Journal of Integer Sequences, Vol. 19 (2016),
Article 16.4.5

Additive Functions on the Greedy and Lazy Fibonacci Expansions

Kalyan Chakraborty
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad, UP 211019
India
kalyan@hri.res.in

Imre Kátai and Bui Minh Phong
Computer Algebra Department
Eötvös Loránd University
Pázmány Péter Sétány I/C
1117 Budapest
Hungary
katai@inf.elte.hu
bui@inf.elte.hu

Abstract

We find all complex-valued functions that are additive with respect to both the greedy and the lazy Fibonacci expansions. We take it a little further by considering the subsets of these functions that are also multiplicative. In the final section we extend these ideas to Tribonacci expansions.

1 Introduction

The purpose of this paper is to study complex-valued functions that are additive with respect to (both greedy and lazy) Fibonacci and Tribonacci expansions. We begin by studying the functions with respect to Fibonacci expansions. The greedy Fibonacci expansion of a natural number was studied by Ostrowski [4], Zeckendorf [7], and Lekkerkerker [3], among many others. These days, authors call this greedy Fibonacci expansion the "Zeckendorf expansion", but Ostrowski [4] introduced it much earlier and in a more general setup. In the later sections we study a subset of the above set of functions, which is also multiplicative in nature. The lazy and greedy Tribonacci expansion of a natural number are defined analogously, and in the final section we extend our study of these functions to the Tribonacci expansions. We refer to the work of Carlitz, Scoville, and Hoggatt [2] for generalizations of Fibonacci representations.

2 $\mathcal{F}$-additive and $\mathcal{F}$-lazy additive functions

2.1 Fibonacci sequence

Let, as usual, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of positive integers, real and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of non-negative integers and $\mathcal{F} = (F_n)_{n \geq 0}$ be the sequence of Fibonacci numbers defined as follows:

$$F_1 = 1, F_2 = 2 \text{ and } F_{m+2} = F_{m+1} + F_m \quad (m \in \mathbb{N}).$$

Every natural number n has at least one expansion of the form

$$n = \sum_{\nu=1}^{k} \epsilon_\nu F_\nu \tag{1}$$

where the coefficients $\epsilon_\nu \in \{0, 1\}$. The expansion (1) is called the greedy or regular expansion of n and is characterized by the property that

$$\epsilon_l \epsilon_{l+1} = 0 \quad (l = 1, \ldots, k-1).$$

Similarly, the lazy expansion

$$n = \sum_{\mu=1}^{h} \delta_\mu F_\mu, \quad \delta_\mu \in \{0, 1\}, \delta_h = 1 \tag{2}$$

is characterized by the property that no two consecutive zeros are preceded by a one. Noting that $0F_\mu + 0F_{\mu+1} + 1F_{\mu+2}$ can be replaced by $1F_\mu + 1F_{\mu+1} + 0F_{\mu+2}$. It is known that every such sequence is the lazy expansion of some integer (cf. Steiner [5, Lemma 1]). Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be a function and we say that f is $\mathcal{F}$-additive for the regular expansion (1) if

$$f(n) = \sum_{\nu=1}^{k} \epsilon_\nu f(F_\nu).$$

Let $\mathcal{A}_{\mathcal{F}}^{(1)}$ be the set of such $\mathcal{F}$-additive functions. If $g : \mathbb{N} \longrightarrow \mathbb{C}$ is such that

$$g(n) = \sum_{\mu=1}^{h} \delta_\mu g(F_\mu)$$

with expansion of n as in (2), then we say that g is $\mathcal{F}$-lazy additive. We denote the set of these functions by $\mathcal{A}_{\mathcal{F}}^{(2)}$ and set $\mathcal{B} = \mathcal{A}_{\mathcal{F}}^{(1)} \cap \mathcal{A}_{\mathcal{F}}^{(2)}$.

Remark 1. We denote an element of $\mathcal{B}$ by f for notational convenience even though we have already written $f \in \mathcal{A}_{\mathcal{F}}^{(1)}$.

Theorem 2. *Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. If $f \in \mathcal{B}$, then*

$$f(F_m) = d_1\alpha^m + d_2\beta^m, \quad (m = 1, 2, \ldots) \tag{3}$$

where $d_i (i = 1, 2)$ are constants.

Conversely, if (3) holds, then $f \in \mathcal{B}$. Also for an arbitrary expansion

$$n = \sum_{\nu=1}^{k} e_\nu F_\nu, \quad e_\nu \in \{0, 1\},$$

one has

$$f(n) = \sum_{\nu=1}^{k} e_\nu f(F_\nu).$$

Proof. Let us assume that $f \in \mathcal{B}$ and $x_k = f(F_k)$. The lazy expansions of $F_i (i = 1, 2, 3)$ are

$$F_i = F_i (i = 1, 2) \text{ and } F_3 = F_1 + F_2.$$

Let $k \geq 4$ and that $S_k = \sum_{n=1}^{k} F_n$. Now it is easy to see

$$F_{k+2} = S_k + 2 \quad (k = 1, 2, \ldots).$$

Thus the largest component in the lazy expansion of F_{k+2} is F_{k+1}. Hence:

$$(\text{lazy expansion of } \ F_{k+2}) = F_{k+1} + (\text{lazy expansion of } \ F_k) \quad k \geq 4.$$

Thus,

$$x_{k+2} = x_{k+1} + x_k \qquad (k = 1, 2, \ldots). \tag{4}$$

Let us consider a sequence $\{x_k\}_{k\geq 1}$ so that (4) holds for its terms. Let

$$n = e_1F_1 + \cdots + e_kF_k, \quad n \geq 3, \quad e_i \in \{0, 1\} \tag{5}$$

and

$$\sigma\left(n|e_1, \ldots, e_k\right) = \sum_{\nu=1}^{k} e_\nu x_\nu. \tag{6}$$

We shall prove that σ (as defined in (6)) does not depend on $e_1, \ldots, e_k$. This can be verified directly for small values of n (e.g., up to $n = 6$), but for large n we can proceed as follows. Let us consider the largest ν, if there is any, for which $e_\nu = e_{\nu+1} = 1$. In such a situation we change

$$1F_\nu + 1F_{\nu+1} \to 1F_{\nu+2}$$

in the representation (5). In the case when $\nu = k-1$, we have

$$\begin{aligned} n &= e_1F_1 + \cdots + e_{k-2}F_{k-2} + 0F_{k-1} + 0F_{k-2} + 1F_{k+1} \\ &= \sum_{\nu=1}^{k+1} e'_\nu x_\nu. \end{aligned}$$

In case when $\nu < k-1$ it is clear that $e_{\nu+2} = 0$. After these substitutions we get

$$n = \sum_{\nu=1}^{k} e'_\nu F_\nu$$

with $e'_h = e_h$ if $h \neq \nu, \nu+1, \nu+2$. Also $e'_\nu = e'_{\nu+1} = 0$ and $e'_{\nu+2} = 1$.

Thus in both the cases

$$\sigma\left(n|e'_1, \ldots, e'_k\right) \quad \text{or} \quad \sigma\left(n|e'_1, \ldots, e'_{k+1}\right) = \sigma\left(n|e_1, \ldots, e_k\right).$$

On repeating this algorithm, after some steps we arrive at the greedy expansion. This completes the proof of Theorem 2. □

2.2 Modified Fibonacci sequence

Let $\mathcal{F}^* = (F^*_n)_{n\geq 0}$ be the modified Fibonacci sequence defined as follows:

$$F^*_0 = 0, F^*_1 = 1 \text{ and } F^*_{m+2} = F^*_{m+1} + F^*_m \quad (m = 0, 1, \ldots).$$

Then

$$F^*_{m+1} = F_m \quad (m = 1, 2, \ldots).$$

The greedy and lazy expansions of positive integers can be defined analogously and Theorem 2 remains true with the condition that $f(F^*_1) = f(F^*_2)$. We give an outline of the proof for the sake of completeness.

Theorem 3. *Let f be $\mathcal{F}^*$-greedy and $\mathcal{F}^*$-lazy additive. Then*

$$f(n) = cn \quad (n \in \mathbb{N}_0)$$

where c is an arbitrary constant.

Proof. We have as before $f(F_m^*) = d_1\alpha^m + d_2\beta^m$. Taking into account that $f(F_1^*) = f(F_2^*)$, we have

$$d_1\alpha + d_2\beta = d_1\alpha^2 + d_2\beta^2,$$

and thus

$$d_1\alpha(\alpha - 1) = -d_2\beta(\beta - 1).$$

Clearly $\alpha(\alpha-1) = \beta(\beta-1) = 1$ and this holds if and only if $d_2 = -d_1$. Also using Binet's formula [6] we have

$$F_m^* = \frac{1}{\sqrt{5}}\alpha^m - \frac{1}{\sqrt{5}}\beta^m \text{ for } m \in \mathbb{N}.$$

Hence we obtain that

$$f(F_m^*) = (d_1\sqrt{5})F_m^* \text{ and } f(n) = (d_1\sqrt{5})n.$$

Theorem 3 is now true with $c = d_1\sqrt{5}$. □

Remark 4. The function $f(n) = cn$ is clearly additive for an arbitrary $n = \sum_{\nu=0}^{k} e_\nu F_\nu^*$, $(e_\nu \in \{0,1\})$ representation since

$$f(n) = \sum_{\nu=0}^{k} e_\nu f(F_\nu^*)$$

holds.

3 $\mathcal{F}$-greedy ($\mathcal{F}^*$-greedy) additive and multiplicative functions

Let $\mathcal{M}$ be the set of all complex-valued multiplicative functions. We shall determine the sets $\mathcal{M} \cap \mathcal{A}_{\mathcal{F}}^{(1)}$ and $\mathcal{M} \cap \mathcal{A}_{\mathcal{F}^*}$, where $\mathcal{A}_{\mathcal{F}^*}$ is the set of all $\mathcal{F}^*$-greedy additive functions.

Theorem 5.

$$\mathcal{M} \cap \mathcal{A}_{\mathcal{F}^*} = \mathcal{M} \cap \mathcal{A}_{\mathcal{F}}^{(1)} = \{id\},$$

where $id(n) = n$ for every $n \in \mathbb{N}$.

Proof. Let $f \in \mathcal{M} \cap \mathcal{A}_{\mathcal{F}^*}$ and so $f(1) = 1$. First we show that

$$f(n) = n \quad \text{for} \quad n \leq 11 \text{ and } n = 144. \tag{7}$$

We list some initial members of $\mathcal{F}^*$ that will be of use.

$F_0^* = 0$, $F_1^* = 1$, $F_2^* = 1$, $F_3^* = 2$, $F_4^* = 3$, $F_5^* = 5$, $F_6^* = 8$, $F_7^* = 13$, $F_8^* = 21$, $F_9^* = 34$, $F_{10}^* = 55$, $F_{11}^* = 89$ and $F_{12}^* = 144$.

Let us call $f(2) = x$ and $f(3) = y$. We have the following greedy expansions:

$$
\begin{aligned}
4 &= 1+3 = F_2^* + F_4^*,\\
6 &= 1+5 = F_2^* + F_5^*,\\
7 &= 2+5 = F_3^* + F_5^*,\\
9 &= 1+8 = F_2^* + F_6^*,\\
10 &= 2+8 = F_3^* + F_6^*,\\
14 &= 1+13 = F_2^* + F_7^*.
\end{aligned}
$$

Hence (utilizing $f \in \mathcal{M} \cap \mathcal{A}_{\mathcal{F}^*}$),

$$
\begin{aligned}
f(4) &= f(3)+1 = y+1,\\
f(5) &= f(6)-1 = f(2)f(3)-1 = xy-1,\\
f(7) &= f(2)+f(5) = xy+x-1,\\
f(8) &= f(2)f(5)-f(2) = x^2y-2x,\\
f(9) &= f(1)+f(8) = x^2y-2x+1,\\
f(13) &= f(14)-1 = f(2)f(7)-1 = x^2y+x^2-x-1.
\end{aligned}
$$

The expansions of 12, 15 and 18 are

$$
\begin{aligned}
12 = 1+3+8 &= F_2^* + F_4^* + F_6^*,\\
15 = 2+13 &= F_3^* + F_7^*,\\
18 = 5+13 &= F_5^* + F_7^*.
\end{aligned}
$$

These give us,

$$
\begin{aligned}
f(3)f(4) = f(12) &= f(1)+f(3)+f(8),\\
f(3)f(5) = f(15) &= f(2)+f(13),\\
f(2)f(3)^2 = f(18) &= f(5)+f(13),
\end{aligned}
$$

and finally the following three relations:

$$
\begin{aligned}
y^2-1-x^2y+2x &= 0,\\
(x+xy-1)(-x+y-1) &= 0,\\
xy^2-xy-x^2y+2-x^2+x &= 0.
\end{aligned}
$$

We can conclude from the last three equations that

$$x = f(2) = 2, \quad y = f(3) = 3.$$

Thus

$$f(n) = n \text{ for } n \le 10.$$

Also, using

$$\begin{aligned} 11 &= 3+8 = F_4^* + F_6^* \\ 165 &= 21+144 = F_8^* + F_{12}^*, \end{aligned}$$

we obtain

$$\begin{aligned} f(11) &= f(3)+f(8) = 3+8 = 11, \\ f(144) &= f(3)f(5)f(11) - f(3)f(7) = 165-21 = 144. \end{aligned}$$

Now we complete the proof of Theorem 5 by proving that if $f(n) = n$ for every $n < N$ (where $N > 11$) then

$$f(N) = N.$$

This assertion clearly holds true in the following two cases:

(1) if N is not a prime-power; or

(2) if $N \notin \mathcal{F}^*$.

Thus let us assume that $N = F_m^*$ and that $N = p^\alpha$ for some $\alpha \in \mathbb{N}$. It follows from the result of Bugeaud et al. [1] that if $\alpha \geq 2$, then $N \in \{1, 8, 144\}$. In all these Thus $\alpha = 1$ and $N > 11$ is a prime. Let

$$2^e \| N+1, 2^\ell \| N+3, \quad e, \ell \in \mathbb{N}.$$

Then $\min(e, \ell) = 1$. Assume that $e = 1$. Since $1 < \frac{N+1}{2} < N$, we have

$$f(N+1) = f(2)f\Big(\frac{N+1}{2}\Big) = 2 \cdot \frac{N+1}{2} = N+1.$$

Now,

$$\begin{aligned} f(N+1) &= f(F_m^* + F_2^*) = \\ &= f(F_m^*) + f(F_2^*) = \\ &= f(F_m^*) + f(1) = \\ &= f(F_m^*) + 1. \end{aligned}$$

However, from above, we know that $f(N+1) = N+1$, so $f(F_m^*) + 1 = N+1$, from which it follows that $f(N) = f(F_m^*) = N$.

If $\ell = 1$, then $1 < \frac{N+3}{2} < N$, and we have

$$f(N+3) = f(2)f\Big(\frac{N+3}{2}\Big) = 2 \cdot \frac{N+3}{2} = N+3.$$

Now,

$$\begin{aligned} f(N+3) &= f(F_m^* + F_4^*) = \\ &= f(F_m^*) + f(F_4^*) = \\ &= f(F_m^*) + f(3) = \\ &= f(F_m^*) + 3, \end{aligned}$$

from which it follows that $f(N) = f(F_m^*) = N$.

Therefore, we have $\mathcal{M} \cap \mathcal{A}_{\mathcal{F}^*} = \{id\}$. The proof of the assertion $\mathcal{M} \cap \mathcal{A}_{\mathcal{F}}^{(1)} = \{id\}$ is similar and we omit it. $\square$

4 A Tribonacci-type sequence

A Tribonacci-type sequence $\mathcal{G} = (G_n)_{n\geq 1}$, which is a generalization of the Fibonacci sequence, is defined as follows:

$$\begin{aligned} G_i &= i, \text{ for } i = 1, 2, 3 \\ G_{m+3} &= G_{m+2} + G_{m+1} + G_m \quad (m = 1, 2, \ldots). \end{aligned} \tag{8}$$

Remark 6. It is worth mentioning that a Tribonacci-type sequence (8) for us is $1, 2, 3, 6, 11, \ldots$ whereas $1, 1, 2, 4, 7, 13, 24, 44, \ldots$ is more familiar.

We aim to prove similar results as in the previous sections for $\mathcal{G}$. Every integer $n \in \mathbb{N}$ can be written as

$$n = \sum_{k=1}^{\ell} e_k G_k, \quad e_k \in \{0, 1\}. \tag{9}$$

In general n has more than one expansion. We can define the greedy and lazy expansion as follows:

$$n = \sum_{k=1}^{\ell} \epsilon_k G_k \qquad \text{(greedy)}$$

and

$$n = \sum_{\mu=1}^{r} \delta_\mu G_\mu, \quad \delta_r = 1 \qquad \text{(lazy)}.$$

It is not hard to see that a sequence $\epsilon_1, \ldots, \epsilon_k$ is a sequence of the digits for a greedy expansion of some $n \in \mathbb{N}$ if $\epsilon_\nu \epsilon_{\nu+1} \epsilon_{\nu+2} \neq 111$ holds for every $\nu = 1, \ldots, k-2$. Similarly, $\delta_1, \ldots, \delta_r \quad (\delta_r = 1)$ is a lazy expansion of some $n \in \mathbb{N}$ if $\delta_\mu \delta_{\mu+1} \delta_{\mu+2} \neq 000$ holds for every $\mu = 1, \ldots, r-2$. We let lazyexp (G_h) denote the lazy expansion of G_h. Then

$$\text{lazyexp}(G_h) = \begin{cases} G_h, & \text{if } h = 1, 2, 3; \\ G_{h-1} + G_{h-2} + G_{h-3}, & \text{if } h = 4, 5. \end{cases}$$

If we write $S_k = G_1 + \cdots + G_k$, then

$$S_1 = G_1 = 1, S_2 = 3, S_3 = 6, S_4 = 10$$

and

$$S_{k+4} = 2S_{k+3} - S_k \text{ for } k \geq 1.$$

If we set

$$L_k = G_{k+1} - S_{k-1} \quad (k \geq 2),$$

then

$$L_{k+1} - L_k = G_{k+2} - S_k - (G_{k+1} - S_{k-1}) = G_{k+2} - G_k - G_{k+1} = G_{k-1}.$$

Thus $L_{k+1} - L_k > 0$. Hence the largest component in the lazy expansion of G_{k+1} is G_k and therefore

$$\mathrm{lazyexp}(G_{k+1}) = G_k + \mathrm{lazyexp}(G_{k-1} + G_{k-2}).$$

If $G_{k-1} + G_{k-2} \leq S_{k-2}$, then $G_{k-1} \leq S_{k-3}$, which cannot occur if $k \geq 5$. Thus

$$\mathrm{lazyexp}(G_{k-1} + G_{k-2}) = G_{k-1} + \mathrm{lazyexp}(G_{k-2}).$$

Let us assume now that g is $\mathcal{G}$-additive for the greedy and lazy expansions, then with $x_k = g(G_k)$, we obtain

$$x_{k+3} = x_{k+2} + x_{k+1} + x_k \quad (k = 1, 2, \ldots).$$

Now repeating the argument used in the proof of Theorem 2, we obtain

Theorem 7. *Let g be a $\mathcal{G}$-additive function for the greedy and for lazy expansion of the integers $n \in \mathbb{N}$. Let*

$$g(n) = \sum_{k=1}^{\ell} \epsilon_k g(G_k)$$

and

$$g(n) = \sum_{\mu=1}^{r} \delta_\mu g(G_\mu).$$

Then $x_h = g(G_h)$ $(h = 1, 2, \ldots)$ satisfies

$$x_{h+3} = x_{h+2} + x_{h+1} + x_h \quad (h = 1, 2, \ldots). \tag{10}$$

On the other hand, if x_h is such a sequence for which (10) holds, then

$$g(n) = \sum_{k=1}^{\ell} e_k g(G_k)$$

holds for an arbitrary expansion (9) of n with $e_k \in \{0, 1\}$.

Finally we show that,

Theorem 8. *Let g be $\mathcal{G}$-additive for the greedy expansion and that g is also a multiplicative function. Then $g(n) = n$ for every $n \in \mathbb{N}$.*

Proof. The proof is a direct consequence of the following assertions:

(A) $g(n) = n$ for $n \leq 6$;

(B) If $g(n) = n$ for every $n < N$, then $g(N) = N$.

Assuming that (A) is proved, assertion (B) is clearly true, if $N \leq 6$, or if $N \notin \mathcal{G}$, or if $N \neq$ prime power. Thus let us deal with the case when $N = G_m = p^\alpha$ for some prime p and $N \geq 7$. If $p = 2$, then $\alpha \geq 3$, $2^{\alpha-1} + 1 < 2^\alpha = N$ and

$$\begin{aligned} N + 2 &= G_m + G_2 \\ &= 2^\alpha + 2 \\ &= 2(2^{\alpha-1} + 1), \end{aligned}$$

which gives

$$\begin{aligned} g(N + 2) &= g(G_m) + 2 \\ &= g(2)g(2^{\alpha-1} + 1) \\ &= 2(2^{\alpha-1} + 1) \\ &= N + 2. \end{aligned}$$

Consequently,

$$g(N) = g(G_m) = N.$$

If $p > 2$, then $2^e \| N + 1 = G_m + G_1$, where $\frac{N+1}{2^e} < N$. If $\frac{N+1}{2^e} > 1$, then $2^e < N$ and so

$$\begin{aligned} g(N + 1) &= g(G_m) + 1 \\ &= g(2^e)g\Big(\frac{N + 1}{2^e}\Big) \\ &= N + 1. \end{aligned}$$

This implies that

$$g(N) = g(G_m) = N.$$

If $\frac{N+1}{2^e} = 1$, then $N = G_m = 2^e - 1$ and $2^{e-1} + 1 < 2^e - 1 = N$. Consequently,

$$\begin{aligned} g(G_m) + 3 &= g(G_m + G_3) \\ &= g(2^e + 2) \\ &= g(2)g(2^{e-1} + 1) \\ &= G_m + 3. \end{aligned}$$

Thus $g(N) = g(G_m) = N$.

It remains to prove assertion (A). Let $\xi = g(2), \eta = g(3)$. It follows from (8) that

$$G_1 = 1, G_2 = 2, G_3 = 3, G_4 = 6, G_5 = 11 \text{ and } G_6 = 20.$$

Now we use the facts that g is a $\mathcal{G}$-additive function for the greedy expansion and g is also a multiplicative function and get

$$\begin{aligned} g(4) &= g(G_3) + g(G_1) = \eta + 1, \\ g(5) &= g(G_3) + g(G_2) = \eta + \xi, \\ g(7) &= g(6) + g(1) = g(2)g(3) + 1 = \xi\eta + 1, \end{aligned}$$

and

$$\begin{aligned} g(11) &= g(12) - g(1) \\ &= g(3)g(4) - 1 \\ &= \eta^2 + \eta - 1. \end{aligned}$$

Again, using

$$\begin{aligned} 10 &= G_4 + G_3 + G_1 \\ &= 6 + 3 + 1, \\ 14 &= G_5 + G_3 \\ &= 11 + 3, \end{aligned}$$

and

$$\begin{aligned} 28 &= G_6 + G_4 + G_2 \\ &= 20 + 6 + 2, \end{aligned}$$

we get the following system of equations:

$$\begin{aligned} \xi^2 - \eta - 1 &= 0 \\ \xi^2\eta + \xi - \eta^2 - 2\eta + 1 &= 0 \\ (\eta + 1)(\xi\eta - 2\xi - \eta + 1) &= 0. \end{aligned}$$

The solutions of this system are $\xi = 2$ and $\eta = 3$. Therefore,

$$g(n) = n \text{ for } n \in \{1, 2, 3, 4, 5, 6, 7, 10, 11\}.$$

This completes the proof of Theorem 8. $\square$

5 Acknowledgments

The authors are indebted to the anonymous referee for many suggestions/comments which have helped immensely to improve the presentation and to bring in more clarity.

References

[1] Y. Bugeaud, M. Mignotte, and S. Siksek, Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers. *Ann. of Math.* **163** (2006), 969–1018.

[2] L. Carlitz, R. Scoville, and V. E. Hoggatt, Jr., Fibonacci representations of higher order. *Fibonacci Quart.* **10** (1972), 43–70.

[3] C. G. Lekkerkerker, Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci. *Simon Stevin*, **29** (1951-1952), 190–195.

[4] A. Ostrowski, Bemerkungen zur Theorie der Diophantischen Approximationen. *Abh. Math. Sem. Hamburg* **1** (1922), 77–98.

[5] W. Steiner, The joint distribution of greedy and lazy Fibonacci expansions, *Fibonacci Quart.*. **43** (2005), 60–69 .

[6] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section, Theory and Applications*, John Wiley & Sons, 1989.

[7] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. Roy. Sci. Liége* **41** (1972), 179–182.

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11A67.
Keywords: Fibonacci sequence, Tribonacci sequence, greedy expansion, lazy expansion.

Received July 7 2015; revised versions received October 18 2015; October 29 2015; December 7 2015; April 18 2016; April 24 2016. Published in *Journal of Integer Sequences*, May 9 2016.

Return to Journal of Integer Sequences home page.