} s_k^i$, $1 \le j \le x_{i-1}$, $1 \le i \le H+1$. Then $u_1^i \ge u_2^i \ge \cdots \ge u_{x_{i-1}}^i$. Let $y_i = \sum_{k=1}^{i} x_k$, $1 \le i \le H+1$, and $y_0 = 0$. We now define R^i by induction, and let the number of entries in R^i be y_i.

Set $R^1 = (0, 0, \ldots, 0)$, in which there are $x_1 = y_1$ zeros. Assume $R^i = (r_1^i, r_2^i, \ldots, r_{y_i}^i)$, $1 \le i \le H$, and let

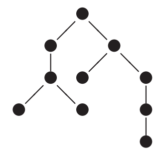
$$R^{i+1} = (r_1^i + x_{i+1}, r_2^i + x_{i+1}, \ldots, r_{y_{i-1}}^i + x_{i+1}, r_{y_{i-1}+1}^i + u_1^{i+1}, r_{y_{i-1}+2}^i + u_2^{i+1}, \ldots, r_{y_{i-1}+x_i}^i + u_{x_i}^{i+1}, 0, 0, \ldots, 0)$$

in which there are x_{i+1} zeros, and thus R^{i+1} has $y_{i-1} + x_i + x_{i+1} = y_{i+1}$ entries.

Theorem 8. *The vector R^{H+1} represents an n-element semiorder of length H with x_{H+1} good elements. This gives a bijective map from $(n+1)$-node ordered trees of height $H+1$ to n-element semiorders of length H.*

Since this map is naturally derived from the recurrence proof, we do not give a rigorous proof on why the map is valid and why it is a bijection. The main idea here is to map an ordered tree of height $H+1$ to a semiorder with $H+1$ levels, where the number of elements on the i^{th} level of the semiorder is equal to the number of nodes of depth i in the tree, $1 \le i \le H+1$. We get a bijection between (a) the connections between nodes of depths i and $i+1$ in the tree, and (b) the set of ordered pairs between elements on levels i and $i+1$ in the semiorder, $1 \le i \le H$. This bijection preserves not only the number of elements but also much additional structure. It presents an effective way to connect semiorders and ordered trees, as well as Dyck paths. In order to illustrate the bijection more clearly, we show by an example how the map works.

Example 9. Assume we have the following ordered tree:



Here the number of nodes is 10 and height is 4. We have $(x_1, x_2, x_3, x_4) = (2, 3, 3, 1)$. So the tree should correspond to a semiorder with 9 elements, length 3, and the number of elements of each depth is given by $(2, 3, 3, 1)$. Write $s^i = (s^i_1, s^i_2, \ldots, s^i_{x_{i-1}})$ and $u^i = (u^i_1, u^i_2, \ldots, u^i_{x_{i-1}})$. Then the vectors representing the connections between two adjacent depth-levels in the ordered tree are

$$s^1 = (2), \;\; s^2 = (1, 2), \;\; s^3 = (2, 0, 1), \;\; s^4 = (0, 0, 1).$$

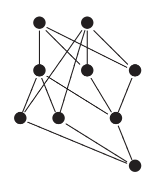
We transform these vectors into vectors that can represent the set of ordered pairs between two adjacent levels of the semiorder. These vectors are

$$u^1 = (2), \;\; u^2 = (3, 2), \;\; u^3 = (3, 1, 1), \;\; u^4 = (1, 1, 1).$$

Now let us construct R^i, $1 \le i \le 4$:

$$R^1 = (0, 0), \;\; R^2 = (3, 2, 0, 0, 0), \;\; R^3 = (6, 5, 3, 1, 1, 0, 0, 0), \;\; R^4 = (7, 6, 4, 2, 2, 1, 1, 1, 0).$$

In fact, R^i depicts the semiorder with only the first i levels, $1 \le i \le 4$, and R^4 is the final semiorder we desired. Its Hasse diagram is as follows:



The inverse map can be done by reversing the steps.

3 Generating Functions and Explicit Formulas

3.1 On unlabeled semiorders

Let f^n_h denote the number of nonisomorphic unlabeled semiorders with n elements and length h, and $f^n_{\le h}$ denote the number of nonisomorphic unlabeled semiorders with n elements and length at most h, so $f^n_h = f^n_{\le h} - f^n_{\le (h-1)}$. Let $F_h(x) = \sum_{n=0}^{\infty} f^n_h x^n$, and $F_{\le h}(x) = \sum_{n=0}^{\infty} f^n_{\le h} x^n$.

De Bruijn, Knuth, and Rice [2] calculated the generating function for the number of fixed-height ordered trees in 1972. Based on this generating function and Theorem 1, we have the following corollary.

Corollary 10. *For $h \geq 0$,*

$$\bullet\ F_h(x) = \sum_{n=0}^{\infty} f_h^n x^n = \frac{x^{h+1}}{p_{h+1}(x)p_h(x)} \tag{6}$$

$$\bullet\ F_{\leq h}(x) = \sum_{n=0}^{\infty} f_{\leq h}^n x^n = \frac{p_h(x)}{p_{h+1}(x)} \tag{7}$$

where

$$p_0(x) = 1, \quad p_1(x) = 1 - x, \quad p_{h+1}(x) = p_h(x) - x \cdot p_{h-1}(x).$$

De Bruijn, Knuth, and Rice [2] also found the explicit formulas for the number of fixed-height ordered trees. Based on their results and Theorem 1, we have the following corollary:

Corollary 11. $f_{\leq h}^1 = f_{\leq h}^0 = 1$. *For $n \geq 2$, $h \geq 0$, we have*

$$f_{\leq h}^n = (h+3)^{-1} \sum_{1 \leq j \leq \frac{h+2}{2}} 4^{n+1} \sin^2\left(\frac{j\pi}{h+3}\right) \cos^{2n}\left(\frac{j\pi}{h+3}\right). \tag{8}$$

3.2 On labeled semiorders

Let g_h^n denote the number of nonisomorphic labeled semiorders with n elements and length h, and $g_{\leq h}^n$ denote the number of nonisomorphic labeled semiorders with n elements and length at most h. Thus $g_h^n = g_{\leq h}^n - g_{\leq (h-1)}^n$. Let $G_h = \sum_{n=0}^{\infty} g_h^n \frac{x^n}{n!}$ and $G_{\leq h} = \sum_{n=0}^{\infty} g_{\leq h}^n \frac{x^n}{n!}$.

We obtain the exponential generating function G_h from the ordinary generating function F_h by the following lemma, which is due to Zhang [7]. We first define equivalence of elements and then state the lemma.

Definition 12. Two elements p and p' of a poset P are *equivalent* if

$$p' < q \Leftrightarrow p < q, \text{ for all } q \in P$$

and

$$p' > q \Leftrightarrow p > q, \text{ for all } q \in P.$$

Lemma 13. *Define the following two operations on an unlabeled poset P.*

- *The* expansion *of P at $p \in P$ is obtained from P by adjoining a new element p' such that p and p' are equivalent.*
- *The* contraction *$c(P)$ of P is a poset $c(P)$ obtained from P by replacing every equivalence class of elements with a single element. Call a poset P a* seed *if $P = c(P)$. Call a seed P* rigid *if P has no nontrivial automorphisms.*

Let C be a family of unlabeled posets such that C is closed under expansion and contraction, and all seeds in C are rigid. Let $F(x) = \sum_{P\in C} x^{\#P}$ and $G(x) = \sum_{P\in C} D_p \frac{x^{\#P}}{(\#P)!}$, where $\#P$ is the number of elements in poset P and D_p is the number of ways to label the elements of P up to isomorphism, i.e. $D_p = \frac{\#P}{\#(aut\ P)}$, where aut P is the automorphism group of P. Then $G(x) = F(1 - e^{-x})$.

The class of semiorders of length h is closed under expansion and contraction. Zhang has observed that all $(\mathbf{2} + \mathbf{2})$-free seeds are rigid. Since semiorders are $(\mathbf{2} + \mathbf{2})$-free, all seeds of semiorders are rigid. As a result, Lemma 13 implies the following corollary.

Corollary 14. *For $h \geq 0$,*

$$G_h(x) = F_h(1 - e^{-x}) = \frac{(1 - e^{-x})^{h+1}}{p_{h+1}(1 - e^{-x})p_h(1 - e^{-x})} \tag{9}$$

and

$$G_{\leq h}(x) = F_{\leq h}(1 - e^{-x}) = F_{\leq h}(1 - e^{-x}) = \frac{p_h(1 - e^{-x})}{p_{h+1}(1 - e^{-x})}. \tag{10}$$

4 Recurrence Relations

The generating functions and explicit formulas for the number of semiorders of fixed length are complicated, but there are some concise recurrence relations underneath. We will discuss two useful recurrence formulas in this section. The first recurrence formula (11) is a standard result that is known for ordered trees [2, p.17], but only a proof using generating functions was given, while the second recurrence formula (12) is not obvious for ordered trees. We will provide concise combinatorial proofs for both formulas. In this way, we can better understand the relations between fixed-length semiorders with different numbers of elements.

4.1 Recurrence formula 1

Theorem 15. *For $n \geq 2$ and $h \geq 1$,*

$$f^n_{\leq h} = \sum_{t=0}^{n-1} f^t_{\leq h} f^{n-1-t}_{\leq h-1}, \tag{11}$$

where $f^0_{\leq h} = 1$.

Proof. Let us prove this theorem by first defining the relative positions of elements on the same level of a semiorder.

Definition 16. For elements a and b on the same level of a semiorder S, we say that element b is *to the right of* element a if b is smaller than more elements than a is, or b and a are smaller than the same number of elements while b is larger than fewer elements than a is.

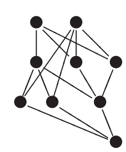
Remark 17. The above definition is unique up to isomorphism. In fact, if semiorder S has n elements and say the integer vector corresponding to semiorder S, as discussed in Section 1, is $(r_1, r_2, \ldots, r_n)$, then element b is *to the right of* element a if b corresponds to r_j, while a corresponds to r_i, $i < j$. For detailed basic properties of semiorders, see [4].

Let us now prove Theorem 15. Let a_1 be the *rightmost* element on the first level of S, and let $T_1 = \{a_1\}$. Once T_i is defined, let T_{i+1} be the set of elements on the $(i+1)^{\text{th}}$ level, each of whose elements is smaller than at least one element in T_i, $1 \le i \le h$. For a given semiorder S, the set T_i is uniquely determined, $1 \le i \le h$. Notice that it is possible that $T_i = \emptyset$, for some i, $1 \le i \le h$, and if $T_i = \emptyset$, then we must have $T_j = \emptyset$ for all $i \le j \le h+1$. Let $A_2 = T_1 \cup T_2 \cup \cdots \cup T_{h+1}$, and $A_1 = A - A_2$, where A is the set of all elements of S. Since $a_1 \in A_2$, we must have $1 \le |A_2| \le n$, $0 \le |A_1| \le n-1$, and $|A_1| + |A_2| = n$.

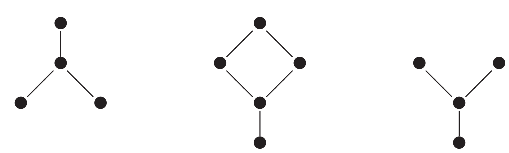
Let us separate S into two semiorders S_1 and S_2. Let S_1 be the induced semiorder with element set A_1. Similarly, let S_2 be the induced semiorder with element set A_2. Let S_3 be the semiorder obtained from S_2 by removing element a_1. Then for a given semiorder S, we have that S_1, S_2, S_3 are uniquely defined. Since S is a semiorder of length at most h, semiorders S_1 and S_2 have length at most h, and thus S_3 has length at most $h-1$.

Assume $|A_1| = t$, so S_3 is a semiorder with $n-1-t$ elements. As a result, for a given n-element semiorder S of length at most h, we can uniquely obtain a pair of semiorders S_1 and S_3, of length at most h and at most $h-1$, and with t and $n-1-t$ elements, respectively, $0 \le t \le n-1$.

For example, if S is as follows,



the corresponding S_1, S_2, S_3 are:



S_1 S_2 S_3

On the other hand, given a pair of semiorders S_1 and S_3, of length at most h and at most $h-1$, and with t and $n-1-t$ elements, respectively, $0 \le t \le n-1$, we can first add an element a_1 to S_3, and let a_1 be larger than all other elements in S_3. Let us call the new semiorder S_2. Then S_2 has $n-t$ elements and length at most h.

Let us construct a new semiorder S by combining S_1 and S_2 as follows. The elements on the i^{th} level of S are the elements on the i^{th} level of S_1 and S_2, $1 \le i \le h+1$, and the order relations in S_1 and S_2 are preserved. In addition, let every element on the $(i-1)^{\text{th}}$ level of S_1 be larger than all elements on the i^{th} level of S_2.

This construction uniquely gives an n-element semiorder S of length at most h, and if we separate S into two semiorders by the method discussed above, we get back S_1 and S_3.

As a result, there is a one-to-one map between n-element semiorders S of length at most h, and pairs of semiorders S_1 and S_3, of length $\leq h$ and $\leq h-1$, and with t and $n-1-t$ elements, respectively, $0 \leq t \leq n-1$.

Therefore, summing up possible t's, we have

$$f_{\leq h}^{n} = \sum_{t=0}^{n-1} f_{\leq h}^{t} f_{\leq h-1}^{n-1-t}.$$

□

4.2 Recurrence formula 2

Theorem 18. *For $n \geq 2$ and $h \geq 1$, we have*

$$f_{\leq h}^{n} = \sum_{k=1}^{\lfloor \frac{h+2}{2} \rfloor} (-1)^{k-1} \binom{h+2-k}{k} f_{\leq h}^{n-k}. \tag{12}$$

Proof. Let us prove this theorem by first defining bad elements of a semiorder and then considering removing one or more bad elements from a given semiorder.

Definition 19. We say an element of a semiorder is *bad* if the following two conditions hold.
• It is on the first level, or it is smaller than all elements on the level immediately above it, and
• it is on the last level, or it is not larger than any element on the level immediately below it.

Remark 20. By the above definition, there are no two adjacent levels which both have bad elements. In addition, by Proposition 5, if two bad elements are on the same level, they must be equivalent. Therefore there is at most one non-equivalent bad element on each level. We thus only consider one bad element on each level.

Proposition 21. *For every semiorder, there exist bad elements.*

Proof. Assume to the contrary that there is no bad element for some semiorder. Say a is the rightmost element on the last level based on Definition 16. Since a is not bad, there must be some element b on the last but one level which is not larger than a. Let $\{a_1, a_2, \ldots, a_s\}$ be the set of all elements on the last but one level which are larger than a. By the definition of element levels, we must have $s \geq 1$.

If there exists some element b_1 on the last level such that $b > b_1$, since a is the rightmost, i.e., a is to the right of b_1, there must exist some i, $1 \leq i \leq s$, such that a_i is not larger than

b_1. Then $\{a_i > a, b > b_1\}$ forms a $(\mathbf{2}+\mathbf{2})$-structure. This is a contradiction. Hence b is not larger than any element on the last level, and thus b is the rightmost element on the last but one level.

Since b is not bad, it must not be on the first level, and there must be some element c on the level immediately above b that is not larger than b. Again due to the fact that b is the rightmost element on its level and that we cannot have a $(\mathbf{2}+\mathbf{2})$-structure, c must be not larger than any element on the level immediately below the level c is on. Continuing, we can find an element on each level that is not larger than any element on the level immediately below it. Since the length of the semiorder is finite, we can finally obtain an element d on the first level such that d is not larger than any element on the second level. Then d is bad, and we get a contradiction. Hence, for every semiorder, bad element exists. □

We are now ready to deduce the recurrence formula (18). For fixed k, consider adjoining k bad elements to an $(n-k)$-element semiorder to get a new semiorder. Specifically, for an $(n-k)$-element semiorder of length at most h, and for k nonadjacent levels among levels $1, 2, \ldots, h+1$, we consider adjoining k elements onto the given levels of the semiorder in the following manner. Say we are adjoining an element onto the l^{th} level.

- If $l = 1$, let the new element be larger than all elements on the i^{th} level, $i \geq 3$, and be not comparable with any other element.
- If $l \geq 2$, and the $(l-1)^{\text{th}}$ level originally has at least one element, we let the new element be smaller than all elements on the $(l-1)^{\text{th}}$ level, be larger than all elements on the i^{th} level, $i \geq l+2$, and be not comparable with any other element.
- If $l \geq 2$, and the $(l-1)^{\text{th}}$ level originally has no element, we *hang* the new element on the l^{th} level, that is, we place it as an isolated vertex on the l^{th} level. We then call the new semiorder an *invalid* semiorder, and if some semiorder r_0 can be obtained from the invalid semiorder by taking out the hanging elements, we call the invalid semiorder the *disguise* of r_0.

By Definition 19, in the above adjoining, all new elements which are not hung are bad elements in the new semiorder. There are $\binom{h+2-k}{k}$ ways to choose k nonadjacent levels among levels $1, 2, \ldots, h+1$. Therefore, including multiplicity and the invalid ones, we can obtain $\binom{h+2-k}{k} \cdot f_{\leq h}^{n-k}$ n-element semiorders from $(n-k)$-element semiorders by adjoining k elements. Let $\mathcal{S}^k$ be the set of all such n-element semiorders, including multiplicity. Then $|\mathcal{S}^k| = \binom{h+2-k}{k} \cdot f_{\leq h}^{n-k}$.

Let $\mathcal{R}$ be the set of all n-element semiorders of length at most h, and $\mathcal{R}'$ be the set of all semiorders with at most $n-1$ elements and length at most h. By Proposition 21, every semiorder has bad elements, so every semiorder $r \in \mathcal{R}$ can be obtained by the above process from some $(n-k)$-element semiorder of length at most h and k given nonadjacent levels, i.e., $r \in \mathcal{S}^k$, for some $1 \leq k \leq \lfloor \frac{h+2}{2} \rfloor$. However, r might be in $\mathcal{S}^k$ for multiple k's, and r may have multiple copies in $\mathcal{S}^k$. Meanwhile, $\mathcal{S}^k$ may contain some semiorders not in $\mathcal{R}$, but are the disguises of some semiorders $r' \in \mathcal{R}'$. Notice that $|\mathcal{R}| = f^n_{\leq h}$. In the following argument, we calculate the number of copies of a semiorder in each $\mathcal{S}^k$ and obtain a formula connecting $|\mathcal{R}|$ and $|\mathcal{S}^k|$, $1 \leq k \leq \lfloor \frac{h+2}{2} \rfloor$.

For a semiorder $r_0 \in \mathcal{R} \cup \mathcal{R}'$, let $\mathcal{S}^k_{r_0}$ be the set of all semiorders in $\mathcal{S}^k$ which are equal to r_0 or a disguise of r_0. Then $\mathcal{S}^k = \bigcup_{r \in \mathcal{R} \cup \mathcal{R}'} \mathcal{S}^k_r$, and

$$\sum_{r \in \mathcal{R} \cup \mathcal{R}'} |\mathcal{S}^k_r| = |\mathcal{S}^k| = \binom{h+2-k}{k} \cdot f^{n-k}_{\leq h}. \tag{13}$$

Next, we show that $\sum_{k=1}^{\lfloor \frac{h+2}{2} \rfloor} (-1)^{k-1} |\mathcal{S}^k_r| = 1$, for every $r \in \mathcal{R}$, and $\sum_{k=1}^{\lfloor \frac{h+2}{2} \rfloor} (-1)^{k-1} |\mathcal{S}^k_{r'}| = 0$, for every $r' \in \mathcal{R}'$.

1. For a semiorder $r \in \mathcal{R}$, assume r has m bad elements. Since we adjoined k elements to an $(n-k)$-element semiorder to obtain the semiorder r, which has n elements, the k new elements should all be added to the levels among the m levels where the bad elements are, and no new element is hung. So $k \leq m$. Further notice that for a given k, $1 \leq k \leq m$, and given k levels among the m levels where the bad elements are, there is a unique $(n-k)$-element semiorder can be used to adjoin k bad elements to the chosen levels to obtain semiorder r. There are $\binom{m}{k}$ ways to choose the k levels, so $|\mathcal{S}^k_r| = \binom{m}{k} \cdot 1$, and

$$\sum_{k=1}^{\lfloor \frac{h+2}{2} \rfloor} (-1)^{k-1} |\mathcal{S}^k_r| = \sum_{k=1}^{m} (-1)^{k-1} |\mathcal{S}^k_r| = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \cdot 1 = 1. \tag{14}$$

2. For a semiorder $r' \in \mathcal{R}'$, assume r' has n' elements, m' of which are bad. Further assume that semiorder r' has length h'. For a given k, $1 \leq k \leq \lfloor \frac{h+2}{2} \rfloor$, if we adjoined k elements to an $(n-k)$-element semiorder to obtain r', we need to adjoin $t' = n' - (n-k)$ elements to levels where the bad elements of r' are, and hang the remaining $n - n'$ elements. Moreover, since we hung $n - n'$ elements, there should be at least $n - n'$ nonadjacent levels among levels $h'+2, h'+3, \ldots, h+1$. As a result, $\lceil \frac{h-h'}{2} \rceil \geq n - n'$.

To obtain the semiorder r', if we are given t' levels among the m' levels where the bad elements of r' are and $n - n'$ nonadjacent levels among levels $h'+2, h'+3, \ldots, h+1$, there is a unique $(n-k)$-element semiorder to which we can adjoin k elements to the chosen levels to obtain the disguise of r'. Notice that there are $\binom{m'}{t'}$ ways to choose t' levels among the m' levels, and $\binom{h-h'+1-(n-n')}{n-n'}$ ways to choose $n - n'$ nonadjacent levels from levels $h'+2, h'+3, \ldots, h+1$. Thus,

$$|\mathcal{S}^k_{r'}| = |S^{t'-n'+n}_{r'}| = \binom{m'}{t'} \cdot \binom{h-h'+1-(n-n')}{n-n'} \cdot 1.$$

Then

$$\begin{aligned}
\sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}|\mathcal{S}_{r'}^{k}| &= \sum_{t'=0}^{m'}(-1)^{t'-n'+n-1}|\mathcal{S}_{r'}^{t'-n'+n}| \\
&= \sum_{t'=0}^{m'}(-1)^{t'-n'+n-1}\binom{m'}{t'}\cdot\binom{h-h'+1-(n-n')}{n-n'}\cdot 1 \\
&= (-1)^{n-n'-1}\cdot\binom{h-h'+1-(n-n')}{n-n'}\sum_{t'=0}^{m'}(-1)^{t'}\binom{m'}{t'} = 0. \qquad (15)
\end{aligned}$$

However, we should be careful with the special case when $t' \geq 1$ and the $(h'+1)^{\text{th}}$ level of r' has bad elements. In this case, the $(h'+1)^{\text{th}}$ level and the $(h'+2)^{\text{th}}$ level might be both chosen when we choose $n-n'$ nonadjacent levels among levels $h'+2, h'+3, \ldots, h+1$ and choose t' levels among the m' levels where bad elements are (note that here the $(h'+1)^{\text{th}}$ level is among the m' levels). Further notice that $\mathcal{S}_{r'}^{k} \in \mathcal{S}^{k}$ and $\mathcal{S}^{k}$ is defined as the set of all n-element semiorders generated by adjoining k elements to k nonadjacent levels to $(n-k)$-element semiorders. Therefore, when we calculate $|\mathcal{S}_{r'}^{k}|$, we need to take out the overcounts of cases when the two adjacent levels $(h'+1)$ and $(h'+2)$ are both chosen. Thus in the special case where $t' \geq 1$ and the $(h'+1)^{\text{th}}$ level of r' has bad elements,

$$\begin{aligned}
|\mathcal{S}_{r'}^{k}| &= |\mathcal{S}_{r'}^{t'-n'+n}| \\
&= \binom{m'}{t'}\cdot\binom{h-h'+1-(n-n')}{n-n'}\cdot 1 - \binom{m'-1}{t'-1}\cdot\binom{h-h'-1-(n-n'-1)}{n-n'-1}\cdot 1
\end{aligned}$$

By similar calculations, we have $\sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}|\mathcal{S}_{r'}^{k}| = 0$.

To conclude the proof, by equations (13), (14), and (15), we have

$$\begin{aligned}
f_{\leq h}^{n} = |\mathcal{R}| = \sum_{r\in\mathcal{R}} 1 + \sum_{r'\in\mathcal{R}'} 0 &= \sum_{r\in\mathcal{R}}\sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}|\mathcal{S}_{r}^{k}| + \sum_{r'\in\mathcal{R}'}\sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}|\mathcal{S}_{r'}^{k}| \\
&= \sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}\sum_{r\in\mathcal{R}\cup\mathcal{R}'}|\mathcal{S}_{r}^{k}| \\
&= \sum_{k=1}^{\lfloor\frac{h+2}{2}\rfloor}(-1)^{k-1}\binom{h+2-k}{k}\cdot f_{\leq h}^{n-k}.
\end{aligned}$$

□

5 The Number of Semiorders of Small Length

We can substitute certain lengths H in the explicit formulas for the number of semiorders. Though the original formulas are very complicated, we can get some simple results for small values of H. In this section, we list these simple results and give bijective proofs, which present a clearer view of the number of fixed-length semiorders.

5.1 $f^n_{\le 1}$, the number of nonisomorphic unlabeled n-element semiorders of length at most one

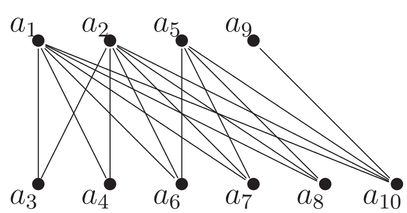
Theorem 22. *For $n \ge 1$, $f^n_{\le 1} = 2^{n-1}$.*

We give a simple bijective proof here.

Proposition 23. *For n elements $a_1, a_2, \ldots, a_n$, put a_1 on the upper level, and each of $a_2, \ldots, a_n$ either on the lower or upper level. Define the order relations in the following way: $a_i > a_j$ if and only if $i < j$, and a_i is on the upper level while a_j is on the lower level.*

We claim that the above defines a bijective map from (a) an arrangement of $n-1$ elements onto two levels in (b) an n-element semiorder of length at most one.

Here is an example of the map. Say $n = 10$, and for $a_2, \ldots, a_{10}$, let $\{a_2, a_5, a_9\}$ be on the upper level, and $\{a_3, a_4, a_6, a_7, a_8, a_{10}\}$ on the lower level. Then the corresponding semiorder looks like:



Proof. We first show that the map gives a semiorder of length at most one. It suffices to show that the poset the map gives is indeed a semiorder. Then the only possible violation is a $(\mathbf{2}+\mathbf{2})$-structure. If there exist four distinct elements a_i, a_j, a_k, a_m, such that $a_i > a_j$, $a_k > a_m$, $a_i \sim a_k$, $a_i \sim a_m$, $a_k \sim a_j$, $a_m \sim a_j$, then a_i, a_k must be on the upper level, while a_j, a_m must be on the lower level, and $i < j$, $k < m$. Since $a_i \sim a_m$, we must have $i > m$; since $a_k \sim a_j$, we must have $k > j$. Then $k > j > i > m > k$, which is a contradiction. Hence the map gives a semiorder of length at most one.

We then claim that the inverse map is also well-defined, and thus the map is bijective. For a given n-element semiorder r with m elements on the upper level, let $\rho(r) = (r_1, r_2, \ldots, r_n)$. Then $r_{m+1} = r_{m+2} = \cdots = r_n = 0$. Say element a_{t_i} corresponds to r_i, $1 \le i \le m$, and then there should be exactly r_i elements on the lower level such that their subscripts are larger than t_i. As a result, note that a_1 is on the upper level, we should also have $a_{r_1-r_2+2}, a_{r_1-r_3+3}, \ldots, a_{r_1-r_m+m}$ on the upper level. In other words, for a given semiorder of length at most one, the elements arranged on the upper level are uniquely determined. Therefore, the inverse map is well-defined.

$\square$

As an example, if we have $(r_1, r_2, \ldots, r_n) = (6, 6, 4, 1, 0, 0, 0, 0, 0, 0)$, then the elements on the upper level must be a_1, a_2, a_5, a_9.

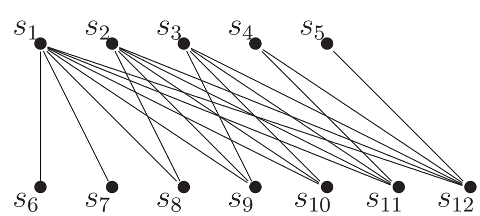
There are 2^{n-1} ways to arrange elements $a_2, \ldots, a_n$ on either upper or lower level, and thus there are 2^{n-1} nonisomorphic unlabeled n-element semiorders of length at most one.

5.2 The number of nonisomorphic trees derived from semiorders of length at most one

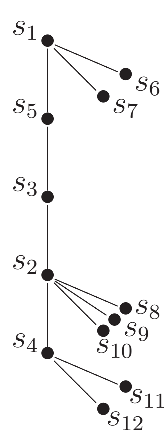
In this subsection, we take a closer look at the unlabeled semiorders of length at most one. For an n-element semiorder S of length at most one, and exactly m elements on the first level, let $\rho(S) = (r_1, r_2, \ldots, r_m, 0, 0, \ldots, 0)$, where there are $n-m$ 0's and $n - m \geq r_1 \geq r_2 \geq \cdots \geq r_m \geq 0$. Let the elements of the semiorder be $s_1, s_2, \ldots, s_n$, with s_i corresponding to r_i, $1 \leq i \leq m$, and then $s_1, s_2, \ldots, s_m$ are on the upper level.

For a permutation $\sigma = (a_1, a_2, \ldots, a_m)$ of $\{1, 2, \ldots, m\}$, if we add the relations $s_{a_1} > s_{a_2} > \cdots > s_{a_m}$ to the original semiorder, we get a tree with the main trunk $s_{a_1} > s_{a_2} > \cdots > s_{a_m}$, and the elements $s_{m+1}, s_{m+2}, \ldots, s_n$ attached to one of the elements on the main trunk in the following manner. For $m + 1 \leq i \leq n$, element s_i is attached to element s_j on the main trunk if and only if $s_i < s_j$, and s_i is incomparable with all elements on the main trunk that are below s_j, $1 \leq j \leq m$. We denote the tree derived from semiorder S and permutation σ by $T(S, \sigma)$.

For example, the Hasse diagram of the semiorder S with $\rho(S) = (7, 5, 4, 2, 1, 0, 0, 0, 0, 0, 0)$ is as follows:



Suppose that $\sigma = (1, 5, 3, 2, 4)$, and then we add the relations $s_1 > s_5 > s_3 > s_2 > s_4$ to the original semiorder. Then $T(S, \sigma)$ is



Here we call $s_1 > s_5 > s_3 > s_2 > s_4$ the main trunk, and say elements s_6, s_7 are attached to s_1, elements s_8, s_9, s_{10} are attached to s_2, and elements s_{11}, s_{12} are attached to s_4.

The idea of transforming a semiorder of length at most one to a tree is suggested by R. Stanley, in the context of finding the number of linear extensions of n-element semiorders of length at most one. Though this idea may not be useful in its original context, we can give a different application.

Theorem 24. *Given an n-element semiorder R_m of length at most one and exactly m elements on the first level, let $\rho(R_m) = (r_1, r_2, \ldots, r_n)$. If $r_i \neq r_j$ for any $1 \leq i < j \leq m$, then the number of nonisomorphic unlabeled trees in $\{T(R_m, \sigma)|\sigma \in S_m\}$ is the Catalan number C_m.*

Proof. A permutation $\sigma = (a_1, a_2, \ldots, a_m)$ of $\{1, 2, \ldots, m\}$ uniquely determines the main trunk. For $m + 1 \leq i \leq n$, assume element s_i is smaller than t_i elements. Then s_i must be smaller than $s_1, s_2, \ldots, s_{t_i}$ and is not comparable with the other elements. Let s_{u_i} be the lowest element on the main trunk among $s_1, s_2, \ldots, s_{t_i}$. Then s_i is attached to s_{u_i} as a leaf, meaning s_i is smaller than s_{u_i} but not comparable with any element on the main trunk below s_{u_i}.

As a result, for an element s_u on the main trunk, s_u has leaves only if it is the lowest element on the main trunk among $s_1, s_2, \ldots, s_t$, for some $1 \leq t \leq m$. In other words, assume $b_1 < b_2 < \cdots < b_k$ are the set of right-to-left minima of the permutation $\sigma = (a_1, a_2, \ldots, a_m)$, and then only the elements $s_{b_1}, s_{b_2}, \ldots, s_{b_k}$ may have leaves. Further notice that the numbers of leaves attached to $s_{b_1}, s_{b_2}, \ldots, s_{b_k}$ are $r_{b_1} - r_{b_2}, r_{b_2} - r_{b_3}, \ldots, r_{b_{k-1}} - r_{b_k}, r_{b_k}$, respectively, and $r_i \neq r_j$ for any $1 \leq i < j \leq m$. Therefore, for a given n-element semiorder R_m of length at most one and exactly m elements on the first level, the value and position of the right-to-left minima of the permutation $\sigma = (a_1, a_2, \ldots, a_m)$ uniquely determines $T(R_m, \sigma)$.

For instance, in the example above, we have $\sigma = (1, 5, 3, 2, 4)$ and $\rho(R_5) = (r_1, r_2, \ldots, r_{12}) = (7, 5, 4, 2, 1, 0, 0, 0, 0, 0, 0, 0)$. The right-to-left-minima of σ and their positions with σ is given by $1, *, *, 2, 4$. Then the corresponding tree $T(R_5, \sigma)$ has five nodes on the main trunk, $r_1 - r_2 = 2$ leaves attached to the first node, $r_2 - r_4 = 3$ leaves attached to the forth node, and $r_4 = 2$ leaves attached to the fifth node.

Therefore, the number of all possible nonisomorphic unlabeled trees in $\{T(R_m, \sigma)|\sigma \in S_m\}$ is equal to the number of ways to specify the values and positions of the right-to-left minima of permutations $\sigma \in S_m$. That is, if we let $RtLM(\sigma) = \{(a, \sigma(a))|1 \leq a \leq m, \sigma(a) \text{ is a right-to-left minima in } \sigma\}$, then $\#\{T(R_m, \sigma)|\sigma \in S_m\} = \#\{RtLM(\sigma)|\sigma \in S_m\}$. We calculate $\#\{RtLM(\sigma)|\sigma \in S_m\}$ in the following lemma.

Lemma 25. *Let $RL(\sigma)$ be the number of right-to-left minima of the permutation σ. For $1 \leq k \leq m$, let $f(m, k) = \#\{RtLM(\sigma)|\sigma \in S_m, RL(\sigma) = k\}$. Then $f(m, k) = N(m, k) = \frac{1}{m}\binom{m}{k}\binom{m}{k-1}$, a Narayana number.*

For example, for $m = 3$ and $k = 2$, $f(3, 2) = \#\{RtLM(\sigma)|\sigma \in S_3, RL(\sigma) = 2\} = \#\{RtLM(\sigma)|\sigma = (23), (12), \text{or } (132)\} = \#\{\{(3, 2), (1, 1)\}, \{(3, 3), (2, 1)\}, \{(3, 2), (2, 1)\}\} = 3$.

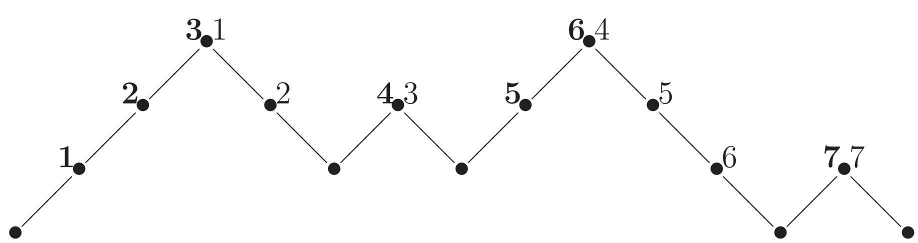
Proof. For $1 \leq k \leq m$, the Narayana number $N(m, k)$ is equal to the number of Dyck paths of semilength m with k peaks, which are the turning points from a $(1, 1)$ step to a $(1, -1)$ step on the path. We prove the lemma by establishing a bijection between (i) Dyck paths of semilength m with k peaks, and (ii) the collection of different $RtLM(\sigma)$'s for $\sigma \in S_m, RL(\sigma) = k$. We define a map from (i) to (ii) as follows:

Given a Dyck path of semilength m with k peaks, let us read the Dyck path from left to right and do the following:

- Label the endpoints of $(1, 1)$ steps from left to right with 1 to m. Since there are m $(1, 1)$ steps, there should be m such endpoints.
- Label the startpoints of $(1, -1)$ steps from left to right with 1 to m. Since there are m $(1, -1)$ steps, there should be m such startpoints.
- Notice that a point on the Dyck path is a peak if and only if it is both an endpoint of a $(1, 1)$ step and a startpoint of a $(1, -1)$ step. Let (i, j) be the coordinate of a peak, if the peak is the i^{th} endpoint and the j^{th} startpoint. Assume the coordinate of the k peaks are $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$. Then $a_1 < a_2 < \cdots < a_k = m$ and $1 = b_1 < b_2 < \cdots < b_k$.
- Obtain a specification of the values and positions of the right-to-left minima of a permutation by putting the number b_i on position a_i, $1 \leq i \leq k$.

Proposition 26. *The above map is valid, i.e., $\{(a_i, b_i), 1 \leq i \leq k\} = RtLM(\sigma)$, for some $\sigma \in S_m, RL(\sigma) = k$, and the above map is a bijection.*

For example, if we have the following Dyck path:



The semilength of the Dyck path is $m = 7$, and it has 4 peaks. The numbers in bold face are the labels for the endpoints of $(1, 1)$ steps, and the numbers in ordinary type are the labels for the startpoints of $(-1, 1)$ steps. Then the coordinates of the 4 peaks are $(3, 1), (4, 3), (6, 4), (7, 7)$. We put 1 on position 3, 3 on position 4, 4 on position 6, 7 on position 7, and then we get a possible specification of the values and positions of the right-to-left minima of a permutation:

$$*, *, 1, 3, *, 4, 7. \tag{16}$$

Proof. • We first show that the above map gives us a valid specification of the values and positions of the right-to-left minima of some permutation $\sigma \in S_m$. We prove the validity by constructing such σ.

Assume the labels of the $m-k$ endpoints of the $(1,1)$ steps which are not peaks, are $c_1 < c_2 < \cdots < c_{m-k}$; assume the labels of the $m-k$ startpoints of the $(1,-1)$ steps which are not peaks are $d_1 < d_2 < \cdots < d_{m-k}$. Then

$$\{a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_{m-k}\} = \{b_1, b_2, \ldots, b_k, d_1, d_2, \ldots, d_{m-k}\} = \{1, 2, \ldots, m\}.$$

Since the path never goes below the x-axis, we must have $c_j < d_j$ for every $1 \le j \le m-k$.

Let σ be a permutation such that $\sigma(a_i) = b_i$, $1 \le i \le k$, and $\sigma(c_j) = d_j$, $1 \le j \le m-k$. We will show that $RtLM(\sigma) = \{(a_i, b_i), 1 \le i \le k\}$.

For every j, $1 \le j \le m-k$, since the Dyck path never goes below the x axis, there must be a peak i on the path between c_j and d_j. Then $c_j < a_i$ and $b_i < d_j$, and thus in σ, $b_i = \sigma(a_i)$ is smaller than $d_j = \sigma(c_j)$, while $a_i > c_j$, i.e., b_i is to the right of d_j. Therefore d_j cannot be a right-to-left minimum.

On the other hand, for every i, $1 \le i \le k$, there does not exist some $1 \le i' \le k$ such that $b_{i'} < b_i$ and $a_{i'} > a_i$. In addition, if there exists some $1 \le j \le m-k$ such that $d_j = \sigma(c_j) < b_i = \sigma(a_i)$ and $c_j < a_i$, then by the above paragraph there exists $1 \le i' \le k$, such that $b_{i'} < d_j < b_i$ and $a_{i'} > c_j > a_i$. We obtain a contradiction. As a result, $RtLM(\sigma) = \{(a_i, b_i), 1 \le i \le k\}$.

For example, let us construct the permutation σ for the above example: $c_1 = 1$, $c_2 = 2$, $c_3 = 5$ and $d_1 = 2$, $d_2 = 5$, $d_3 = 6$. We obtain $\sigma = (2, 5, 1, 3, 6, 4, 7)$, and this permutation exactly corresponds to the right-to-left minima as shown in Example 16.

• We now show that the inverse of the map is well-defined, and thus the map is bijective.

For a given specification $\{(a_i, b_i), 1 \le i \le k\} = RtLM(\sigma)$, for some $\sigma \in S_m$, we have $a_1 < a_2 < \cdots < a_k = m$ and $1 = b_1 < b_2 < \cdots < b_k$. We construct the corresponding Dyck path as follows: when we walk along the path from left to right, we first walk up a_1 steps and then turn down, and walk down $b_2 - b_1$ steps and then turn up. We continue to walk up $a_2 - a_1$ steps and then turn down, and walk down $b_3 - b_2$ steps and then turn up. In general, we walk up $a_i - a_{i-1}$ steps and then turn down, and walk down $b_{i+1} - b_i$ steps, $2 \le i \le m-1$. In the end, we walk up $a_m - a_{m-1}$ steps and walk down $m + 1 - b_m$ steps. During the walk, we walk up in total $a_1 + (a_2 - a_1) + \cdots + a_m - a_{m-1} = a_m = m$ steps, and walk down in total $(b_2 - b_1) + \cdots + (b_m - b_{m-1}) + m + 1 - b_m = m + 1 - b_1 = m$ steps, and we make k turns from up to down. Since $\{(a_i, b_i), 1 \le i \le k\}$ is a collection of right-to-left minima of some permutation, for any $1 \le i \le k-1$, $a_i \le a_{i+1} - 1 \le b_{i+1} - 1 = b_{i+1} - b_1$, so we never walk below the x-axis on the path. Therefore we get a unique Dyck path with semilength m and k peaks. Hence the inverse map is well-defined, so Lemma 25 is proved.

$\square$

$\square$

To conclude the proof of Theorem 24, note that the Catalan number $C_m = \sum_{k=1}^{m} N(m,k)$. Hence, we have that the number of nonisomorphic unlabeled trees in $\{T(R_m,\sigma)|\sigma \in S_m\}$ is equal to $\#\{RtLM(\sigma)|\sigma \in S_m\} = \sum_{k=1}^{m} f(m,k) = \sum_{k=1}^{m} N(m,k) = C_m = \frac{1}{m+1}\binom{2m}{m}$. $\square$

Remark 27. Theorem 24, along with Lemma 25, gives another combinatorial explanation of the Catalan number.

5.3 The generating function for the number of nonisomorphic labeled n-element semiorders of length at most one

Recall that an ordered partition of a set is a partition of the set into some pairwise disjoint nonempty subsets, together with a linear ordering of these subsets. From the generating function (9) for $G_{\leq h}(x)$, we get $G_{\leq 1}(x) = (1-e^{-x})/(2e^{-x}-1) = (e^x-1)/(2-e^x)$, which is exactly the exponential generating function for the number of ordered partitions [5, p. 472]. As a result, we can get the following theorem:

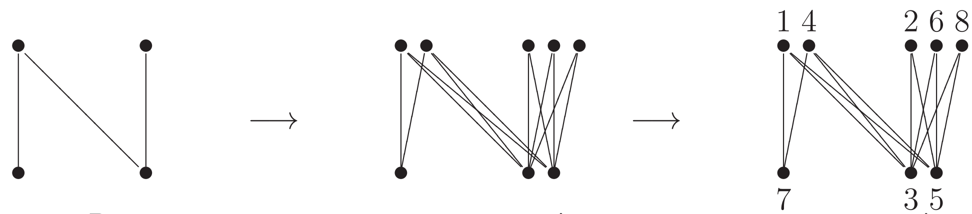
Theorem 28. *The number of nonisomorphic labeled n-element semiorders of length at most one is equal to the number of ordered partitions of $[n]$.*

We give a simple bijective proof to Theorem 28.

Proposition 29. *For an ordered partition $(A_1,\ldots,A_k)$ of $[n]$, let $|A_i| = a_i$, $1 \leq i \leq k$, so $n = a_1 + a_2 + \cdots + a_k$. Define the semiorder R by $\rho(R) = (m, m-1, \ldots, 1, 0, \ldots, 0)$, where $m = \lfloor \frac{k}{2} \rfloor$, and there are $\lceil \frac{k}{2} \rceil$ 0's. Then R has k elements. Say the elements are $t_1, t_2, \ldots, t_k$, with t_i corresponding to the i^{th} entry of R's integer vector. Let R' be another semiorder such that its contraction $c(R')$ (defined in Lemma 13) is R, and in R', the sizes of the equivalence classes are $a_1, a_2, \ldots, a_k$, respectively, with a_i corresponding to equivalence class t_i. Label the elements in the i^{th} equivalence class with the corresponding numbers in A_i, $1 \leq i \leq k$.*

We claim that the above defines a bijective map from ordered partitions of $[n]$ to n-element labeled semiorders R' of length at most one.

For example, if we have an ordered partition $\{1,4\}\{2,6,8\}\{7\}\{3,5\}$, then $k=4$, $m=2$, and $(a_1,a_2,a_3,a_4) = (2,3,1,2)$. We have $\rho(R) = (2,1,0,0)$, and then the map works as follows:



R unlabeled R' labeled R'

Proof. We first show that the map takes an ordered partition of $[n]$ to a labeled n-element semiorder of length at most one. The size of every equivalence class of R is one, and thus semiorder R is a valid contraction. The length of R is at most one, and thus R' also has length at most one. In addition, R has k elements, so R' has k equivalence classes. Therefore, we can construct the equivalence classes of R to have sizes $a_1, a_2, \ldots, a_k$. Moreover, within an equivalence class with a_i elements, $1 \leq i \leq k$, since we only consider nonisomorphic semiorders, it does not matter which of the a_i numbers in A_i is assigned to which element in this equivalence class. Hence the way to label elements is unique up to isomorphism. Thus each ordered partition of $[n]$ uniquely corresponds to a labeled n-element semiorder of length at most one.

Next we show that the inverse map is well-defined and uniquely determines an ordered partition of $[n]$. Let the labeled n-element semiorder R' of length at most one have k equivalence classes. Let us group up the labels within every equivalence class, so

$$[n] = \bigcup \{\text{labels in each equivalence class}\}.$$

To obtain an ordered partition of $[n]$, it suffices to find the way to order the k equivalence classes of R'. The contraction $c(R')$ must have length at most one with $\rho(c(R')) = (\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor - 1, \ldots, 1, 0, \ldots, 0)$, where there are $\lceil \frac{k}{2} \rceil$ 0's.

Order the k elements of $c(R')$ such that the i^{th} element corresponds to the i^{th} entry of $\rho(c(R'))$. Afterwards, we can order the k equivalence classes of R' correspondingly. Thus we get a unique ordered partition of $[n]$, so the inverse map is well-defined. $\square$

5.4 The number of nonisomorphic unlabeled n-element semiorders of length at most three

Theorem 30. *For $n \geq 1$, we have*

$$f^n_{\leq 3} = \frac{3^{n-1} + 1}{2}. \tag{17}$$

Corollary 31. *For $n \geq 2$, $f^n_3 = 3f^{n-1}_3 + f^{n-2}_{\leq 2} - 1 = 3f^{n-1}_3 + f^{n-2}_2 + f^{n-2}_1$.*

Proof. By Theorem 18, $f^n_{\leq 2} = 3f^{n-1}_{\leq 2} - f^{n-2}_{\leq 2}$. By equation (17), $f^n_{\leq 3} = 3f^{n-1}_{\leq 3} - 1$. Therefore,

$$\begin{aligned} f^n_3 = f^n_{\leq 3} - f^n_{\leq 2} &= 3f^{n-1}_{\leq 3} - 1 - (3f^{n-1}_{\leq 2} - f^{n-2}_{\leq 2}) \\ &= 3(f^{n-1}_{\leq 3} - f^{n-1}_{\leq 2}) + f^{n-2}_{\leq 2} - 1 \\ &= 3f^{n-1}_3 + f^{n-2}_2 + f^{n-2}_1. \end{aligned}$$

□

Remark 32. Theorem 30 can be directly derived from equation (8), or from the recurrence formula in Theorem 18. Like Theorem 22, there might also be a more straightforward bijective proof for Theorem 30, which may give us a more intuitive way to understand why $f^n_{\leq 3}$ grows roughly exponentially with a base 3. This leaves an open question for this paper.

This paper studies the number of unlabeled semiorders of size n and length H and gives an explicit formula for this number by establishing a bijection between semiorders and ordered trees. With this result, we derive a series of related results including the number of fixed-length labeled semiorders, some recurrence relations of semiorders of fixed length, and some interesting properties of the number of semiorders of small length. Bijections are widely used in this paper and as can be seen, bijective proofs can sometimes provide us pretty neat ways to view the semiorder and its relationship with some other sets of combinatorial objects. Further research may combine the results of this paper with some other bijective results of $(\mathbf{2}+\mathbf{2})$-free posets and find more interesting results about semiorders.

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