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A Note on a Theorem of Guo, Mező, and Qi

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Abstract

In a recent paper, Guo, Mező, and Qi proved an identity representing the Bernoulli polynomials at non-negative integer points m in terms of the m-Stirling numbers of the second kind. In this note, using a new representation of the Bernoulli polynomials in the context of the Zeon algebra, we give an alternative proof of the aforementioned identity.

1 Introduction

In an interesting recent paper [9], Guo, Mező, and Qi found the following identity

$$B_n(m) = \sum_{l=0}^{n} (-1)^l \frac{l!}{l+1} S_m(n+m, l+m) \tag{1}$$

relating the Bernoulli polynomials $B_n(m)$ at non-negative integer points m with m-Stirling numbers of the second kind $S_m(n+m, l+m)$. Eq. (1) is a generalization of the identity [8, p. 560]

$$B_n = \sum_{m=0}^{n} (-1)^m \frac{m!}{m+1} S(n,m).$$

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Note that $B_n \equiv B_n(0)$ and $S \equiv S_0$ are the usual Bernoulli numbers [16, Chap. 2] and Stirling numbers of the second kind [16, Chap. 1], respectively.

In this work, we will give another proof of (1) by showing that (1) is a straightforward consequence of a new Zeon representation [7, 11], [10, Chap. 5] of the Bernoulli polynomials.

We believe the approach here is of interest because it gives a straightforward demonstration of Guo, Mező, and Qi result, and, as a consequence, it provides another instance where computations involving Zeons and/or Grassmann variables provide direct and interesting results [1, 2, 5, 11, 12, 13, 15]. For more on Grassmann variables we refer the reader to the books of Berezin [3, Chap. 1], DeWitt [6, Chap. 1], and Rogers [14, Chap. 3].

For completeness, we recall some basic definitions and results already stated in previous work [11]. Throughout this work we let $\mathbb{R}$ denote the real numbers, $\mathbb{N}$ the positive integers, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the non-negative integers.

2 Zeon Algebra and Grassmann-Berezin Integral

Definition 1. The *Zeon algebra* $\mathcal{Z}_n \supset \mathbb{R}$ is defined as the associative algebra generated by the collection $\{\varepsilon_i\}_{i=1}^n$ $(n < \infty)$ and the scalar $1 \in \mathbb{R}$, such that $1\varepsilon_i = \varepsilon_i = \varepsilon_i 1$, $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ $\forall$ i, j and $\varepsilon_i^2 = 0$ $\forall$ i.

For $\{i, j, \ldots, k\} \subset \{1, 2, \ldots, n\}$ and $\varepsilon_{ij\cdots k} \equiv \varepsilon_i\varepsilon_j \cdots \varepsilon_k$ the most general element with n generators ε_i can be written as (with the convention of sum over repeated indices implicit)

$$\phi_n = a + a_i\varepsilon_i + a_{ij}\varepsilon_{ij} + \cdots + a_{12\cdots n}\varepsilon_{12\cdots n} = \sum_{\mathbf{i}\in 2^{[n]}} a_{\mathbf{i}}\varepsilon_{\mathbf{i}}, \tag{2}$$

with a, a_i, a_{ij}, $\ldots$, $a_{12\cdots n} \in \mathbb{R}$, $2^{[n]}$ being the power set of $[n] := \{1, 2, \ldots, n\}$, and $1 \leq i < j < \cdots \leq n$. We define the soul of ϕ_n by $s(\phi_n) := \phi_n - a$ [6, Chap. 1].

Definition 2. The *Grassmann-Berezin integral* on $\mathcal{Z}_n$, denoted by $\int$, is the linear functional $\int : \mathcal{Z}_n \to \mathbb{R}$ such that (we use throughout this work the compact notation $d\nu_n := d\varepsilon_n \cdots d\varepsilon_1$)

$$d\varepsilon_i d\varepsilon_j = d\varepsilon_j d\varepsilon_i, \quad \int \phi_n\big(\hat{\varepsilon}_i\big) d\varepsilon_i = 0 \text{ and} \int \phi_n\big(\hat{\varepsilon}_i\big)\varepsilon_i d\varepsilon_i = \phi_n\big(\hat{\varepsilon}_i\big),$$

where $\phi_n\big(\hat{\varepsilon}_i\big)$ means any element of $\mathcal{Z}_n$ with no dependence on ε_i. Multiple integrals are iterated integrals, i.e.,

$$\int f(\phi_n) d\nu_n = \int \cdots \left(\int \left(\int f(\phi_n) d\varepsilon_n \right) d\varepsilon_{n-1} \right) \cdots d\varepsilon_1.$$

Many functions of ordinary calculus admit extensions to the realm of Zeon algebra [11]. For instance, if $\phi_n = a + s(\phi_n)$ in Eq. (2), we have

$$e^{\phi_n} := e^a \sum_{m=0}^{n} \frac{s^m(\phi_n)}{m!} \tag{3}$$

and only a finite number of terms is present in the sum on the right-hand side of Eq. (3), since $s^m(\phi_n) = 0$ for $m > n$.

Likewise, we have

$$\frac{1}{\phi_n} = \frac{1}{a + s(\phi_n)} := \frac{1}{a}\sum_{m=0}^{n}\left(-\frac{1}{a}\right)^m s^m(\phi_n). \tag{4}$$

3 Proof of Eq. (1)

We are now ready to prove Eq. (1). We take $\varphi_n := \varepsilon_1 + \cdots + \varepsilon_n \in \mathcal{Z}_n$ from now on. We start with

$$B_n(x) = \sum_{m=0}^{n}\frac{(-1)^m}{m+1}\int e^{x\varphi_n}\left(e^{\varphi_n} - 1\right)^m d\nu_n. \tag{5}$$

We will proceed by showing that

$$\sum_{m=0}^{n}\binom{n+1}{m}B_m = 0 \tag{6}$$

with $n \in \mathbb{N}$, $B_0 \equiv 1$, $\binom{n}{m} := n!/\big(m!(n-m)!\big)$, and

$$B_n(x) = \sum_{m=0}^{n}\binom{n}{m}B_m x^{n-m}. \tag{7}$$

Eqs. (6) and (7) can be regarded as the definitions of the Bernoulli numbers [8, Eq. (6.79)] and the Bernoulli polynomials [8, Eq. (7.80)], respectively.

We will first show by induction on n that

$$\varphi_n = \sum_{m=0}^{n-1}\frac{(-1)^m}{m+1}\left(e^{\varphi_n} - 1\right)^{m+1}. \tag{8}$$

Indeed, it is easy to see that both sides of Eq. (8) give $\varphi_1 \equiv \varepsilon_1$ for $n = 1$. Next, using

Eqs. (3) and (4), we have

$$
\begin{aligned}
\varphi_{n+1} &= \varphi_n + \varepsilon_{n+1} \\
&= \varphi_n + \varepsilon_{n+1} \frac{e^{\varphi_n}}{1 + (e^{\varphi_n} - 1)} \\
&= \varphi_n + \varepsilon_{n+1} \sum_{m=0}^{n} (-1)^m e^{\varphi_n} \left(e^{\varphi_n} - 1\right)^m \\
&= \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \left(e^{\varphi_n} - 1\right)^{m+1} + \varepsilon_{n+1} \sum_{m=0}^{n} (-1)^m e^{\varphi_n} \left(e^{\varphi_n} - 1\right)^m \\
&= \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \binom{m+1}{0} \left(e^{\varphi_n} - 1\right)^{m+1} + \varepsilon_{n+1} \sum_{m=0}^{n} \frac{(-1)^m}{m+1} \binom{m+1}{1} e^{\varphi_n} \left(e^{\varphi_n} - 1\right)^m \\
&= \sum_{m=0}^{n} \frac{(-1)^m}{m+1} \left(e^{\varphi_n} + \varepsilon_{n+1} e^{\varphi_n} - 1\right)^{m+1} \\
&= \sum_{m=0}^{n} \frac{(-1)^m}{m+1} \left(e^{\varphi_{n+1}} - 1\right)^{m+1},
\end{aligned}
$$

and the result follows, i.e., Eq. (8) is true for all $n \geq 1$.

Now we can prove Eq. (6). Starting with Eq. (8) and using Eq. (3) we have

$$
\varphi_n = \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \sum_{l=1}^{n} \frac{\varphi_n^l}{l!} \left(e^{\varphi_n} - 1\right)^m .
$$

Integrating, we get ($n \geq 2$)

$$
\begin{aligned}
0 = \int \varphi_n d\nu_n &= \sum_{l=1}^{n} \sum_{1 \leq k_1, k_2, \ldots, k_l \leq n} \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \int \varepsilon_{k_1 k_2 \cdots k_l} \left(e^{\varphi_n} - 1\right)^m d\nu_n \\
&= \sum_{l=1}^{n} \binom{n}{l} \sum_{m=0}^{n-l} \frac{(-1)^m}{m+1} \int \left(e^{\varphi_{n-l}} - 1\right)^m d\nu_{n-l} \\
&= \sum_{l=1}^{n} \binom{n}{l} B_{n-l}
\end{aligned}
$$

and making the change of variables $n - l \mapsto l$ we obtain Eq. (6).

We will now show Eq. (7). Indeed, from Eq. (5) we have

$$\begin{aligned}
B_n(x) &= \sum_{l=0}^{n}\sum_{m=0}^{n}\frac{(-1)^l}{l+1}\frac{x^m}{m!}\int \varphi_n^m \left(e^{\varphi_n}-1\right)^l d\nu_n \\
&= \sum_{l=0}^{n}\frac{(-1)^l}{l+1}\int \left(e^{\varphi_n}-1\right)^l d\nu_n \\
&\qquad + \sum_{l=0}^{n}\sum_{m=1}^{n}\sum_{1\le l_1,l_2,\ldots,l_m\le n}\frac{(-1)^l}{l+1}x^m\int \varepsilon_{l_1 l_2\cdots l_m}\left(e^{\varphi_n}-1\right)^l d\nu_n \\
&= \sum_{m=0}^{n}\binom{n}{m}x^m\sum_{l=0}^{n-m}\frac{(-1)^l}{l+1}\int \left(e^{\varphi_{n-m}}-1\right)^l d\nu_{n-m} \\
&= \sum_{m=0}^{n}\binom{n}{m}x^m B_{n-m}
\end{aligned}$$

and making the change of variables $n-m\mapsto m$ we obtain Eq. (7).

We recall the generating function for the m-Stirling numbers [4, Thm. 16]

$$\sum_{n=l}^{\infty} S_m(n+m,l+m)\frac{x^n}{n!} = \frac{1}{l!}e^{mx}\left(e^x-1\right)^l \tag{9}$$

with $m\in\mathbb{N}_0$. The Zeon representation of $S_m(n+m,l+m)$ comes from the generating function in Eq. (9) taking, as in previous work [11], $x\to\varphi_n\in\mathcal{Z}_n$ and doing a Grassmann-Berezin integration over the Zeon algebra to get the representation

$$S_m(n+m,l+m) = \sum_{k=l}^{n} S_m(k+m,l+m)\underbrace{\int \frac{\varphi_n^k}{k!}d\nu_n}_{\delta_{k,n}} = \frac{1}{l!}\int e^{m\varphi_n}\left(e^{\varphi_n}-1\right)^l d\nu_n \tag{10}$$

with $\delta_{k,n}$ meaning the Kronecker delta. We note that the representation in Eq. (10) is a generalization of the representation of the usual Stirling numbers of the second kind [15, Prop. 2.1] obtained by setting $m=0$ in Eq. (10). Therefore, by setting $x\equiv m$ in Eq. (5), we conclude that

$$B_n(m) = \sum_{l=0}^{n}\frac{(-1)^l}{l+1}\int e^{m\varphi_n}\left(e^{\varphi_n}-1\right)^l d\nu_n,$$

which is equivalent to Eq. (1) using the Zeon representation of the numbers $S_m(n+m,l+m)$ in Eq. (10).

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