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A Proof of Dixon's Identity

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Abstract

We give a proof of Dixon's binomial coefficient identity using recurrence equations and induction.

1 Introduction

In 1912, Dixon [1] established the following famous identity

$$\sum_{k=-a}^{a}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}=\frac{(a+b+c)!}{a!\cdot b!\cdot c!}, \tag{1}$$

where a, b, c are nonnegative integers.

There are many proofs of Eq. (1). In 1916, MacMahon proved his master theorem. In 1962, Good [4] found a short proof of Eq. (1) using MacMahon's master theorem. Gessel and Stanton [3] gave a short proof using Laurent series in 1985. In 1990, Ekhad [2] gave a very short proof using induction. In 2003, Guo [5] gave a short proof using polynomials.

We give an elementary proof of Eq. (1) using recurrence equations and induction. In order to obtain recurrence equations, we will use some so-called auxiliary sums.

2 Proof of Eq. (1)

Proof. Let $\mathbb{N}$ denote the set of positive integers, and let $\mathbb{N}_0$ denote the set of nonnegative integers. We let $S(a,b,c)$ denote the left side of Eq. (1), where $a,b,c\in\mathbb{N}_0$. We introduce the auxiliary sums $P(a,b,c)$, $Q(a,b,c)$ and $R(a,b,c)$ as follows

$$P(a,b,c) = \sum_{k=-a}^{a} (-1)^k (a^2 - k^2) \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k}, \tag{2}$$

$$Q(a,b,c) = \sum_{k=-a}^{a} (-1)^k (b^2 - k^2) \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k}, \tag{3}$$

$$R(a,b,c) = \sum_{k=-a}^{a} (-1)^k (c^2 - k^2) \binom{a+b}{a+k} \binom{b+c}{b+k} \binom{c+a}{c+k}. \tag{4}$$

Let $a,b,c \in \mathbb{N}$. We will use the well-known binomial identity $k\binom{n}{k} = n\binom{n-1}{k-1}$. Then

$$\begin{aligned}
P(a,b,c) &= \sum_{k=-a+1}^{a-1} (-1)^k (a+k) \binom{a+b}{a+k} \binom{b+c}{b+k} (a-k) \binom{c+a}{a-k} \\
&= \sum_{k=-a+1}^{a-1} (-1)^k (a+b) \binom{a+b-1}{a+k-1} \binom{b+c}{b+k} (c+a) \binom{c+a-1}{a-k-1} \\
&= (a+b)(a+c) \sum_{k=-a+1}^{a-1} (-1)^k \binom{a-1+b}{a-1+k} \binom{b+c}{b+k} \binom{c+a-1}{c+k}.
\end{aligned}$$

From the last equation above, it follows that

$$P(a,b,c) = (a+b)(a+c) S(a-1,b,c). \tag{5}$$

Similarly, we obtain that

$$Q(a,b,c) = (a+b)(b+c) S(a,b-1,c), \tag{6}$$

and

$$R(a,b,c) = (a+c)(b+c) S(a,b,c-1). \tag{7}$$

From Eqns. (2) and (3), we have that

$$P(a,b,c) - Q(a,b,c) = (a^2 - b^2) S(a,b,c). \tag{8}$$

Let $a \neq b$. From Eqns. (5), (6) and (8), it follows that

$$S(a,b,c) = \frac{1}{a-b}\big((a+c)S(a-1,b,c) - (b+c)S(a,b-1,c)\big). \tag{9}$$

Similarly, if $a \neq c$, then it follows that

$$S(a,b,c) = \frac{1}{a-c}\big((a+b)S(a-1,b,c) - (b+c)S(a,b,c-1)\big). \tag{10}$$

We treat the case when $a = b = c$ separately.

$$\begin{aligned}
S(a,a,a) &= \sum_{k=-a}^{a} (-1)^k \binom{2a}{a+k}^3 \\
&= \sum_{k=-a}^{a} (-1)^k \left(\binom{2a-1}{a+k} + \binom{2a-1}{a-1+k} \right)^3 \\
&= \sum_{k=-a}^{a} (-1)^k \left(\binom{2a-1}{a+k}^3 + 3\binom{2a}{a+k}\binom{2a-1}{a+k}\binom{2a-1}{a-1+k} + \binom{2a-1}{a-1+k}^3 \right) \\
&= \sum_{k=-a}^{a} (-1)^k \binom{2a-1}{a+k}^3 + \sum_{k=-a}^{a} (-1)^k \binom{2a-1}{a-1+k}^3 + 3S(a,a,a-1) \\
&= \sum_{k=-a}^{a-1} (-1)^k \binom{2a-1}{a+k}^3 + \sum_{k=-a+1}^{a} (-1)^k \binom{2a-1}{a-1+k}^3 + 3S(a,a,a-1) \\
&= \sum_{k=-a}^{a-1} (-1)^k \binom{2a-1}{a+k}^3 + \sum_{t=-a}^{a-1} (-1)^{t+1} \binom{2a-1}{a+t}^3 + 3S(a,a,a-1) \\
&= 3S(a,a,a-1).
\end{aligned}$$

Therefore,

$$S(a,a,a) = 3S(a,a,a-1). \tag{11}$$

Now we give a proof of Eq. (1) using induction:

Eq. (1) is true if $a = b = c = 0$. Let $n \in \mathbb{N}_0$ be fixed. Assume that Eq. (1) holds for all nonnegative integers a, b and c, such that $a + b + c = n$. Let a, b and c be nonnegative integers such that $a + b + c = n + 1$. Then we have three cases:

Case 1: If, at least, one of numbers a, b or c is equal to zero, then Eq. (1) obviously holds. Therefore, we may assume that $a, b, c \in \mathbb{N}$.

Case 2: If $a = b = c$, then we use Eq. (11). From the induction hypothesis, we have that

$$S(a,a,a-1) = \frac{(3a-1)!}{a! \cdot a! \cdot (a-1)!} \; .$$

$$\begin{aligned}
\text{Then } \; S(a,a,a) &= 3 \cdot \frac{(3a-1)!}{a! \cdot a! \cdot (a-1)!} \\
&= 3a \cdot \frac{(3a-1)!}{a! \cdot a! \cdot (a-1)! \cdot a} \\
&= \frac{(3a)!}{a! \cdot a! \cdot a!}, \text{ as desired.}
\end{aligned}$$

Case 3: If $a \neq b$, then we use Eq. (9). From the induction hypothesis, we have that

$$S(a-1,b,c) = \frac{(a+b+c-1)!}{(a-1)! \cdot b! \cdot c!}, \text{ and } S(a,b-1,c) = \frac{(a+b+c-1)!}{a! \cdot (b-1)! \cdot c!}.$$

$$\begin{aligned}
S(a,b,c) &= \frac{1}{a-b}\big((a+c)S(a-1,b,c) - (b+c)S(a,b-1,c)\big) \\
&= \frac{1}{a-b}\big((a+c)\frac{(a+b+c-1)!}{(a-1)! \cdot b! \cdot c!} - (b+c)\frac{(a+b+c-1)!}{a! \cdot (b-1)! \cdot c!}\big) \\
&= \frac{1}{a-b} \cdot \frac{(a+b+c-1)!}{(a-1)! \cdot (b-1)! \cdot c!}\big(\frac{a+c}{b} - \frac{b+c}{a}\big) \\
&= \frac{(a+b+c-1)! \cdot (a^2 - b^2 + c(a-b))}{(a-b) \cdot a! \cdot b! \cdot c!} \\
&= \frac{(a+b+c-1)! \cdot (a-b)(a+b+c)}{(a-b) \cdot a! \cdot b! \cdot c!} \\
&= \frac{(a+b+c)!}{a! \cdot b! \cdot c!}, \text{ as desired.}
\end{aligned}$$

If $a = b$ then $a \neq c$. We do similarly as before using Eq. (10). This proves Case 3. By induction, Eq. (1) follows. □

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