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Two Properties of Catalan-Larcombe-French Numbers

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Abstract

Let (P_n) be the Catalan-Larcombe-French numbers. The numbers P_n occur in the theory of elliptic integrals, and are related to the arithmetic-geometric-mean. In this paper we investigate the properties of the related sequence $S_n = P_n/2^n$ instead, since S_n is an Apéry-like sequence. We prove a congruence and an inequality for P_n.

1 Introduction

Let (P_n) be the sequence given by

$$P_0 = 1,\ P_1 = 8 \quad \text{and} \quad (n+1)^2 P_{n+1} = 8(3n^2+3n+1)P_n - 128n^2 P_{n-1}\ (n \geq 1). \tag{1}$$

The numbers P_n are called Catalan-Larcombe-French numbers since Catalan first defined P_n in [1], and Larcombe and French [6] proved that

$$P_n = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-4)^k \binom{2n-2k}{n-k}^2 \binom{n-k}{k} = \sum_{k=0}^{n} \frac{\binom{2k}{k}^2 \binom{2n-2k}{n-k}^2}{\binom{n}{k}},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x. The numbers P_n are related to the arithmetic-geometric-mean. See [6] and A053175 in Sloane's "On-Line Encyclopedia of Integer Sequences".

Let (S_n) be defined by

$$S_0 = 1,\ S_1 = 4 \quad \text{and} \quad (n+1)^2 S_{n+1} = 4(3n^2+3n+1)S_n - 32n^2 S_{n-1}\ (n \geq 1). \tag{2}$$

Comparing (2) with (1), we see that

$$S_n = \frac{P_n}{2^n}.$$

Zagier noted that

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k}^2 \binom{n}{2k} 4^{n-2k}.$$

As observed by Jovović [7] in 2003 ,

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} \quad (n = 0, 1, 2, \ldots).$$

Recently Z. W. Sun stated that

$$S_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{k}{n-k}(-4)^{n-k} = \frac{1}{(-2)^n} \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{k}{n-k}(-4)^k.$$

The first few values of S_n are shown below:

$$\begin{aligned} &S_0 = 1,\ S_1 = 4,\ S_2 = 20,\ S_3 = 112,\ S_4 = 676,\ S_5 = 4304,\ S_6 = 28496, \\ &S_7 = 194240,\ S_8 = 1353508,\ S_9 = 9593104,\ S_{10} = 68906320, \\ &S_{11} = 500281280,\ S_{12} = 3664176400,\ S_{13} = 27033720640. \end{aligned}$$

Let p be an odd prime. Jarvis, Larcombe, and French [3] proved that if $n = a_r p^r + \cdots + a_1 p + a_0$ with $a_0, a_1, \ldots, a_r \in \{0, 1, \ldots, p-1\}$, then

$$P_n \equiv P_{a_r} \cdots P_{a_1} P_{a_0} \pmod{p}.$$

Jarvis and Verrill [5] showed that

$$P_n \equiv (-1)^{\frac{p-1}{2}} 128^n P_{p-1-n} \pmod{p} \quad \text{for} \quad n = 0, 1, \ldots, p-1$$

and

$$P_{mp^r} \equiv P_{mp^{r-1}} \pmod{p^r} \quad \text{for} \quad m, r \in \mathbb{Z}^+,$$

where $\mathbb{Z}^+$ is the set of positive integers.

For a prime p let $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p. Let p be an odd prime, $n \in \mathbb{Z}_p$ and $n \not\equiv 0, -16 \pmod{p}$. The second author [11] proved that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{S_k}{(n+16)^k} \equiv \Big(\frac{n(n+16)}{p}\Big) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{n^{2k}} \pmod{p},$$

where $(\frac{a}{p})$ is the Legendre symbol.

In 1894 Franel [2] introduced the following Franel numbers (f_n):

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots).$$

The first few Franel numbers are as below:

$$f_0 = 1,\ f_1 = 2,\ f_3 = 10,\ f_4 = 56,\ f_5 = 346,\ f_6 = 2252,\ f_7 = 15184.$$

Franel [2] noted that the sequence (f_n) satisfies the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n^2 + 7n + 2) f_n + 8n^2 f_{n-1}\ (n \geq 1).$$

Let $r \in \mathbb{Z}^+$ and p be a prime with $p \equiv 5, 7 \pmod{8}$. The second author [11] conjectured that

$$S_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r} \quad \text{and} \quad f_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r}. \tag{3}$$

In this paper we prove (3) in the case $r = 2$. We also prove the second author's conjecture [11]:

$$\Big(1 + \frac{1}{m(m-1)}\Big) S_m^2 > S_{m+1} S_{m-1} \quad \text{for} \quad m = 2, 3, \ldots.$$

2 Basic lemmas

Lemma 1 (Lucas' theorem [8]). *Let p be an odd prime. Suppose $a = a_r p^r + \cdots + a_1 p + a_0$ and $b = b_r p^r + \cdots + b_1 p + b_0$, where $a_r, \ldots, a_0, b_r, \ldots, b_0 \in \{0, 1, \ldots, p-1\}$. Then*

$$\binom{a}{b} \equiv \binom{a_r}{b_r} \cdots \binom{a_0}{b_0} \pmod{p}.$$

Lucas' theorem is often formulated as follows.

Lemma 2 ([8])**.** *Let p be an odd prime and $a, b \in \mathbb{Z}^+$. Suppose $a_0, b_0 \in \{0, 1, \ldots, p-1\}$. Then*

$$\binom{ap+a_0}{bp+b_0} \equiv \binom{a}{b}\binom{a_0}{b_0} \pmod{p}.$$

Lemma 3 ([4, Lemma 2.7])**.** *For any positive integer n we have*

$$S_n = 2\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{2k}{k}\binom{2n-2k}{n-k}.$$

Lemma 4 ([9])**.** *Let p be an odd prime. Suppose $n = n_1p + n_0$ and $k = k_1p + k_0$ with $k_1, n_1 \in \mathbb{Z}^+$ and $k_0, n_0 \in \{0, 1, \ldots, p-1\}$. Then*

$$\binom{n}{k} \equiv \binom{n_1}{k_1}\left((1+n_1)\binom{n_0}{k_0} - (n_1+k_1)\binom{n_0-p}{k_0} - k_1\binom{n_0-p}{k_0+p}\right) \pmod{p^2}.$$

Lemma 5. *Let p be an odd prime. Then*

$$\sum_{t=0}^{(p-1)/2}(-1)^t\left(\binom{\frac{p-1}{2}+t}{t} - \binom{p+\frac{p-1}{2}+t}{p+t}\right)\binom{-\frac{1}{2}}{t}^2 \equiv 0 \pmod{p^2}.$$

Proof. For $0 \le t \le (p-1)/2$, from Lemma 2 we have

$$\begin{aligned}\binom{\frac{p-1}{2}-p}{p+t} &= (-1)^{t+1}\binom{p+\frac{p+1}{2}+t-1}{p+t} \equiv (-1)^{t+1}\binom{\frac{p+1}{2}+t-1}{t} \\ &= -\binom{\frac{p-1}{2}-p}{t} \pmod{p}\end{aligned}$$

and so

$$\binom{\frac{p-1}{2}-p}{t} + \binom{\frac{p-1}{2}-p}{p+t} = (-1)^t\left(\binom{\frac{p-1}{2}+t}{t} - \binom{p+\frac{p-1}{2}+t}{p+t}\right) \equiv 0 \pmod{p}.$$

We first assume $p \equiv 1 \pmod{4}$. Applying Lemma 4 we get

$$\begin{aligned}&\binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}} - \binom{p+\frac{3(p-1)}{4}}{p+\frac{p-1}{4}} \equiv \binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}} - \left(2\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}} - \binom{\frac{3(p-1)}{4}-p}{\frac{p-1}{2}}\right) \\ &= -\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}} + (-1)^{\frac{p-1}{2}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}} = 0 \pmod{p^2}\end{aligned}$$

and

$$
\begin{aligned}
&\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}+\binom{p-1-t}{\frac{p-1}{2}-t}-\binom{p+p-1-t}{p+\frac{p-1}{2}-t}\\
&\equiv-\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{\frac{p-1}{2}-p+t}{\frac{p-1}{2}}-\binom{p-1-t}{\frac{p-1}{2}}+\binom{-1-t}{\frac{p-1}{2}}\\
&=\Big((-1)^{\frac{p-1}{2}}-1\Big)\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\Big((-1)^{\frac{p-1}{2}}-1\Big)\binom{p-1-t}{\frac{p-1}{2}}\\
&=0 \pmod{p^2}.
\end{aligned}
$$

Also,

$$
(-1)^t\binom{-\frac{1}{2}}{t}^2-(-1)^{\frac{p-1}{2}-t}\binom{-\frac{1}{2}}{\frac{p-1}{2}-t}^2\equiv(-1)^t\Big(\binom{\frac{p-1}{2}}{t}^2-\binom{\frac{p-1}{2}}{\frac{p-1}{2}-t}^2\Big)=0 \pmod{p}.
$$

By the above four congruences, we have

$$
\begin{aligned}
&\sum_{t=0}^{(p-1)/2}(-1)^t\Big(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}\Big)\binom{-\frac{1}{2}}{t}^2\\
&=\sum_{t=0}^{(p-5)/4}(-1)^t\binom{-\frac{1}{2}}{t}^2\Big(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}\Big)\\
&\quad+(-1)^{\frac{p-1}{4}}\binom{-\frac{1}{2}}{\frac{p-1}{4}}^2\Big(\binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}-\binom{p+\frac{3(p-1)}{4}}{p+\frac{p-1}{4}}\Big)\\
&\quad+\sum_{t=0}^{(p-5)/4}(-1)^{\frac{p-1}{2}-t}\binom{-\frac{1}{2}}{\frac{p-1}{2}-t}^2\Big(\binom{p-1-t}{\frac{p-1}{2}-t}-\binom{p+p-1-t}{p+\frac{p-1}{2}-t}\Big)\\
&\equiv\sum_{t=0}^{(p-5)/4}\Big((-1)^t\binom{-\frac{1}{2}}{t}^2-(-1)^{\frac{p-1}{2}-t}\binom{-\frac{1}{2}}{\frac{p-1}{2}-t}^2\Big)\Big(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}\Big)\\
&\quad+(-1)^{\frac{p-1}{4}}\binom{-\frac{1}{2}}{\frac{p-1}{4}}^2\Big(\binom{\frac{3(p-1)}{4}}{\frac{p-1}{4}}-\binom{p+\frac{3(p-1)}{4}}{p+\frac{p-1}{4}}\Big)\equiv 0 \pmod{p^2}.
\end{aligned}
$$

Thus the result is true for $p\equiv 1 \pmod 4$.

Now we assume $p\equiv 3 \pmod 4$. By Lemma 4,

$$
\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}\equiv-\Big(\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{p-1-t}{\frac{p-1}{2}}\Big) \pmod{p^2}.
$$

As

$$\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{p-1-t}{\frac{p-1}{2}}\equiv\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{-1-t}{\frac{p-1}{2}}$$
$$=\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+(-1)^{\frac{p-1}{2}}\binom{t+\frac{p-1}{2}}{\frac{p-1}{2}}=0 \pmod p$$

and

$$(-1)^t\binom{-\frac{1}{2}}{t}^2+(-1)^{\frac{p-1}{2}-t}\binom{-\frac{1}{2}}{\frac{p-1}{2}-t}^2\equiv(-1)^t\left(\binom{\frac{p-1}{2}}{t}^2-\binom{\frac{p-1}{2}}{\frac{p-1}{2}-t}^2\right)=0 \pmod p,$$

we obtain

$$\sum_{t=0}^{(p-1)/2}(-1)^t\left(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{p+t}\right)\binom{-\frac{1}{2}}{t}^2$$
$$\equiv-\sum_{t=0}^{(p-1)/2}(-1)^t\left(\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{p-1-t}{\frac{p-1}{2}}\right)\binom{-\frac{1}{2}}{t}^2$$
$$=-\sum_{t=0}^{(p-3)/4}\left(\binom{\frac{p-1}{2}+t}{\frac{p-1}{2}}+\binom{p-1-t}{\frac{p-1}{2}}\right)\left((-1)^t\binom{-\frac{1}{2}}{t}^2+(-1)^{\frac{p-1}{2}-t}\binom{-\frac{1}{2}}{\frac{p-1}{2}-t}^2\right)$$
$$\equiv 0 \pmod{p^2}.$$

Hence the result is also true in this case. The proof is now complete. □

Lemma 6 ([10, Theorem 3.3])**.** *Let p be a prime with $p\equiv 5,7 \pmod 8$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv 0 \pmod{p^2}.$$

Lemma 7 ([4, Lemma 2.8])**.** *Let $m\in\mathbb{Z}$ and $k,p\in\mathbb{Z}^+$. Then*

$$\binom{mp^r-1}{k}=(-1)^{k-\lfloor\frac{k}{p}\rfloor}\binom{mp^{r-1}-1}{\lfloor k/p\rfloor}\prod_{i=1,\ p\nmid i}^{k}\left(1-\frac{mp^r}{i}\right).$$

3 Congruences for $S_{\frac{p^2-1}{2}}$ and $f_{\frac{p^2-1}{2}} \pmod{p^2}$

Theorem 8. *Let p be a prime with $p\equiv 5,7 \pmod 8$. Then*

$$S_{\frac{p^2-1}{2}}\equiv f_{\frac{p^2-1}{2}}\equiv 0 \pmod{p^2}.$$

Moreover,

$$S_{\frac{p^2-1}{2}}\equiv f_{\frac{p^2-1}{2}} \pmod{p^3}.$$

Proof. For $\frac{p-1}{2} < t < p$ and $0 \le s \le \frac{p-1}{2}$, from Lemma 2 we see that

$$\binom{p^2-1}{\frac{p^2-1}{2}} \equiv \binom{p-1}{\frac{p-1}{2}}^2 \equiv 1 \pmod p,$$

$$\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t} \equiv \binom{\frac{p-1}{2}}{s}\binom{\frac{p-1}{2}}{t} = 0 \pmod p,$$

$$\binom{2sp+2t}{sp+t} = \binom{(2s+1)p+2t-p}{sp+t} \equiv \binom{2s+1}{s}\binom{2t-p}{t} = 0 \pmod p$$

and

$$\begin{aligned}\binom{p^2-1-2sp-2t}{\frac{p^2-1-2sp-2t}{2}} &= \binom{(p-2s-2)p+2p-2t-1}{(\frac{p-1}{2}-s-1)p+p+\frac{p-1}{2}-t}\\ &\equiv \binom{p-2s-2}{\frac{p-1}{2}-s-1}\binom{2p-2t-1}{p+\frac{p-1}{2}-t} = 0 \pmod p.\end{aligned}$$

Now we assert that

$$\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3 \equiv 0 \pmod{p^2} \quad \text{for} \quad s=0,1,2,\dots. \tag{4}$$

We prove the result by induction on s. For $0 \le t \le (p-1)/2$ we see that

$$\binom{\frac{p^2-1}{2}}{t} \equiv \binom{-\frac{1}{2}}{t} = \frac{\binom{2t}{t}}{(-4)^t} \pmod{p^2}.$$

From Lemma 6 we know that the result is true for $s=0$. Suppose that (4) holds for $s=k$. For $s=k+1$, applying Lemma 4 we have

$$\begin{aligned}&\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{(k+1)p+t}^3\\ &\equiv \binom{\frac{p-1}{2}}{k+1}^3\sum_{t=0}^{\frac{p-1}{2}}\Big(\frac{p+1}{2}\binom{\frac{p-1}{2}}{t} - (\frac{p-1}{2}+k)\binom{\frac{p-1}{2}-p}{t}\\ &\quad - k\binom{\frac{p-1}{2}-p}{t+p} - \Big(\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}\Big)\Big)^3 \pmod{p^2}.\end{aligned}$$

Hence $\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{(k+1)p+t}^3 \equiv 0 \pmod{p^2}$ for $k \ge \frac{p-1}{2}$. For $k < \frac{p-1}{2}$, by the inductive hypothesis and Lemma 4 we have

$$\sum_{t=0}^{(p-1)/2}\Big(\frac{p+1}{2}\binom{\frac{p-1}{2}}{t} - (\frac{p-1}{2}+k)\binom{\frac{p-1}{2}-p}{t} - k\binom{\frac{p-1}{2}-p}{t+p}\Big)^3 \equiv 0 \pmod{p^2}.$$

Also, $\binom{\frac{p-1}{2}}{t}\equiv\binom{\frac{p-1}{2}-p}{t}\equiv\binom{-\frac12}{t} \pmod p$ and $\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}=(-1)^t\Big(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{t+p}\Big)\equiv 0 \pmod p$ for $t\in\{0,1,\ldots,\frac{p-1}{2}\}$. By Lemma 5,

$$\begin{aligned}
&\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{(k+1)p+t}^3\\
&\equiv\binom{\frac{p-1}{2}}{k+1}^3\Big(\sum_{t=0}^{(p-1)/2}\Big(\frac{p+1}{2}\binom{\frac{p-1}{2}}{t}-\big(\frac{p-1}{2}+k\big)\binom{\frac{p-1}{2}-p}{t}-k\binom{\frac{p-1}{2}-p}{t+p}\Big)^3\\
&\quad+3\sum_{t=0}^{(p-1)/2}\Big(\frac{p+1}{2}\binom{\frac{p-1}{2}}{t}-\big(\frac{p-1}{2}+k\big)\binom{\frac{p-1}{2}-p}{t}-k\binom{\frac{p-1}{2}-p}{t+p}\Big)\\
&\qquad\quad\times\Big(\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}\Big)^2\\
&\quad-3\sum_{t=0}^{(p-1)/2}\Big(\frac{p+1}{2}\binom{\frac{p-1}{2}}{t}-\big(\frac{p-1}{2}+k\big)\binom{\frac{p-1}{2}-p}{t}-k\binom{\frac{p-1}{2}-p}{t+p}\Big)^2\\
&\qquad\quad\times\Big(\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}\Big)\\
&\quad-\sum_{t=0}^{(p-1)/2}\Big(\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}\Big)^3\Big)\\
&\equiv-3\binom{\frac{p-1}{2}}{k+1}^3\sum_{t=0}^{(p-1)/2}\binom{-\frac12}{t}^2\Big(\binom{\frac{p-1}{2}-p}{t}+\binom{\frac{p-1}{2}-p}{t+p}\Big)\\
&=-3\binom{\frac{p-1}{2}}{k+1}^3\sum_{t=0}^{(p-1)/2}(-1)^t\binom{-\frac12}{t}^2\Big(\binom{\frac{p-1}{2}+t}{t}-\binom{p+\frac{p-1}{2}+t}{t+p}\Big)\\
&\equiv 0 \pmod{p^2}.
\end{aligned}$$

Hence

$$f_{\frac{p^2-1}{2}}\equiv\sum_{s=0}^{(p-1)/2}\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\equiv 0 \pmod{p^2}.$$

Set $H_0=H_0(1,1)=0$, $H_k=\sum_{i=1}^k\frac1k$ and $H_k(1,1)=\sum_{1\le i<j\le k}\frac{1}{ij}$ for $k\in\mathbb{Z}^+$. For $0\le s\le (p-1)/2$, it is easily seen that $H_{p-1}\equiv 0 \pmod p$, $\binom{p-1}{2s}\equiv 1-pH_{2s}+p^2H_{2s}(1,1) \pmod{p^3}$ and so $\frac{1}{\binom{p-1}{2s}}\equiv 1+pH_{2s}+p^2\big(H_{2s}^2-H_{2s}(1,1)\big) \pmod{p^3}$. By Lemma 7, for $0\le t\le (p-1)/2$ we see that

$$\binom{p^2-1}{2sp+2t}=\binom{p-1}{2s}\prod_{i=1,p\nmid i}^{2sp+2t}\Big(1-\frac{p^2}{i}\Big)\equiv\binom{p-1}{2s}\Big(1-p^2\sum_{i=1,p\nmid i}^{2sp+2t}\frac1i\Big) \pmod{p^3}.$$

Applying (4), Lemma 6 and the identity

$$\binom{a-b}{c-d}\binom{b}{d}=\binom{a}{c}\binom{c}{d}\binom{a-c}{b-d}\Big/\binom{a}{b}$$

we derive that

$$\begin{aligned}
S_{\frac{p^2-1}{2}} &\equiv \sum_{s=0}^{(p-1)/2}\sum_{t=0}^{(p-1)/2}\binom{\frac{p^2-1}{2}}{sp+t}\binom{2sp+2t}{sp+t}\binom{p^2-1-2sp-2t}{\frac{p^2-1}{2}-sp-t}\\
&=\binom{p^2-1}{\frac{p^2-1}{2}}\sum_{s=0}^{(p-1)/2}\sum_{t=0}^{(p-1)/2}\frac{\binom{(p^2-1)/2}{sp+t}^3}{\binom{p^2-1}{2sp+2t}}\\
&\equiv \binom{p^2-1}{\frac{p^2-1}{2}}\sum_{s=0}^{(p-1)/2}\frac{1}{\binom{p-1}{2s}}\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\Big(1+p^2\sum_{i=1,p\nmid i}^{2sp+2t}\frac{1}{i}\Big)\\
&\equiv \binom{p^2-1}{\frac{p^2-1}{2}}\sum_{s=0}^{(p-1)/2}\frac{1}{\binom{p-1}{2s}}\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\\
&\quad+p^2\binom{p^2-1}{\frac{p^2-1}{2}}\sum_{s=0}^{(p-1)/2}\frac{\binom{2s}{s}^3}{(-64)^s}\sum_{t=0}^{(p-1)/2}\frac{\binom{2t}{t}^3}{(-64)^t}H_{2t}\\
&\equiv \binom{p^2-1}{\frac{p^2-1}{2}}\Big(\sum_{s=0}^{(p-1)/2}\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\\
&\quad+p\sum_{s=0}^{(p-1)/2}\big(H_{2s}+p(H_{2s}^2-H_{2s}(1,1))\big)\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\Big)\\
&\equiv \sum_{s=0}^{(p-1)/2}\sum_{t=0}^{(p-1)/2}\binom{\frac{p-1}{2}p+\frac{p-1}{2}}{sp+t}^3\\
&\equiv f_{\frac{p^2-1}{2}}\pmod{p^3}.
\end{aligned}$$

Summarizing the above proves the theorem. □

4 An inequality involving (S_m)

Theorem 9. *For $m=2,3,4,\ldots$ we have*

$$\Big(1+\frac{1}{m(m-1)}\Big)S_m^2>S_{m+1}S_{m-1}.$$

Proof. It is easily seen that

$$\Big(1+\frac{1}{(m-1)(m-2)}\Big)S_{m-1}^2>S_mS_{m-2}\quad\text{for}\quad m=3,4,\ldots,13.$$

Now suppose $m \geq 14$ and $\big(1 + \frac{1}{(m-1)(m-2)}\big) S_{m-1}^2 > S_m S_{m-2}$. By (2), Lemma 3 and the inductive hypothesis we have

$$\begin{aligned}
&\Big(1 + \frac{1}{m(m-1)}\Big) S_m^2 - S_{m+1} S_{m-1} \\
&= \Big(1 + \frac{1}{m(m-1)}\Big) S_m^2 - \frac{4(3m^2+3m+1)}{(m+1)^2} S_m S_{m-1} + \frac{32m^2}{(m+1)^2} S_{m-1}^2 \\
&> \Big(\frac{m^2-m+1}{m(m-1)} S_m - \frac{4(3m^2+3m+1)}{(m+1)^2} S_{m-1} + \frac{32m^2(m-1)(m-2)}{(m+1)^2(m^2-3m+3)} S_{m-2}\Big) S_m \\
&= \Big((20m^5 - 60m^4 + 52m^3 + 28m^2 - 36m + 12) S_{m-1} \\
&\quad + (-128m^5 + 320m^4 - 256m^3 - 32m^2 + 192m - 96) S_{m-2}\Big) \\
&\quad \times \frac{S_m}{(m+1)^2(m^2-3m+3)m^3(m-1)} \\
&= \frac{16 S_m}{(m+1)^2(m^2-3m+3)m^3(m-1)} \sum_{k=0}^{m-2} \binom{m-2}{k} \binom{2k}{k} \binom{2m-4-2k}{m-2-k} F(m,k),
\end{aligned}$$

where

$$\begin{aligned}
F(m,k) = {} & (5m^5 - 15m^4 + 13m^3 + 7m^2 - 9m + 3) \frac{2k+1}{k+1} \\
& - 8m^5 + 20m^4 - 16m^3 - 2m^2 + 12m - 6.
\end{aligned}$$

For $m \geq 14$ it is easily seen that $3 < \frac{(2m-7)(2m-5)}{(m-3)(m-2)} < 4$, $5m^5 - 15m^4 + 13m^3 + 7m^2 - 9m + 3 > 0$, $-8m^5 + 20m^4 - 16m^3 - 2m^2 + 12m - 6 < 0$, $6m^7 - 75m^6 + 223m^5 - 283m^4 - 61m^3 + 427m^2 - 87m - 42 > 0$, and $F(m,k+1) > F(m,k)$ for $k = 0, 1, \ldots, m-3$. Thus, $F(m,m-3) + F(m,1) > F(m,5) + F(m,1) > 0$ and

$$\begin{aligned}
F(m,k) \geq F(m,2) = {} & \frac{5}{3}(5m^5 - 15m^4 + 13m^3 + 7m^2 - 9m + 3) \\
& - 8m^5 + 20m^4 - 16m^3 - 2m^2 + 12m - 6 > 0 \quad \text{for} \quad k \geq 2.
\end{aligned}$$

From the above we derive that

$$
\begin{aligned}
&(1+\frac{1}{m(m-1)})S_m^2 - S_{m+1}S_{m-1} \\
&> \frac{16S_m}{(m+1)^2(m^2-3m+3)m^3(m-1)}\Big(\sum_{k=0}^{2}\binom{m-2}{k}\binom{2k}{k}\binom{2m-4-2k}{m-2-k}F(m,k) \\
&\quad + \sum_{k=m-4}^{m-2}\binom{m-2}{k}\binom{2k}{k}\binom{2m-4-2k}{m-2-k}F(m,k)\Big) \\
&= \frac{16S_m}{(m+1)^2(m^2-3m+3)m^3(m-1)}\Big(\binom{2m-4}{m-2}\big(F(m,m-2)+F(m,0)\big) \\
&\quad + 3(m-2)(m-3)\binom{2m-8}{m-4}\big(F(m,m-4)+F(m,2)\big) \\
&\quad + 2(m-2)\binom{2m-6}{m-3}\big(F(m,1)+F(m,m-3)\big)\Big) \\
&> \Big(3(m^2-5m+6)F(m,m-4) + \frac{4(2m-7)(2m-5)}{(m-3)(m-2)}F(m,0)\Big) \\
&\quad \times \frac{16S_m\binom{2m-8}{m-4}}{(m+1)^2(m^2-3m+3)m^3(m-1)} \\
&= \frac{S_m\binom{2m-8}{m-4}}{(m+1)^2(m^2-3m+3)m^3(m-1)}\Big(\big(6(m-2)(2m-7)+\frac{8(2m-7)(2m-5)}{(m-3)(m-2)}\big) \\
&\quad \times (40m^5-120m^4+104m^3+56m^2-72m+24) \\
&\quad + (-128m^5+320m^4-256m^3-32m^2+192m-96) \\
&\quad \times \big(3(m-2)(m-3)+\frac{4(2m-7)(2m-5)}{(m-3)(m-2)}\big)\Big) \\
&> \Big(\big(6(m-2)(2m-7)+24\big)(40m^5-120m^4+104m^3+56m^2-72m+24) \\
&\quad + \big(3(m-2)(m-3)+16\big)(-128m^5+320m^4-256m^3-32m^2+192m-96)\Big) \\
&\quad \times \frac{S_m\binom{2m-8}{m-4}}{(m+1)^2(m^2-3m+3)m^3(m-1)} \\
&= (6m^7-75m^6+223m^5-283m^4-61m^3+427m^2-87m-42) \\
&\quad \times \frac{16S_m\binom{2m-8}{m-4}}{(m+1)^2(m^2-3m+3)m^3(m-1)} \\
&> 0.
\end{aligned}
$$

Hence the inequality is proved by induction. □

Corollary 10. *For $m = 2, 3, 4, \ldots$ we have*

$$\left(1 + \frac{1}{m(m-1)}\right)P_m^2 > P_{m+1}P_{m-1}.$$

Proof. Since $P_m = 2^m S_m$, the result follows from Theorem 9. □

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