Journal of Integer Sequences, Vol. 19 (2016),
Article 16.6.2

Integer Sequences Connected to the Laplace Continued Fraction and Ramanujan's Identity

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Abstract

We consider integer sequences connected to the famous Laplace continued fraction for the function $R(t) = \int_t^\infty \varphi(x)\mathrm{d}x/\varphi(t)$, where $\varphi(t) = e^{-t^2/2}/\sqrt{2\pi}$ is the standard normal density. We compute the generating functions for these sequences and study their relation to the Hermite and Bessel polynomials. Using the master equation for the generating functions, we find a new proof of the Ramanujan identity.

1 Introduction

We consider two infinite matrices, $\mathfrak{p} = \|p_{k,m}\|_{k,m\geq 0}$ and $\mathfrak{q} = \|q_{k,m}\|_{k,m\geq 0}$, defined as follows. If $m > k$ or $k - m \equiv 1 \pmod 2$, then

$$p_{k,m} = q_{k,m} = 0, \quad k, m = 0, 1, \ldots.$$

If $k = m + 2n$ and $n \geq 0$ then

$$p_{k,m} = \frac{k!}{m!\; 2^n\; n!}, \tag{1}$$

and

$$q_{k,m} = \frac{\left(\frac{k+m}{2}\right)!}{m!} 2^{-n} \sum_{j=0}^{n} \binom{k+1}{j}. \tag{2}$$

The matrices $\mathfrak{p}$ and $\mathfrak{q}$ are connected to the Laplace continued fraction [11]

$$\mathcal{L}(t) := \cfrac{1}{t + \cfrac{1}{t + \cfrac{2}{t + \cfrac{3}{t + \cfrac{4}{\ddots}}}}}, \quad t > 0, \tag{3}$$

for the function $R(t) := \bar{\Phi}(t)/\varphi(t)$, where $\varphi(t)$ is the standard normal density and $\bar{\Phi}(t) = \int_t^\infty \varphi(s)\mathrm{d}s$ is the tail of the standard normal distribution. The function, $R(t)$, is often called the Mills ratio, after John Mills who tabulated it [13] in 1929. The Mills ratio appears in the probability theory, [5, 6, 12, 15] in the context of asymptotic expansions for the normal probability integral, in statistical analysis [8, 13] and in the area of numerical analysis [3, 9, 14, 16] where the functional inequalities and irrational approximations for $R(t)$ are discussed.

In this paper we study the matrices $\mathfrak{p}$ and $\mathfrak{q}$. In Section 2, we show that certain integer sequences in [17] are encapsulated in these matrices. In particular, the triangular array A180048 is described by (2). In Section 2, we also describe connections between the matrix $\mathfrak{p}$ and the coefficients of the Hermite and Bessel polynomials [1, 2].

In Section 3, we introduce and study the generating polynomials. Despite the fact that the recurrence relations for the polynomials P_k and Q_k have been known for a very long time[1], their coefficients were not systematically studied until recently. The first analysis of these coefficients, published in [9], appeared only in 2006, to the best of our knowledge[2].

The statements in the first three sections are elementary. Their proofs are left to the reader. In Section 4, we derive the master equation linking together the generating functions of the Laplace polynomials and the Laplace continued fraction, $\mathcal{L}(t)$. In Section 5, the master equation is used for a short derivation of the famous identity discovered by Ramanujan:

$$\cfrac{1}{1 + \cfrac{2}{1 + \cfrac{3}{\ddots}}} = -1 + \frac{1}{\sqrt{\frac{e\pi}{2}} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)!!}}.$$

[1] Some of these relations were derived by Jacobi in [7].

[2] Our definition of the coefficients of the polynomials $Q_k(t)$ is different from that in [9].

2 Matrices $\mathfrak{p}$ and $\mathfrak{q}$

We shall start with the following recurrence equations for the elements of $\mathfrak{p}$ and $\mathfrak{q}$.

Proposition 1. *The elements of the matrices* $\mathfrak{p}$ *and* $\mathfrak{q}$ *satisfy*

$$\begin{aligned} m \cdot p_{k,m} &= k \cdot p_{k-1,m-1}, & k \geq 1, 1 \leq m \leq k, \quad (4)\\ p_{k+1,m} &= p_{k,m-1} + (m+1) \cdot p_{k,m+1}, & k \geq 0, 1 \leq m \leq k, \quad (5)\\ q_{k,m} &= p_{k,m} + (m+1) \cdot q_{k-1,m+1}, & k \geq 1, 1 \leq m \leq k, \quad (6)\\ q_{k,m} &= q_{k-1,m-1} + k \cdot q_{k-2,m}, & k \geq 2, 1 \leq m \leq k. \quad (7) \end{aligned}$$

Proof. The proof is based on the following lemma.

Lemma 2. *Write*

$$S_i^{(j)}(k) = \sum_{l=i}^{j} \binom{k}{l}, \quad k > j.$$

Then

$$S_0^{(n-1)}(k) - 2S_0^{(n-2)}(k-1) = \binom{k-1}{n-1}. \quad (8)$$

The derivation of (4) is straightforward. The derivation of (5), (6) and (7) is based on Lemma 2, which in the case of (7) is used twice. □

The elements in the first nine rows and columns of the matrix $\mathfrak{p}$ are given in Table 1.

k	m								
	0	1	2	3	4	5	6	7	8
0	1	0	0	...					0
1	0	1	0	...					0
2	1	0	1	0	...				0
3	0	3	0	1	0	...			0
4	3	0	6	0	1	0	...		0
5	0	15	0	10	0	1	0	...	0
6	15	0	45	0	15	0	1	0	0
7	0	105	0	105	0	21	0	1	0
8	105	0	420	0	210	0	28	0	1

Table 1: Matrix $\mathfrak{p} = \|p_{k,m}\|$.

The matrix $\mathfrak{p}$ encapsulates many remarkable integer sequences. The rows of the matrix describe the coefficients of the Laplace polynomials [10], $P_k(t) = \sum_{m=0}^{k} p_{k,m} t^m$. These polynomials will be discussed in the next section. The columns of $\mathfrak{p}$ are integer sequences that can be found in [17]. In particular, the first column is $p_{2n,0} = (2n-1)!!$, $(n = 1, 2, \dots)$, which is A001147. In the case $m = 1$, the column $p_{2n-1,1} = p_{2n,0}$. If $m = 2$ then $p_{2n+2,2} = (n+1)\cdot(2n+1)!!$, which is A001879. In the case $m = 3$ we have the sequence A000457.

The diagonals of $\mathfrak{p}$ also represent some remarkable integer sequences. The diagonal $p_{k,2N-k}$ represents the coefficients of the Nth Bessel polynomial. Indeed, the Bessel polynomials are

$$y_N(t) = \sum_{j=0}^{N} \frac{(N+j)!}{(N-j)!\cdot j!}\left(\frac{t}{2}\right)^j, \quad N = 0, 1, \dots.$$

Their coefficients, $B_{N,j} := (N+j)!/((N-j)!j!2^j)$, are equal to $p_{N+j,N-j}$, for $j = 0, 1, \dots, N$. Thus the Bessel polynomials can be written as $y_N(t) = \sum_{j=0}^{N} p_{N+j,N-j} t^j$.

The diagonals $k - m = 2n$ also represent some well-known sequences. If $n = 0$ then we have the sequence A000012. If $n = 1$, $p_{m+2,m} = (m+1)(m+2)/2$ is the sequence of triangular numbers A000217. If $n = 2$, $p_{m+4,m}$ is the sequence of tritriangular numbers A050534.

Analytical properties of the matrix $\mathfrak{q}$ are equally interesting. The elements in the first eight rows of the matrix $\mathfrak{q}$ are given in Table 2. For small m, Formula (2) can be simplified.

k	m								
	0	1	2	3	4	5	6	7	8
0	1	0	0	...					0
1	0	1	0	...					0
2	2	0	1	0	...				0
3	0	5	0	1	0	...			0
4	8	0	9	0	1	0	...		0
5	0	33	0	14	0	1	0	...	0
6	48	0	87	0	20	0	1	0	0
7	0	279	0	185	0	27	0	1	0

Table 2: Elements of the matrix $\mathfrak{q} = \|q_{k,m}\|$.

Proposition 3. *For $n = 0, 1, 2, \dots,$ we have*

$$q_{2n,0} = (2n)!!, \tag{9}$$
$$q_{2n+1,1} = (2n+2)!! - (2n+1)!!, \tag{10}$$
$$q_{2n+2,2} = \frac{1}{2}(2n+4)!! - (2n+3)!!, \tag{11}$$
$$q_{2n+3,3} = \frac{1}{3!}(2n+6)!! - \frac{1}{2!}(2n+5)!! + \frac{1}{3!}(2n+3)!!, \tag{12}$$
$$q_{2n+4,4} = \frac{1}{4!}(2n+8)!! - \frac{1}{3!}(2n+7)!! + \frac{1}{3!}(2n+5)!!. \tag{13}$$

According to Proposition 3, the first column of the matrix $\mathfrak{q}$, described by (9), corresponds to the sequence A000165. The second column, satisfying Equation (10), describes the sequence A129890. The third column, satisfying Equation (11), represents the integer sequence A035101. This sequence is related to the Catalan numbers (see [17]) but our formula (11) looks simpler. The sequence $q_{2n+3,3}$ represents the integer sequence A263384. The sequence $q_{2n+4,4}$, described by (13), is less known.

The elements of the matrix $\mathfrak{q} = \|q_{k,m}\|$ can be represented as a linear combination of the diagonal elements of the matrix $\mathfrak{p}$. Let $k \equiv m \pmod 2$, and $n = (k-m)/2$. Then [10]

$$m! \cdot q_{k,m} = \sum_{j=0}^{n} (m+j)! \cdot p_{k-j,m+j}.$$

3 Polynomials $P_k(t)$ and $Q_k(t)$

3.1 Laplace and Jacobi polynomials

Consider the polynomials $P_k(t) := \sum_{m=0}^{k} p_{k,m} t^m$, and $Q_k(t) := \sum_{m=0}^{k} q_{k,m} t^m$, where $p_{k,m}$ and $q_{k,m}$ are defined by (1) and (2), respectively. In this section, we shall connect these polynomials with the convergents of the Laplace continued fraction, $\mathcal{L}(t)$. For this reason, we call $P_k(t)$ and $Q_k(t)$ the *Laplace polynomials* in what follows.

Proposition 4 ([7, 9, 10, 14])**.** *For $k \geq 1$, the Laplace polynomials satisfy the following recurrent equations*

$$P_{k+1}(t) = tP_k(t) + P'_k(t), \tag{14}$$
$$Q_k(t) = P_k(t) + Q'_{k-1}(t), \tag{15}$$
$$P_{k+1}(t) = tP_k(t) + kP_{k-1}(t), \tag{16}$$
$$Q_{k+1}(t) = tQ_k(t) + (k+1)Q_{k-1}(t), \tag{17}$$

where $P_0(t) = Q_0(t) = 1$.

Proposition 4 allows one to find the polynomials $P_k(t)$ and $Q_k(t)$ for any integer $k \geq 1$. The first eight polynomials are given in Table 3.

k	$P_k(t)$	$Q_{k-1}(t)$
1	t	1
2	t^2+1	t
3	t^3+3t	t^2+2
4	t^4+6t^2+3	t^3+5t
5	t^5+10t^3+15t	t^4+9t^2+8
6	$t^6+15t^4+45t^2+15$	t^5+14t^3+33t
7	$t^7+21t^5+105t^3+105t$	$t^6+20t^4+87t^2+48$
8	$t^8+28t^6+210t^4+420t^2+105$	$t^7+27t^5+185t^3+279t$

Table 3: Laplace polynomials $P_k(t)$ and $Q_{k-1}(t)$.

For $t > 0$, we consider the rational functions $\mathcal{L}_k(t) = Q_{k-1}(t)/P_k(t), (k = 1, 2, \ldots)$. The function $\mathcal{L}_k(t)$ is the kth convergent of the continued fraction[3], $\mathcal{L}(t)$:

$$\lim_{k\to\infty} \mathcal{L}_k(t) = \mathcal{L}(t) = R(t), \quad t > 0.$$

The function $R(t)$ satisfies the differential equation

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = t \cdot R(t) - 1. \tag{18}$$

Let us now consider the derivatives of the function $R(t)$. From (18) we derive

$$\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} = t \cdot \frac{\mathrm{d}^{k-1} R(t)}{\mathrm{d}t^{k-1}} + (k-1) \cdot \frac{d^{k-2} R(t)}{\mathrm{d}t^{k-2}}. \quad k = 2, 3, \ldots. \tag{19}$$

It was proved in [7, 9, 10] that [4]

$$\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} = R(t) \cdot P_k(t) - Q_{k-1}(t), \qquad k = 1, 2, \ldots. \tag{20}$$

3.2 Laplace and Hermite polynomials

The polynomials $P_k(t)$ are closely connected to the Hermite polynomials [1, 2]. Denote the differentiation operator by $\mathfrak{D}$: $\mathfrak{D}g(t) = \frac{\mathrm{d}g(t)}{\mathrm{d}t}$. Then, as usual, $\mathfrak{D}^n g(t) = \frac{\mathrm{d}^n g(t)}{\mathrm{d}t^n}$, $(n = 1, 2, \ldots)$. Recall that the Hermite polynomials can be defined as

$$H_k(t) := (-1)^k e^{t^2/2}\, \mathfrak{D}^k e^{-t^2/2}.$$

[3] See [9] and [10] regarding this statement

[4] Equation (20) follows from (18), (19) and Proposition 4.

Lemma 5. *The Laplace polynomials, $P_k(t)$, satisfy*

$$P_k(t) = e^{-t^2/2}\,\mathfrak{D}^k e^{t^2/2}, \quad k = 0, 1, 2, \ldots. \tag{21}$$

Proof. We shall prove this lemma by induction. For $k = 1$

$$e^{-t^2/2}\,\mathfrak{D} e^{t^2/2} = t = P_1(t).$$

Suppose that $e^{-t^2/2}\,\mathfrak{D}^n e^{t^2/2} = P_n(t)$. Let us verify that

$$P_{n+1}(t) = e^{-t^2/2}\,\mathfrak{D}^{n+1} e^{t^2/2}.$$

Indeed,

$$\begin{aligned}
\mathfrak{D}^{n+1} e^{t^2/2} &= \mathfrak{D}\Big(\mathfrak{D}^n e^{t^2/2}\Big) \\
&= \mathfrak{D}\Big(e^{t^2/2}\cdot P_n(t)\Big) \\
&= e^{t^2/2}\Big(tP_n(t) + P_n'(t)\Big).
\end{aligned}$$

Therefore

$$e^{-t^2/2}\,\mathfrak{D}^{n+1} e^{t^2/2} = tP_n(t) + P_n'(t).$$

Since Equation (14) and the initial condition, $P_1(t) = t$, uniquely determine the sequence of polynomials generated by the recurrence (14), we obtain

$$e^{-t^2/2}\,\mathfrak{D}^{n+1} e^{t^2/2} = P_{n+1}(t),$$

as was to be proved. □

Lemma 5 implies (see [9, 10])

$$P_k(t) = (-i)^k H_k(it). \tag{22}$$

This relation allows us to reformulate the classical results obtained for the Hermite polynomials in terms of the Laplace polynomials. In particular, one can easily derive from (22) and the generating function of the Hermite polynomials,

$$\mathcal{H}(t,s) := \sum_{k=0}^{\infty} H_k(t)\frac{s^k}{k!} = e^{st - s^2/2}\,,$$

the generating function $\mathcal{P}(s,t) = \sum_{k=0}^{\infty} P_k(t)\frac{s^k}{k!}$.

Lemma 6 ([9, 10])**.** *The generating function, $\mathcal{P}(s,t)$, is*

$$\mathcal{P}(s,t) = \exp\Big(st + \frac{s^2}{2}\Big). \tag{23}$$

4 Master equation

Let us now compute the generating function of the Laplace polynomials $Q_k(t)$. Write

$$\mathcal{Q}(s,t) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q_{k,m} t^m \frac{s^{k+1}}{(k+1)!}.$$

Lemma 7. *The generating function $\mathcal{Q}(s,t)$ is*

$$\mathcal{Q}(s,t) = \sqrt{2\pi}\, e^{(s+t)^2/2} \cdot \Big(\bar{\Phi}(t) - \bar{\Phi}(s+t)\Big). \tag{24}$$

Proof. We have,

$$\mathcal{Q}(s,t) = \sum_{k=0}^{\infty} Q_k(t) \frac{s^{k+1}}{(k+1)!} \quad \text{and} \quad \mathcal{P}(s,t) = \sum_{k=0}^{\infty} P_k(t) \frac{s^k}{k!}.$$

The Taylor series expansion for the function $R(t)$ can be written as

$$R(s+t) = \sum_{k=0}^{\infty} \frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} \frac{s^k}{k!}.$$

Recall that the derivatives of the function $R(t)$ satisfy Equation (20):

$$\frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} = P_k(t) R(t) - Q_{k-1}(t).$$

We have

$$R(s+t) = \sqrt{2\pi} e^{(s+t)^2/2} \bar{\Phi}(s+t)\,.$$

Therefore

$$\begin{aligned}
\sqrt{2\pi}\, e^{(s+t)^2/2} \cdot \bar{\Phi}(s+t) &= \sum_{k=0}^{\infty} \frac{\mathrm{d}^k R(t)}{\mathrm{d}t^k} \frac{s^k}{k!} \\
&= R(t) \sum_{k=0}^{\infty} P_k(t) \frac{s^k}{k!} - \sum_{k=0}^{\infty} Q_{k-1}(t) \frac{s^k}{k!} \\
&= R(t) \cdot e^{st+s^2/2} - \mathcal{Q}(s,t) \\
&= \sqrt{2\pi}\, e^{t^2/2} \cdot \bar{\Phi}(t) \cdot e^{st+s^2/2} - \mathcal{Q}(s,t).
\end{aligned}$$

Finally, we obtain

$$\mathcal{Q}(s,t) = \sqrt{2\pi}\, e^{(s+t)^2/2} \cdot \Big(\bar{\Phi}(t) - \bar{\Phi}(s+t)\Big).$$

Equation (24) is thus proved. □

Theorem 8. *The generating functions, $\mathcal{P}(s,t)$ and $\mathcal{Q}(s,t)$, satisfy*

$$R(s+t) + \mathcal{Q}(s,t) = \mathcal{P}(s,t)R(t). \tag{25}$$

Proof. We have $\sqrt{2\pi}\,\exp(t^2/2)\cdot\bar{\Phi}(t) = R(t)$. From Lemma 7 we derive

$$\begin{aligned}\mathcal{Q}(s,t) &= \sqrt{2\pi}\,e^{(s+t)^2/2}\cdot\Big(\bar{\Phi}(t)-\bar{\Phi}(s+t)\Big)\\ &= e^{st+s^2/2}\cdot R(t) - R(t+s).\end{aligned}$$

The latter relation is equivalent to (25). □

5 Identities

In [10] we derived a new, to the best of our knowledge, identity

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n}\frac{(n+m)!}{m!j!(m+2n+1-j)!}\;2^{-n} = \sqrt{2\pi}e^2\Big(\Phi(2)-\Phi(1)\Big),$$

having an unusual combination of constants in the right hand side. In fact, the whole derivation is based on the substitution $s=t=1$ into (25) in Theorem 8. Choosing integer values for s and t we obtain a series of elegant combinatorial identities similar to those discussed in [10]. Another interesting, unexpected connection to the Laplace continued fraction is the Ramanujan identity [4, Entry 43, p. 166]:

$$\cfrac{1}{1+\cfrac{2}{1+\cfrac{3}{1+\cfrac{4}{1+\cfrac{5}{\ddots}}}}} = \frac{1}{\sqrt{\frac{e\pi}{2}}-\sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)!!}} - 1. \tag{26}$$

This identity can be derived from Equation (25). We will need the following generalization of (26).

Proposition 9. *For $s>0$, consider the continued fraction*

$$\mathcal{S}(s) = \cfrac{1}{s+\cfrac{2}{s+\cfrac{3}{s+\cfrac{4}{s+\cfrac{5}{\ddots}}}}}.$$

Then

$$\mathcal{S}(s) = \frac{1}{e^{s^2/2}\sqrt{\frac{\pi}{2}} - \sum\limits_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}} - s. \tag{27}$$

Proof. From (25) we derive

$$R(s) + \mathcal{Q}(s,0) = e^{s^2/2} \cdot R(0), \tag{28}$$

and

$$\begin{aligned} \mathcal{Q}(s,0) &= \sum_{n=0}^{\infty} q_{2n,0} \frac{s^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (2n)!! \frac{s^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}. \end{aligned} \tag{29}$$

Inspecting the continued fractions (3) and $\mathcal{S}(s)$ we find

$$\mathcal{S}(s) = -s + \frac{1}{\mathcal{L}(s)}, \tag{30}$$

and, therefore, for $s > 0$

$$\mathcal{S}(s) = -s + \frac{1}{R(s)}.$$

Taking into account that $R(0) = \sqrt{\pi/2}$ we obtain from the latter equation, (28) and (29)

$$\frac{1}{s + \mathcal{S}(s)} = e^{s^2/2}\sqrt{\frac{\pi}{2}} - \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}. \tag{31}$$

Equation (31) is equivalent to (27). □

Substituting $s = 1$ into (27) we obtain the Ramanujan identity (26).

Remark 10. The series $\sum\limits_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!!}$ converges to the function

$$e^{s^2/2} \int_0^s e^{-u^2/2} \mathrm{d}u = (\Phi(s) - 0.5)/\varphi(s),$$

where $\varphi(s) = \exp(-s^2/2)/\sqrt{2\pi}$. The coefficients of the convergents of the continued fraction $\mathcal{S}(s)$ are described by the triangular array A180048.

6 Acknowledgments

I am very grateful to Ian Iscoe, Sebastian Jaimungal, Alexey Kuznetsov, Tom Salisbury, Eugene Seneta, Isaac Sonin, Hans Tuenter, Alek Vainshtein, Vladimir Vinogradov, and Vladimir Zhuravlev for interesting comments and stimulating discussions. I would also like to thank the referee for the suggestions and critical comments. This paper would never appear without the OEIS.

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2000 *Mathematics Subject Classification*: Primary 11Y05; Secondary 11Y55.
Keywords: continued fraction, integer sequence, Ramanujan identity.

(Concerned with sequences A000012, A000165, A000217, A000457, A001147, A001879, A035101, A050534, A129890, A180048, and A263384.)

Received November 11 2015; revised versions received April 5 2016; June 24 2016; June 27 2016. Published in *Journal of Integer Sequences*, June 27 2016.

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