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Reciprocal Sums of the Tribonacci Numbers

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Abstract

In this article, we consider infinite sums of the reciprocals of the tribonacci numbers. Then, by applying the floor function to the reciprocals of these sums, we obtain a new identity involving the tribonacci numbers. Further, we give the formulas for an alternating sum of the reciprocals of the tribonacci numbers and a sum of reciprocal hypertribonacci numbers.

1 Introduction

As is well known, the sequences of Fibonacci numbers A000045 $\{F_n\}_{n=0}^{\infty}$ and tribonacci numbers A000073 $\{T_n\}_{n=0}^{\infty}$ are defined, respectively, by

$$F_{n+1} = F_n + F_{n-1}, \ F_0 = 0, \ F_1 = 1$$
$$T_{n+2} = T_{n+1} + T_n + T_{n-1}, \quad T_0 = 0, \ T_1 = T_2 = 1.$$

Ohtsuka and Nakamura [2] derived a formula for infinite sums of reciprocal Fibonacci numbers, as follows:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{1}$$

where $\lfloor \cdot \rfloor$ is the floor function. Holliday and Komatsu [1] generalized a formula (1) for the generalized Fibonacci numbers. Similar properties were investigated in several different ways; see [5, 9, 10, 11]. In [5], the author gave a similar formula (1) for alternating sums of reciprocal Fibonacci numbers as

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{F_k} \right)^{-1} \right\rfloor = (-1)^n F_{n+1} - 1 \qquad (n > 1), \tag{2}$$

and the generalized Fibonacci numbers are shown in [6]. Liu and Zhao [7] showed the formula for the infinite sums of reciprocal hyperfibonacci numbers A136431 as:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=1}^{k} F_i} \right)^{-1} \right\rfloor = F_n - 1 \qquad (n \geq 3). \tag{3}$$

Komatsu [3] studied the reciprocals of the tribonacci numbers and proved the following formula:

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right\| = T_n - T_{n-1},$$

where $\| \cdot \|$ is the nearest integer. Komatsu and Laohakosol [4] extended the above identity on the tribonacci numbers to the higher order linear recurrence.

In this paper, we consider the floor function, to develop similar formulas to (1), (2), and (3) for the tribonacci numbers.

2 Results

We begin by noting that the sequence of tribonacci numbers can be defined for negative values of n by using the definition and the given initial conditions. The first forty negative terms are shown in the following table.

n	T_{-n}
1	0
2	1
3	−1
4	0
5	2
6	−3
7	1
8	4
9	−8
10	5

n	T_{-n}
11	7
12	−20
13	18
14	9
15	−47
16	56
17	0
18	−103
19	159
20	−56

n	T_{-n}
21	−206
22	421
23	−271
24	−356
25	1048
26	−963
27	−441
28	2452
29	−2974
30	81

n	T_{-n}
31	5345
32	−8400
33	3136
34	10609
35	−22145
36	14672
37	18082
38	−54899
39	51489
40	21492

We first provide three lemmas which will be used in the proofs of the main theorems.

Lemma 1. *Let n be a positive integer. Then*

(i) $T_n^2 - T_{n-1}T_{n+1} = T_{-(n+1)}$

(ii) $\sum_{i=1}^{n} T_i = (T_{n+2} + T_n - 1)/2$

(iii) $T_n > T_{-(n+3)} \qquad (n \geq 3)$.

Proof. For any integer k, Shannon and Horadam [8] gave the following matrix equation:

$$\begin{bmatrix} T_{n+k} \\ T_{n+k-1} \\ T_{n+k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} T_k \\ T_{k-1} \\ T_{k-2} \end{bmatrix} = A^n \begin{bmatrix} T_k \\ T_{k-1} \\ T_{k-2} \end{bmatrix}.$$

We have

$$\begin{aligned} T_n^2 - T_{n-1}T_{n+1} &= \begin{vmatrix} T_{n+1} & T_{n+2} & 1 \\ T_n & T_{n+1} & 0 \\ T_{n-1} & T_n & 0 \end{vmatrix} \\ &= \left| A^n \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \quad A^n \begin{bmatrix} T_2 \\ T_1 \\ T_0 \end{bmatrix} \quad A^n A^{-n} \begin{bmatrix} T_1 \\ T_0 \\ T_{-1} \end{bmatrix} \right| \\ &= |A^n| \begin{vmatrix} 1 & 1 & T_{-n+1} \\ 0 & 1 & T_{-n} \\ 0 & 0 & T_{-n-1} \end{vmatrix} \\ &= T_{-(n+1)}, \end{aligned}$$

which completes the proof of part (i). Since $T_{n+2} - T_{n+1} = T_n + T_{n-1}$, evaluating a partial sum by using a telescopic sum, we get part (ii). For part (iii), a straightforward proof may be carried out by induction. □

Lemma 2. *Let $n > 1$ be a positive integer. Then*

(i) $$\frac{1}{T_n - T_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{1}{T_k} < \frac{1}{T_n - T_{n-1} - 1}$$

(ii) $$\frac{1}{(-1)^n(T_n + T_{n-1}) + 1} < \sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} < \frac{1}{(-1)^n(T_n + T_{n-1}) - 1}.$$

Proof. Since the proofs of both parts (i) and (ii) are quite similar, we only give a proof for part (i). Using Lemma 1 (i), we have

$$\begin{aligned}\frac{1}{T_n - T_{n-1} - 1} - \frac{1}{T_{n+1} - T_n - 1} - \frac{1}{T_n} &= \frac{T_{n-1} + 1}{T_n(T_n - T_{n-1} - 1)} - \frac{1}{T_{n+1} - T_n - 1} \\ &= \frac{T_{n-1}T_{n+1} - T_n^2 + T_{n+1} - T_{n-1} - 1}{T_n(T_n - T_{n-1} - 1)(T_{n+1} - T_n - 1)} \\ &= \frac{T_{n+1} - T_{n-1} - T_{-(n+1)} - 1}{T_n(T_n - T_{n-1} - 1)(T_{n+1} - T_n - 1)} \\ &= \frac{T_n + T_{n-2} - T_{-(n+1)} - 1}{T_n(T_n - T_{n-1} - 1)(T_{n+1} - T_n - 1)}.\end{aligned}$$

By Lemma 1 (iii), the numerator of the right-hand side of the above identity is positive,

$$\frac{1}{T_n - T_{n-1} - 1} > \frac{1}{T_n} + \frac{1}{T_{n+1} - T_n - 1}.$$

By applying the above inequality repeatedly, we obtain

$$\begin{aligned}\frac{1}{T_n - T_{n-1} - 1} &> \frac{1}{T_n} + \frac{1}{T_{n+1} - T_n - 1} \\ &> \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2} - T_{n+1} - 1} \\ &> \frac{1}{T_n} + \frac{1}{T_{n+1}} + \frac{1}{T_{n+2}} + \cdots,\end{aligned}$$

so

$$\sum_{k=n}^{\infty} \frac{1}{T_k} < \frac{1}{T_n - T_{n-1} - 1}.$$

Similarly, we can show that

$$\frac{1}{T_n - T_{n-1} + 1} < \frac{1}{T_n} + \frac{1}{T_{n+1} - T_n + 1}.$$

Repeating the above inequality, we get the left inequality of part (i). □

Lemma 3. *Let $n > 2$ be a positive integer. Then*

(i) If $T_{-(n+1)} < 0$, then $\sum_{k=n}^{\infty} \frac{1}{T_k} < \frac{1}{T_n - T_{n-1}}$.

(ii) If $T_{-(n+1)} > 0$, then $\sum_{k=n}^{\infty} \frac{1}{T_k} > \frac{1}{T_n - T_{n-1}}$.

Proof. By Lemma 1 (i), we have

$$\begin{aligned}\frac{1}{T_n}-\frac{1}{T_n-T_{n-1}}+\frac{1}{T_{n+1}-T_n}&=\frac{-T_{n-1}}{T_n(T_n-T_{n-1})}+\frac{1}{T_{n+1}-T_n}\\&=\frac{T_n^2-T_{n+1}T_{n-1}}{T_n(T_n-T_{n-1})(T_{n+1}-T_n)}\\&=\frac{T_{-(n+1)}}{T_n(T_n-T_{n-1})(T_{n+1}-T_n)}.\end{aligned}$$

If $T_{-(n+1)}<0$, then

$$\frac{1}{T_n-T_{n-1}}>\frac{1}{T_n}+\frac{1}{T_{n+1}-T_n},$$

repeating the above inequality, we get

$$\sum_{k=n}^{\infty}\frac{1}{T_k}<\frac{1}{T_n-T_{n-1}}.$$

We can prove part (ii) in a similar way. □

Now we are ready to state and prove the main results.

Theorem 4. *Let n be a positive integer. Then*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{T_k}\right)^{-1}\right\rfloor=\begin{cases}T_n-T_{n-1}, & \text{if } T_{-(n+1)}<0;\\ T_n-T_{n-1}-1, & \text{if } T_{-(n+1)}>0.\end{cases}$$

Proof. If $n=1$, then $T_{-2}=1>0$ and

$$\sum_{k=1}^{\infty}\frac{1}{T_n}>\frac{1}{T_1}=1,$$

so

$$0<\left(\sum_{k=1}^{\infty}\frac{1}{T_k}\right)^{-1}<1.$$

Therefore,

$$\left\lfloor\left(\sum_{k=1}^{\infty}\frac{1}{T_k}\right)^{-1}\right\rfloor=0=T_1-T_0-1.$$

The case $n=2$ can be similarly verified.
If $n>2$, by combining the inequalities of Lemma 2 (i) and Lemma 3, we get

$$\frac{1}{T_n-T_{n-1}+1}<\sum_{k=n}^{\infty}\frac{1}{T_n}<\frac{1}{T_n-T_{n-1}}\qquad(T_{-(n+1)}<0),$$

and

$$\frac{1}{T_n - T_{n-1}} < \sum_{k=n}^{\infty} \frac{1}{T_n} < \frac{1}{T_n - T_{n-1} - 1} \qquad (T_{-(n+1)} > 0).$$

Hence,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right\rfloor = \begin{cases} T_n - T_{n-1}, & \text{if } T_{-(n+1)} < 0; \\ T_n - T_{n-1} - 1, & \text{if } T_{-(n+1)} > 0. \end{cases}$$

□

We now calculate the analogous values to Theorem 4 for $n = 3$ (as $T_{-4} = 0$ is not included). Since

$$\sum_{k=3}^{\infty} \frac{1}{T_k} > \sum_{k=1}^{\infty} \frac{1}{2^k} = 1,$$

we have

$$\left\lfloor \left(\sum_{k=3}^{\infty} \frac{1}{T_k} \right)^{-1} \right\rfloor = 0.$$

Observe for $1 \leq n \leq 19$ that, if $3|n$, the value of T_{-n} is negative. We have the following corollary.

Corollary 5. *Let $n < 19$ be a positive integer with $n \neq 16$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_k} \right)^{-1} \right\rfloor = \begin{cases} T_n - T_{n-1}, & \text{if } 3|(n+1); \\ T_n - T_{n-1} - 1, & \text{if } 3 \nmid (n+1). \end{cases}$$

Theorem 6. *Let $n > 1$ be a positive integer. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} \right)^{-1} \right\rfloor = \begin{cases} (-1)^n (T_n + T_{n-1}), & \text{if } (-1)^n T_{-(n+1)} > 0; \\ (-1)^n (T_n + T_{n-1}) - 1, & \text{if } (-1)^n T_{-(n+1)} < 0. \end{cases}$$

Proof. Observe that

$$\frac{1}{(-1)^n (T_n + T_{n-1})} - \frac{(-1)^n}{T_n} - \frac{1}{(-1)^{n+1}(T_{n+1} + T_n)} = \frac{(-1)^n T_{-(n+1)}}{T_n (T_n + T_{n-1})(T_{n+1} + T_n)}.$$

If $(-1)^n T_{-(n+1)} > 0$, then

$$\frac{1}{(-1)^n (T_n + T_{n-1})} > \frac{(-1)^n}{T_n} + \frac{1}{(-1)^{n+1}(T_{n+1} + T_n)},$$

we obtain

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} < \frac{1}{(-1)^n (T_n + T_{n-1})}.$$

Similarly, if $(-1)^n T_{-(n+1)} < 0$, then

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{T_k} > \frac{1}{(-1)^n (T_n + T_{n-1})}.$$

Combining the inequality of Lemma 2 (ii) and the above inequalities, we obtain the desired result. □

Theorem 7. *Let $n \geq 4$ be a positive integer. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=1}^{k} T_i} \right)^{-1} \right\rfloor = T_n - 1.$$

Proof. We shall prove our theorem directly, using Lemma 1 (ii), it is equivalent to

$$\frac{1}{T_n} < \sum_{k=n}^{\infty} \frac{2}{T_{k+2} + T_k - 1} < \frac{1}{T_n - 1} \qquad (n \geq 4).$$

Consider

$$\begin{aligned}
\frac{1}{T_n - 1} - \frac{2}{T_{n+2} + T_n - 1} - \frac{1}{T_{n+1} + 1} &= \frac{T_{n+2}T_{n+1} - T_n T_{n+1} + T_{n+1} + 3T_n - T_n T_{n+2} - T_n^2 - 2}{(T_n - 1)(T_{n+2} + T_n - 1)(T_{n+1} - 1)} \\
&= \frac{T_{n+1}^2 + T_{n-1}T_{n+1} + T_{n+1} + 3T_n - T_n T_{n+2} - T_n^2 - 2}{(T_n - 1)(T_{n+2} + T_n - 1)(T_{n+1} - 1)} \\
&= \frac{T_{-(n+2)} - T_{-(n+1)} + T_{n+1} + 3T_n - 2}{(T_n - 1)(T_{n+2} + T_n - 1)(T_{n+1} - 1)} \\
&> 0,
\end{aligned}$$

we have

$$\frac{2}{T_{n+2} + T_n - 1} < \frac{1}{T_n - 1} - \frac{1}{T_{n+1}}.$$

Thus,

$$\sum_{k=n}^{\infty} \frac{2}{T_{k+2} + T_k - 1} < \frac{1}{T_n - 1}.$$

In a similar way, we have

$$\begin{aligned}
\frac{1}{T_n} - \frac{2}{T_{n+2}+T_n-1} - \frac{1}{T_{n+1}} &= \frac{T_{n+2}T_{n+1} - T_nT_{n+1} - T_{n+1} + T_n - T_nT_{n+2} - T_n^2}{T_n(T_{n+2}+T_n-1)T_{n+1}} \\
&= \frac{T_{n+1}^2 - T_nT_{n+2} - T_{n+1} + T_{n-1}T_{n+1} - T_n^2 + T_n}{T_n(T_{n+2}+T_n-1)T_{n+1}} \\
&= \frac{T_{-(n+2)} - T_{n+1} - T_{-(n+1)} + T_n}{T_n(T_{n+2}+T_n-1)T_{n+1}} \\
&= \frac{T_{-(n+2)} - (T_n + T_{n-1} + T_{n-2}) - T_{-(n+1)} + T_n}{T_n(T_{n+2}+T_n-1)T_{n+1}} \\
&= \frac{T_{-(n+2)} - T_{n-1} - T_{n-2} - T_{-(n+1)}}{T_n(T_{n+2}+T_n-1)T_{n+1}} \\
&< 0 \qquad\qquad (n \geq 4).
\end{aligned}$$

Therefore,

$$\sum_{k=n}^{\infty} \frac{2}{T_{k+2}+T_k-1} > \frac{1}{T_n} \qquad (n \geq 4).$$

□

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References

[1] S. H. Holliday and T. Komatsu, On the sum of reciprocal generalized Fibonacci numbers, *Integers* **11A** (2011) A11.

[2] H. Ohtsuka and S. Nakamura, On the sum of reciprocal sums of Fibonacci numbers, *Fibonacci Quart.* **46/47** (2008/2009), 153–159.

[3] T. Komatsu, On the sum of reciprocal tribonacci numbers, *Ars. Comb.* **98** (2011), 447–459.

[4] T. Komatsu and V. Laohakosol, On the sum of reciprocals of numbers satisfying a recurrence relation of order s, *J. Integer Sequences* **13** (2010), Article 10.5.8.

[5] K. Kuhapatanakul, On the sums of reciprocal generalized Fibonacci numbers, *J. Integer Seq.*, **16** (2013), Article 13.7.1.

[6] K. Kuhapatanakul, Alternating sums of reciprocal generalized Fibonacci numbers, *SpringerPlus* **3** (485) (2014). Available at http://www.springerplus.com/content/3/1/485.

[7] R. Liu and F. Z. Zhao, On the sums of reciprocal hyperfibonacci numbers and hyperlucas numbers, *J. Integer Seq.*, **15** (2012), Article 12.4.5.

[8] A. G. Shannon and A. F. Horadam, Some properties of third-order recurrence relations. *Fibonacci Quart.* **10** (1972), 135–145.

[9] Z. Wenpeng and W. Tingting, The infinite sum of reciprocal Pell numbers, *Appl. Math. Comput.* **218** (2012), 6164–6167.

[10] Z. Wu and W. Zhang, The sums of the reciprocals of the Fibonacci polynomials and Lucas polynomials, *J. Inequal. Appl.* (2012), Article 2012:134.

[11] G. J. Zhang, The infinite sum of reciprocal of Fibonacci numbers, *J. Math. Res. Expo.* **31** (2011), 1030–1034.

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