Journal of Integer Sequences, Vol. 17 (2014),
Article 14.1.8

On a Conjecture of Farhi

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Abstract

Recently, Farhi showed that every natural number $N \not\equiv 2 \pmod{24}$ can be written as the sum of three numbers of the form $\left\lfloor \frac{n^2}{3} \right\rfloor$ $(n \in \mathbb{N})$. He conjectured that this result remains true even if $N \equiv 2 \pmod{24}$. In this note, we prove this statement.

1 Introduction

Throughout this note, we let $\mathbb{N}$ and $\mathbb{Z}$, respectively, denote the set of the non-negative integers and the set of the integers. We let $\lfloor \cdot \rfloor$ and $\langle \cdot \rangle$ denote the integer-part and the fractional-part functions. Let X be a set. We denote the cardinality of X by $\#X$. We also recall that $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

Recently, Farhi [1] showed that every natural number $N \not\equiv 2 \pmod{24}$ can be written as the sum of three numbers of the form $\left\lfloor \frac{n^2}{3} \right\rfloor$ $(n \in \mathbb{N})$. He conjectured that this result remains true even if $N \equiv 2 \pmod{24}$. We recall his conjecture.

Conjecture 1. Every natural number can be written as the sum of three numbers of the form $\left\lfloor \frac{n^2}{3} \right\rfloor$ $(n \in \mathbb{N})$.

In fact, he proposed a more general conjecture.

Conjecture 2. Let $k \geq 2$ be an integer. There then exists a positive integer $a(k)$ that satisfies the following property: every natural number can be written as the sum of $k+1$ numbers of the form $\left\lfloor \frac{n^k}{a(k)} \right\rfloor$ $(n \in \mathbb{N})$.

In this note, we prove Conjecture 1.

2 Proof of Conjecture 1

We recall Legendre's theorem [3, pp. 331–339], which is a necessary tool for our proof:

Theorem 3. *Every natural number not of the form $4^h(8k+7)(h,k \in \mathbb{N})$ can be represented as the sum of three squares of natural numbers.*

We note that since $4^h(8k+7)$ is congruent to $0, 4$ or 7 modulo 8, every natural number not congruent to $0, 4$ or 7 modulo 8 can be represented as the sum of three squares of natural numbers. We will use this result later.

Let $r_3(n)$ be the number of representations of the positive integer n as the sum of three squares of integers. The following theorem provides an interesting formula for $r_3(n)$, which can be proven using the theory of modular functions.

Theorem 4 (see [2])**.** *For any positive integer n, we have*

$$r_3(n) = \frac{16}{\pi}\sqrt{n}\chi_2(n)K(-4n)\prod_{p^2|n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{b-1}}+\frac{1}{p^b}\left(1-\left(\frac{-p^{-2b}n}{p}\right)\frac{1}{p}\right)^{-1}\right),$$

where $b = b(p)$ is the largest integer such that $p^{2b} \mid n$,

$$K(-4n) = \sum_{m=1}^{\infty}\left(\frac{-4n}{m}\right)\frac{1}{m},$$

and if 4^a is the highest power of 4 dividing n, then

$$\chi_2(n) = \begin{cases} 0, & \text{if } 4^{-a}n \equiv 7 \pmod 8; \\ \frac{1}{2^a}, & \text{if } 4^{-a}n \equiv 3 \pmod 8; \\ \frac{3}{2^{a+1}}, & \text{if } 4^{-a}n \equiv 1,2,5,6 \pmod 8). \end{cases}$$

□

We will require the following technical lemma.

Lemma 5. *For any positive integer $n \equiv 1 \pmod 8$, we have*

$$r_3(9n) > \frac{3}{2}\, r_3(n).$$

Proof. We have

$$r_3(9n) = \frac{16}{\pi}\sqrt{9n}\chi_2(9n)K(-36n)\times$$
$$\prod_{p^2|9n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{b'-1}}+\frac{1}{p^{b'}}\left(1-\left(\frac{-9\,p^{-2b'}n}{p}\right)\frac{1}{p}\right)^{-1}\right),$$

where $b' = b'(p)$ denotes the largest integer for which $p^{2b'} \mid 9n$. Since $n \equiv 1 \pmod 8$, it follows that $4^0 = 1$ is the highest power of 4 dividing n. This result implies that $\chi_2(n) = \frac{3}{2}$. Similarly, we have $9n \equiv 1 \pmod 8$. Thus, $4^0 = 1$ is the highest power of 4 dividing $9n$, which gives $\chi_2(9n) = \chi_2(n) = \frac{3}{2}$. Conversely, it follows from [2, p. 84] that

$$K(-36n) = K(-4\times 3^2 \times n) = \left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)K(-4n).$$

Since $n \equiv 1 \pmod 8$, it follows from Legendre's theorem that n can be represented as the sum of three squares of natural numbers. Thus, $r_3(n) \neq 0$. Dividing through by $r_3(n)$ then yields an identity equivalent to

$$\frac{r_3(9n)}{r_3(n)} = \frac{3}{\left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \frac{\prod_{p^2|9n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{b'-1}}+\frac{1}{p^{b'}}\left(1-\left(\frac{-9\,p^{-2b'}n}{p}\right)\frac{1}{p}\right)^{-1}\right)}{\prod_{p^2|n}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{b-1}}+\frac{1}{p^{b}}\left(1-\left(\frac{-p^{-2b}n}{p}\right)\frac{1}{p}\right)^{-1}\right)}.$$

Let $p \neq 3$ with $p^2 \mid n$. Thus, $b' = b'(p)$ is the largest integer for which $p^{2b'} \mid n$. Therefore, one obtains $b' = b'(p) = b(p) = b$. Furthermore, we have

$$\left(\frac{-9\,p^{-2b'}n}{p}\right) = \left(\frac{3^2}{p}\right)\left(\frac{-p^{-2b'}n}{p}\right) = \left(\frac{-p^{-2b'}n}{p}\right) = \left(\frac{-p^{-2b}n}{p}\right).$$

For every $p \neq 3$ with $p^2 \mid n$, we then have $1+\frac{1}{p}+\cdots+\frac{1}{p^{b'-1}}+\frac{1}{p^{b'}}\left(1-\left(\frac{-9\,p^{-2b'}n}{p}\right)\frac{1}{p}\right)^{-1} = 1+\frac{1}{p}+\cdots+\frac{1}{p^{b-1}}+\frac{1}{p^{b}}\left(1-\left(\frac{-p^{-2b}n}{p}\right)\frac{1}{p}\right)^{-1}$. Thus, two cases are evident: if $3^2 \mid n$, then

$$\frac{r_3(9n)}{r_3(n)} = \frac{3}{\left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \frac{1+\frac{1}{3}+\cdots+\frac{1}{3^{b'-1}}+\frac{1}{3^{b'}}\left(1-\left(\frac{-9\times 3^{-2b'}n}{3}\right)\frac{1}{3}\right)^{-1}}{1+\frac{1}{3}+\cdots+\frac{1}{3^{b-1}}+\frac{1}{3^{b}}\left(1-\left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}},$$

Otherwise, 3^2 does not divide n, so

$$\frac{r_3(9n)}{r_3(n)} = \frac{3}{\left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \left(1+\cdots+\frac{1}{3^{b'-1}}+\frac{1}{3^{b'}}\left(1-\left(\frac{-9\times 3^{-2b'}n}{3}\right)\frac{1}{3}\right)^{-1}\right).$$

We now show that in all cases, $r_3(9n) > \frac{3}{2}\, r_3(n)$.

- If 3^2 does not divide n, $b^{'} = b^{'}(3) = 1$ is implied to be the largest integer for which $3^{2b^{'}} \mid 9n$. One obtains

$$\frac{r_3(9n)}{r_3(n)} = \frac{3}{\left(1 - \left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \left(1 + \frac{1}{3}\left(1 - \left(\frac{-n}{3}\right)\frac{1}{3}\right)^{-1}\right).$$

  We have $\left(1 - \left(\frac{-4n}{3}\right)\frac{1}{3}\right) = 1, \frac{2}{3}$ or $\frac{4}{3}$ and so $\frac{3}{\left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} > \frac{3}{2}$, which gives the result $r_3(9n) > \frac{3}{2}\, r_3(n)$.

- If $3^2 \mid n$, then b (respectively $b^{'}$) is the largest integer for which $3^{2b} \mid n$ (respectively $3^{2b^{'}} \mid 9n$). Hence,

$$\begin{aligned}\frac{r_3(9n)}{r_3(n)} &= \frac{3}{\left(1 - \left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \frac{1 + \frac{1}{3} + \cdots + \frac{1}{3^b} + \frac{1}{3^{b+1}}\left(1 - \left(\frac{-9\times 3^{-2(b+1)}n}{3}\right)\frac{1}{3}\right)^{-1}}{1 + \frac{1}{3} + \cdots + \frac{1}{3^{b-1}} + \frac{1}{3^b}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}} \\ &= \frac{3}{\left(1 - \left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} \times \frac{1 + \frac{1}{3} + \cdots + \frac{1}{3^b} + \frac{1}{3^{b+1}}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}}{1 + \frac{1}{3} + \cdots + \frac{1}{3^{b-1}} + \frac{1}{3^b}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}}.\end{aligned}$$

  We have $\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right) = 1, \frac{2}{3}$ or $\frac{4}{3}$. One obtains the following in all cases:

$$\frac{1}{3^b} + \frac{1}{3^{b+1}}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1} \geq \frac{1}{3^b}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}.$$

  This result implies $1 + \frac{1}{3} + \cdots + \frac{1}{3^b} + \frac{1}{3^{b+1}}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1} \geq 1 + \frac{1}{3} + \cdots + \frac{1}{3^{b-1}} + \frac{1}{3^b}\left(1 - \left(\frac{-3^{-2b}n}{3}\right)\frac{1}{3}\right)^{-1}$. Conversely, $\frac{3}{\left(1-\left(\frac{-4n}{3}\right)\frac{1}{3}\right)^{-1}} > \frac{3}{2}$. Thus, we obtain the desired result, $r_3(9n) > \frac{3}{2}\, r_3(n)$.

□

Theorem 6. *Every natural number $N \equiv 2 \pmod{24}$ can be written as the sum of three numbers of the form $\left\lfloor \frac{n^2}{3} \right\rfloor$ $(n \in \mathbb{N})$.*

Proof. We may write $N = 2 + 24k$ with $k \in \mathbb{N}$. Thus, $3N + 3 = 9(1 + 8k)$. We now define two sets S_1 and S_2 as follows:

$$S_1 = \left\{(a, b, c) \in \mathbb{Z}^3 : a^2 + b^2 + c^2 = 1 + 8k\right\},$$
$$S_2 = \left\{(a, b, c) \in \mathbb{Z}^3 : a^2 + b^2 + c^2 = 9(1 + 8k)\right\}.$$

By the definition of r_3, we have $\#S_2 = r_3(9(1+8k))$ and $\#S_1 = r_3(1+8k)$. Since $1+8k \equiv 1 \pmod 8$, we apply Lemma 5 to obtain $r_3(9(1+8k)) > \frac{3}{2}\, r_3(1+8k) \geq r_3(1+8k)$. One obtains $r_3(9(1+8k)) > r_3(1+8k)$, which is equivalent to $\#S_2 > \#S_1$. We note that this last result is the key to the proof. Let us define the map

$$\begin{array}{llcl} f: & S_1 & \longrightarrow & S_2 \\ & (a,b,c) & \longmapsto & (3a,3b,3c). \end{array}$$

We see easily that f is well defined and injective. Since $\#S_2 > \#S_1$, we can find $(a,b,c) \in S_2$ such that $(a,b,c) \notin f(S_1)$. Furthermore, we have $a^2+b^2+c^2 = 9(1+8k) \equiv 0 \pmod 3$, then either $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod 3$ or $a^2 \equiv b^2 \equiv c^2 \equiv 0 \pmod 3$. The last case cannot hold because one of the elements, a, b and c, is not divisible by 3 ($(a,b,c) \notin f(S_1)$). Thus, $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod 3$ and we have

$$\begin{aligned} N+1 &= 3(1+8k) \\ &= \frac{a^2}{3} + \frac{b^2}{3} + \frac{c^2}{3} \\ &= \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor + \left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle. \end{aligned}$$

Since $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod 3$, then $\left\langle \frac{a^2}{3} \right\rangle + \left\langle \frac{b^2}{3} \right\rangle + \left\langle \frac{c^2}{3} \right\rangle = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, which gives $N = \left\lfloor \frac{a^2}{3} \right\rfloor + \left\lfloor \frac{b^2}{3} \right\rfloor + \left\lfloor \frac{c^2}{3} \right\rfloor$. We replace $(a,b,c) \in \mathbb{Z}^3$ by $(|a|,|b|,|c|) \in \mathbb{N}^3$ to obtain the desired solution. The conjecture is proven. □

3 Acknowledgments

The authors would like to thank the editor and the anonymous reader for their invaluable comments.

This work was supported by Académie Hassan 2 under the project "Mathématiques et Applications: cryptographie" and URAC6-CNRST.

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2010 *Mathematics Subject Classification*: Primary 11B13.
Keywords: additive base, Legendre's theorem, representation of an integer as the sum of three squares.

Received July 25 2013; revised versions received July 31 2013; November 22 2013; December 30 2013. Published in *Journal of Integer Sequences*, December 30 2013. Revision, January 3 2014.

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