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A (p, q)-Analogue of the r-Whitney-Lah Numbers

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Abstract

In this paper, we consider a (p, q)-generalization of the r-Whitney-Lah numbers that reduces to these recently introduced numbers when $p = q = 1$. We develop a combinatorial interpretation for our generalized numbers in terms of a pair of statistics on an extension of the set of r-Lah distributions wherein certain elements are assigned a color. We obtain generalizations of some earlier results for the r-Whitney-Lah sequence, including explicit formulas and various recurrences, as well as ascertain some new results for this sequence. We provide combinatorial proofs of some additional formulas in the case when $q = 1$, among them one that generalizes an identity expressing the r-Whitney-Lah numbers in terms of the r-Lah numbers. Finally, we introduce the (p, q)-Whitney-Lah matrix and study some of its properties.

1 Introduction

Given variables x and m and a positive integer k, define the generalized rising and falling factorials of order k by

$$[x|m]_k = x(x+m)\cdots(x+(k-1)m),$$
$$(x|m)_k = x(x-m)\cdots(x-(k-1)m),$$

with $[x|m]_0 = (x|m)_0 = 1$. The *r-Whitney-Lah numbers*, which we denote by $L(n,k) = L(n,k;r,m)$, were introduced by Cheon and Jung [3] and are connection constants in the polynomial identities

$$[x+2r|m]_n = \sum_{k=0}^{n} L(n,k)(x|m)_k, \qquad n \geq 0. \tag{1}$$

In the $m = 1$ case, the numbers $L(n,k;r,1)$ coincide with the r-Lah numbers [16, 18], which are also known as *unsigned r-restricted Lah numbers*; see sequence A143497 in OEIS [21]. These numbers enumerate partitions of the set $[n+r]$ into $k+r$ ordered lists in which the members of $[r]$ belong to different lists. When $r = 0$ and $m = 1$, the $L(n,k;r,m)$ reduce to the classical Lah numbers [8], which occur as A008297 in OEIS [21].

The r-Whitney-Lah numbers may also be defined, equivalently, as

$$L(n,k) = \sum_{j=k}^{n} w(n,j)W(j,k), \qquad 0 \leq k \leq n, \tag{2}$$

where $w(n,j)$ and $W(j,k)$ are the r-Whitney numbers of the first and second kind, respectively (see, e.g., [1, 3, 13, 14]). This generalizes a well-known formula for the classical Lah numbers in terms of the Stirling numbers of the first and second kind. See [18, Theorem 3.11] for the comparable relation involving r-Lah numbers. Here, we consider a two-variable polynomial generalization $L_{p,q}(n,k)$ of the r-Whitney-Lah sequence which reduces to it when $p = q = 1$. Starting with an extension of formula (2), we derive the recurrence for $L_{p,q}(n,k)$ and other properties that generalize results from [3] for $L(n,k)$. Note that when $r = 0$ and $m = p = 1$, the $L_{p,q}(n,k)$ coincide with previously studied q-Lah numbers [9]. See also [4, 5] for related generalized Stirling polynomials.

We develop a combinatorial interpretation of our $L_{p,q}(n,k)$ in terms of two statistics defined on extended r-Lah distributions in which certain elements are assigned a color. This allows one to ascertain further identities satisfied by $L_{p,q}(n,k)$ and to supply them with combinatorial explanations. When $p = q = 1$, our proofs specialize to provide counting arguments of various identities involving $L(n,k)$ which were shown [3] by algebraic methods using the Riordan group. In particular, we find a combinatorial proof of a generalization of the identity

$$L(n,k;r,m) = \sum_{i=k}^{n} \binom{n}{i} [2r(1-m)|m]_{n-i} m^{i-k} L(i,k;r,1), \qquad n,k \geq 0, \tag{3}$$

see [3, Theorem 4.2]. Note that formula (3) expresses the r-Whitney-Lah numbers in terms of the r-Lah numbers. Due to the negative terms occurring in the sum (once its summands are expanded), to explain (3), we make use of a sign-changing involution defined on a certain extension of the set of r-Lah distributions whose set of survivors has cardinality given by $L(n,k;r,m)$.

The organization of this paper is as follows. In the next section, we define $L_{p,q}(n,k)$ as a convolution of generalized r-Whitney numbers and determine its defining recurrence. In the third section, we develop a combinatorial interpretation for $L_{p,q}(n,k)$ and prove additional properties of these polynomials. Further attention is paid to the special cases $q=0$ and $q=-1$. In the fourth section, additional identities are proven for $L_{p,q}(n,k)$ by combinatorial arguments in the case when $q=1$, including a generalization of (3). In the final section, we introduce the (p,q)-Whitney-Lah matrix and study some of its properties.

We make use of the following terminology and notation. A block within a partition of a finite set in which the elements may be written in any order is described as *contents-ordered*. An element that is the smallest within a content-ordered block is said to be *minimal*; all other elements are *non-minimal*. If m and n are positive integers, then let $[m,n]=\{m,m+1,\ldots,n\}$ if $m\leq n$, with $[m,n]=\varnothing$ if $m>n$. The special case $[1,n]$ is denoted by $[n]$, with $[0]=\varnothing$. Throughout, we let I stand for the set $[r+1,r+n]$. Given a variable q, let $[i]_q=1+q+\cdots+q^{i-1}$ for a positive integer i and $[0]_q=0$. Finally, the q-binomial coefficient $\binom{n}{k}_q$ is defined as $\binom{n}{k}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$ for $0\leq k\leq n$ and is zero otherwise, where $[k]_q!=\prod_{i=1}^{k}[i]_q$ if $k\geq 1$ and $[0]_q!=1$.

2 Definition and recurrence

The (p,q)-Whitney numbers of the first and second kind (see [12, 20], where they were introduced) are connection constants in the identities

$$m^n\prod_{i=0}^{n-1}(x+[i]_q)=\sum_{k=0}^{n}w_{p,q}(n,k)(mx-[r]_p)^k,\qquad n\geq 0,\tag{4}$$

and

$$(mx+[r]_p)^n=\sum_{k=0}^{n}W_{p,q}(n,k)m^k\prod_{i=0}^{k-1}(x-[i]_q),\qquad n\geq 0.\tag{5}$$

Note that when $p=q=1$, the $w_{p,q}(n,k)$ and $W_{p,q}(n,k)$ reduce, respectively, to the r-Whitney numbers of the first and second kind. See [10] for a related q-analogue studied from an algebraic standpoint.

In analogy to (2), define $L_{p,q}(n,k)=L_{p,q}(n,k;r,m)$ by

$$L_{p,q}(n,k):=\sum_{j=k}^{n}w_{p,q}(n,j)W_{p,q}(j,k),\qquad 0\leq k\leq n.\tag{6}$$

Note that the $L_{p,q}(n,k)$ reduce to $L(n,k)$ when $p = q = 1$. Thus, one may refer to these numbers as *generalized* r*-Whitney-Lah numbers*. The $L_{p,q}(n,k)$ satisfy the following two-term recurrence.

Theorem 1. *If* $n, k \geq 1$, *then*

$$L_{p,q}(n,k) = L_{p,q}(n-1,k-1) + (2[r]_p + m([n-1]_q + [k]_q))L_{p,q}(n-1,k), \tag{7}$$

with $L_{p,q}(n,0) = \prod_{i=0}^{n-1}(2[r]_p + m[i]_q)$ *and* $L_{p,q}(0,k) = \delta_{k,0}$ *for all* $n, k \geq 0$.

Proof. The initial condition $L_{p,q}(0,k) = \delta_{k,0}$ is clear. Replacing x with $\frac{x+2[r]_p}{m}$ in (4), and using (5), gives

$$\begin{aligned}\prod_{i=0}^{n-1}(x + 2[r]_p + m[i]_q) &= \sum_{j=0}^{n} w_{p,q}(n,j)(x + [r]_p)^j \\ &= \sum_{j=0}^{n} w_{p,q}(n,j) \sum_{k=0}^{j} W_{p,q}(j,k) \prod_{i=0}^{k-1}(x - m[i]_q) \\ &= \sum_{k=0}^{n} \prod_{i=0}^{k-1}(x - m[i]_q) \sum_{j=k}^{n} w_{p,q}(n,j) W_{p,q}(j,k) \\ &= \sum_{k=0}^{n} L_{p,q}(n,k) \prod_{i=0}^{k-1}(x - m[i]_q). \end{aligned} \tag{8}$$

For $n \geq 1$, we then have

$$\begin{aligned}\sum_{k=0}^{n} L_{p,q}(n,k) \prod_{i=0}^{k-1}(x - m[i]_q) &= \prod_{i=0}^{n-1}(x + 2[r]_p + m[i]_q) \\ &= (x + 2[r]_p + m[n-1]_q) \prod_{i=0}^{n-2}(x + 2[r]_p + m[i]_q) \\ &= (x + 2[r]_p + m[n-1]_q) \sum_{k=0}^{n-1} L_{p,q}(n-1,k) \prod_{i=0}^{k-1}(x - m[i]_q) \\ &= \sum_{k=0}^{n-1} L_{p,q}(n-1,k) \prod_{i=0}^{k}(x - m[i]_q) \\ &\quad + \sum_{k=0}^{n-1}(2[r]_p + m([n-1]_q + [k]_q)L_{p,q}(n-1,k) \prod_{i=0}^{k-1}(x - m[i]_q) \\ &= \sum_{k=0}^{n}(L_{p,q}(n-1,k-1) + (2[r]_p + m([n-1]_q + [k]_q))L_{p,q}(n-1,k)) \prod_{i=0}^{k-1}(x - m[i]_q),\end{aligned}$$

upon replacing k by $k-1$ in the first sum. Equating coefficients of $\prod_{i=0}^{k-1}(x-m[i]_q)$ gives (7). Taking $x=0$ in (8) gives the formula for $L_{p,q}(n,0)$. □

Let $[x|m,q]_k = x(x+m[1]_q)\cdots(x+m[k-1]_q)$ and $(x|m,q)_k = x(x-m[1]_q)\cdots(x-m[k-1]_q)$ denote the generalized q-rising and q-falling factorials of order k. Then the proof of the prior theorem shows the following result, which generalizes (1).

Corollary 2. *The $L_{p,q}(n,k)$ are connection constants in the polynomial identities*

$$[x+2[r]_p|m,q]_n = \sum_{k=0}^{n} L_{p,q}(n,k)(x|m,q)_k, \qquad n \geq 0. \tag{9}$$

3 Combinatorial interpretation and properties

We now provide a combinatorial interpretation for the array $L_{p,q}(n,k)$. To do so, we first need to define a property possessed by certain elements within a contents-ordered block.

Definition 3. Suppose that a contents-ordered block contains a single element $s \in [r]$ along with some members of I. Then $x \in I$ is said to satisfy the nearness property if there exists no $y \in I$ with $y < x$ lying between x and s within the block. Such an x is said to be near s.

For example, if $n=7$, $r=3$, and $B=\{6,8,4,2,9,7\}$, then $s=2$ and 4, 9 and 7 are near 2. Given $0 \leq r \leq m$, by an r-partition of $[m]$, we mean a partition of the set $[m]$ in which the elements of $[r]$ belong to distinct blocks. We now define a set enumerated by $L_{p,q}(n,k)$ in the case when $p=q=1$.

Definition 4. Given $r \geq 0$ and $m \geq 1$, let $\mathcal{L}(n,k) = \mathcal{L}_{r,m}(n,k)$ denote the set of r-partitions of $[n+r]$ into $k+r$ contents-ordered blocks in which non-minimal elements not satisfying the nearness property are assigned one of m colors.

That $|\mathcal{L}(n,k)| = L(n,k)$ follows from comparing recurrences. Members of $\mathcal{L}(n,k)$ are referred to as (r,m)-*Lah distributions*. Within a member of $\mathcal{L}(n,k)$, we refer to the blocks containing an element of $[r]$ as *special* and to the remaining blocks comprised exclusively of elements of I as *non-special*. (The members of $[r]$ themselves will also at times be described as *special*.) Note that by definition all non-minimal elements within non-special blocks are assigned one of m colors, since only elements belonging to special blocks can satisfy the nearness property. We now define a couple of statistics on $\mathcal{L}(n,k)$.

Definition 5. Given $1 \leq i \leq r$, let s_i denote the number of elements of I satisfying the nearness property within the special block of $\lambda \in \mathcal{L}(n,k)$ containing i. Define $\alpha(\lambda) = \sum_{i=1}^{r}(i-1)s_i$.

We now define a property of certain elements within non-special blocks.

Definition 6. Suppose $\lambda \in \mathcal{L}(n,k)$ and that $x \in I$ belongs to a non-special block B of λ, with x not the first element of B. Then the predecessor of x is the first element of I to the left of x within B and smaller than x, provided such an element exists.

We must change the definition somewhat when discussing special blocks.

Definition 7. Suppose $x \in I$ belongs to a special block containing $s \in [r]$ and that x is not near s. We define the predecessor of x as we have already if x lies to the right of s within the block. Otherwise, the predecessor of x is obtained by applying the previous definition to the word obtained by writing all elements to the left of s within the block in reverse order.

For example, if $n = 10$, $r = 2$, and $B = \{8, 6, 9, 12, 7, 1, 4, 5, 10\}$ is the special block containing 1, then the elements 8, 9, 12, 5 and 10 do not satisfy the nearness property and have predecessors 6, 7, 7, 4 and 5, respectively. Note that within special blocks, only elements not satisfying the nearness property have predecessors, while within non-special blocks, only elements not corresponding to left-to-right minima have them.

Definition 8. Given $\lambda \in \mathcal{L}(n,k)$, let S_λ be the set of elements of I that have predecessors, and given $x \in S_\lambda$, let $pred(x)$ denote its predecessor. Let t_i be the number of left-to-right minima within the i-th non-special block of λ, where non-special blocks are arranged in ascending order of their minimal elements. Let

$$\beta_1(\lambda) = \sum_{x \in S_\lambda} (pred(x) - r - 1)$$

and

$$\beta_2(\lambda) = \sum_{i=1}^{k} (i-1)t_i,$$

and define $\beta(\lambda) = \beta_1(\lambda) + \beta_2(\lambda)$.

We consider the joint distribution of the α and β statistics on $\mathcal{L}(n,k)$.

Definition 9. If $n, k \geq 0$, then let

$$M_{p,q}(n,k) := \sum_{\lambda \in \mathcal{L}(n,k)} p^{\alpha(\lambda)} q^{\beta(\lambda)}.$$

It is seen that $M_{p,q}(n,k)$ satisfies the same recurrence as the one given above for $L_{p,q}(n,k)$ in (7), upon considering whether or not the element $n+r$ when added to a prior block satisfies the nearness property, and if not, the value of its predecessor (if it exists). Since the initial values are the same, we obtain the following combinatorial interpretation for $L_{p,q}(n,k)$.

Theorem 10. *If* $n, k \geq 0$, *then* $L_{p,q}(n,k) = M_{p,q}(n,k)$.

We now describe a second combinatorial interpretation for $L_{p,q}(n,k)$. Given $0 \leq k \leq j \leq n$, consider ordered pairs (γ, δ) such that $\gamma \in \Omega_{r,m}(n,j)$ and δ is an arrangement of the cycles of γ according to some member of $\Pi_{r,m}(j,k)$, where the sets $\Omega_{r,m}(n,j)$ and $\Pi_{r,m}(j,k)$ are as previously defined [20, Definitions 3.3 and 3.9]. Let $\mathcal{U}^{(j)}(n,k)$ denote the set of all such ordered pairs (γ, δ) and let $\mathcal{U}(n,k) = \cup_{j=k}^{n} \mathcal{U}^{(j)}(n,k)$. As in the combinatorial proof of [20, Theorem 3.11], let $(\gamma, \delta) \in \mathcal{U}(n,k)$ have weight $p^{v_1(\gamma)+w_1(\delta)} q^{v_2(\gamma)+w_2(\delta)}$, where the v_i and w_i statistics for $i = 1, 2$ are as defined in [20, Definitions 3.5 and 3.10]. By formula (6), we have that $L_{p,q}(n,k)$ is the sum of the weights of all members of $\mathcal{U}(n,k)$. Indeed, this combinatorial interpretation for $L_{p,q}(n,k)$ is equivalent to the one given in Theorem 10.

Proposition 11. *There is a weight-preserving bijection between $\mathcal{U}(n,k)$ and $\mathcal{L}(n,k)$.*

Proof. Given $\lambda = (\gamma, \delta) \in \mathcal{U}(n,k)$, write each cycle of γ belonging to a non-special block of δ with the smallest element *first* and arrange cycles left-to-right in descending order of smallest elements. On the other hand, we write each non-special cycle of γ belonging to a special block of δ with the smallest element *last* and arrange the non-special cycles left-to-right in ascending order of smallest elements, followed by the special cycle, written with its smallest element *first*. We then erase the parentheses enclosing the cycles of γ within all blocks of δ and let λ' denote the resulting member of $\mathcal{L}(n,k)$. Note that the number of objects within λ assigned one of m possible colors is the same as the number of letters assigned a color within λ'. Thus, the mapping $\lambda \mapsto \lambda'$ is seen to be bijection between $\mathcal{U}(n,k)$ and $\mathcal{L}(n,k)$. Assume that members of $\mathcal{L}(n,k)$ are weighted according to Definition 9 above. One may verify that λ has the same p- and q-weights as λ' for all λ, as required. □

We have the following further recurrence relations for $L_{p,q}(n,k)$.

Theorem 12. *If $n > k \geq 0$, then*

$$L_{p,q}(n,k) = \sum_{j=k-1}^{n-1} \prod_{i=j+1}^{n-1} (2[r]_p + m([k]_q + [i]_q)) L_{p,q}(j, k-1) \tag{10}$$

and

$$L_{p,q}(n,k) = \sum_{j=0}^{k} (2[r]_p + m([j]_q + [n-k+j-1]_q)) L_{p,q}(n-k+j-1, j). \tag{11}$$

Proof. To show (10), consider the size $r+j+1$ of the smallest element of the last non-special block within a member of $\mathcal{L}(n,k)$, where $k-1 \leq j \leq n-1$. Then there are $L_{p,q}(j,k-1)$ possibilities concerning the placement of the elements of $[r+j]$, with $r+j+1$ starting a new block. For each $i \in [j+2, n]$, there are $2[r]_p + m([k]_q + [i-1]_q)$ possibilities concerning placement of the element $r+i$, whence there are $\prod_{i=j+1}^{n-1}(2[r]_p + m([k]_q + [i]_q))$ possibilities for all such elements. Summing over j gives (10).

To show (11), consider the size, $r+n-k+j$, of the largest element of I not occupying its own block. Note that we must have $0 \leq j \leq k$. Then each element of $[r+n-k+j+1, r+n]$

within a member of $\mathcal{L}(n,k)$ must belong to its own block, leaving $k-(k-j)=j$ non-special blocks. Thus, there are $L_{p,q}(n-k+j-1,j)$ possibilities concerning arrangement of the elements of $[r+n-k+j-1]$. At the time that the element $r+n-k+j$ is placed, there are j non-special blocks currently occupied. Since this element is not to be the smallest within its block, there are $2[r]_p+m([j]_q+[n-k+j-1]_q)$ possibilities concerning its placement. Summing over all j gives (11). $\square$

Note that the $p=q=m=1$ case of (10) occurs as [18, Theorem 3.3]. Let $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ be sequences of complex numbers (or indeterminates), with the b_i distinct. Let the array $\{u(n,k)\}_{n,k\geq 0}$ be defined by the recurrence

$$u(n,k)=u(n-1,k-1)+(a_{n-1}+b_k)u(n-1,k), \qquad n,k\geq 1,$$

with boundary conditions $u(n,0)=\prod_{i=0}^{n-1}(a_i+b_0)$ and $u(0,k)=\delta_{k,0}$ for all $n,k\geq 0$. By [11, Theorem 1.1], we have the formula

$$u(n,k)=\sum_{j=0}^{k}\left(\frac{\prod_{i=0}^{n-1}(a_i+b_j)}{\prod_{\substack{i=0\\ i\neq j}}^{k}(b_j-b_i)}\right), \qquad n,k\geq 0. \tag{12}$$

From this, one can obtain an explicit formula for $L_{p,q}(n,k)$.

Theorem 13. *If $n,k\geq 0$, then*

$$L_{p,q}(n,k)=\frac{1}{m^k q^{\binom{k}{2}}[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\binom{k}{j}_q\prod_{i=0}^{n-1}(m[i]_q+m[j]_q+2[r]_p). \tag{13}$$

Proof. Taking $a_i=m[i]_q+2[r]_p$ and $b_i=m[i]_q$ in (12) implies

$$\begin{aligned} L_{p,q}(n,k)&=\sum_{j=0}^{k}\frac{\prod_{i=0}^{n-1}(m[i]_q+m[j]_q+2[r]_p)}{\prod_{i=0}^{j-1}m([j]_q-[i]_q)\cdot\prod_{i=j+1}^{k}m([j]_q-[i]_q)}\\ &=\sum_{j=0}^{j}\frac{\prod_{i=0}^{n-1}(m[i]_q+m[j]_q+2[r]_p)}{m^j q^{\binom{j}{2}}\prod_{i=0}^{j-1}[j-i]_q\cdot(-m)^{k-j}q^{j(k-j)}\prod_{i=j+1}^{k}[i-j]_q}\\ &=\frac{1}{m^k[k]_q!}\sum_{j=0}^{k}\frac{(-1)^{k-j}}{q^{\binom{j}{2}+j(k-j)}}\binom{k}{j}_q\prod_{i=0}^{n-1}(m[i]_q+m[j]_q+2[r]_p), \end{aligned}$$

which gives (13), by the fact $\binom{k}{2}=\binom{j}{2}+\binom{k-j}{2}+j(k-j)$. $\square$

Formula (13) may be simplified further when $q=1$.

Corollary 14. *If $n,k\geq 0$, then*

$$L_{p,1}(n,k)=\frac{m^{n-k}n!}{k!}\binom{n+\frac{2[r]_p}{m}-1}{n-k}. \tag{14}$$

Proof. Letting $q = 1$ in (13) gives

$$\begin{aligned}
L_{p,1}(n,k) &= \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}(mi+mj+2[r]_p)\\
&= \frac{m^{n-k}}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\prod_{i=0}^{n-1}\left(i+j+\frac{2[r]_p}{m}\right)\\
&= \frac{m^{n-k}n!}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{j+n+\frac{2[r]_p}{m}-1}{n} = \frac{m^{n-k}n!}{k!}\binom{n+\frac{2[r]_p}{m}-1}{n-k},
\end{aligned}$$

where we have made use of the binomial identity [7, Identity 5.24]. □

Remark 15. If $m,n \geq 1$ and $p,r \geq 0$, one can show using (14) that the sequence $L_{p,1}(n,k)$ for $0 \leq k \leq n$ is strictly log-concave (and therefore unimodal). Taking $p = m = 1$ in (14) yields [18, Theorem 3.7], which was given a bijective proof.

Using (14), one can obtain the following exponential generating function formula.

Corollary 16. *If $k \geq 0$, then*

$$\sum_{n\geq k} L_{p,1}(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-mx}\right)^k\left(\frac{1}{1-mx}\right)^{\frac{2[r]_p}{m}}. \tag{15}$$

We have the following explicit expression in the case when $q = 0$.

Proposition 17. *If $n \geq k \geq 1$, then*

$$L_{p,0}(n,k) = \binom{n-1}{k-1}(2[r]_p+2m)^{n-k}+2[r]_p\sum_{i=k}^{n-1}\binom{i-1}{k-1}(2[r]_p+2m)^{i-k}(2[r]_p+m)^{n-i-1}. \tag{16}$$

Proof. Taking $q = 0$ in (7), and noting $[i]_q\mid_{q=0} = [i > 0]$, gives the recurrence

$$L_{p,0}(n,k) = L_{p,0}(n-1,k-1) + 2([r]_p+m)L_{p,0}(n-1,k), \qquad n > k \geq 1, \tag{17}$$

with $L_{p,0}(n,0) = 2[r]_p(2[r]_p+m)^{n-1}$ for $n > 0$ and $L_{p,0}(n,n) = 1$ for $n \geq 0$. If $k \geq 0$, then let $f_k(x) = \sum_{n\geq k} L_{p,0}(n,k)x^n$. Note that $f_0(x) = \frac{1-mx}{1-(2[r]_p+m)x}$, by the initial values. Multiplying both sides of (17) by x^n, and summing over $n > k$, yields

$$f_k(x) - x^k = x(f_{k-1}(x) - x^{k-1}) + 2([r]_p+m)xf_k(x),$$

whence

$$f_k(x) = \frac{x}{1-2([r]_p+m)x}f_{k-1}(x), \qquad k \geq 1.$$

This implies

$$
\begin{aligned}
f_k(x) &= \frac{1-mx}{1-(2[r]_p+m)x} \cdot \frac{x^k}{(1-2([r]_p+m)x)^k} \\
&= \frac{1-mx}{1-(2[r]_p+m)x} \sum_{i\geq k} \binom{i-1}{k-1} (2[r]_p+2m)^{i-k} x^i \\
&= (1-mx) \sum_{n\geq k} x^n \sum_{i=k}^{n} \binom{i-1}{k-1} (2[r]_p+2m)^{i-k} (2[r]_p+m)^{n-i}.
\end{aligned}
$$

Extracting the coefficient of x^n gives

$$
\begin{aligned}
L_{p,0}(n,k) &= [x^n] f_k(x) \\
&= \sum_{i=k}^{n} \binom{i-1}{k-1} (2[r]_p+2m)^{i-k} (2[r]_p+m)^{n-i} \\
&\quad - m \sum_{i=k}^{n-1} \binom{i-1}{k-1} (2[r]_p+2m)^{i-k} (2[r]_p+m)^{n-i-1} \\
&= \binom{n-1}{k-1} (2[r]_p+2m)^{n-k} + 2[r]_p \sum_{i=k}^{n-1} \binom{i-1}{k-1} (2[r]_p+2m)^{i-k} (2[r]_p+m)^{n-i-1},
\end{aligned}
$$

as desired. □

It is possible to give a bijective proof of the prior result using our combinatorial interpretation for $L_{p,q}(n,k)$.

Combinatorial proof of Proposition 17.

We argue directly that the right-hand side of (16) gives the weight of all members of $\mathcal{L}(n,k)$ having β statistic value zero. In order for a member of $\mathcal{L}(n,k)$ to have zero β value, both its β_1 and β_2 values must be zero. For that to occur, each element of I either (i) is minimal, (ii) satisfies the nearness property, (iii) has $r+1$ as its predecessor, or (iv) is non-minimal, but is a left-to-right minimum in the leftmost non-special block. Note that exactly k elements of I must satisfy condition (i) within any member of $\mathcal{L}(n,k)$. First suppose that the element $r+1$ is among them. In that case, there are $\binom{n-1}{k-1}$ ways in which to choose the other elements that satisfy (i). For each of the remaining $n-k$ elements x of I, there are $2[r]_p$ possibilities if (ii) is to be satisfied, since in this case x comes either directly after or before some member of $[r]$ at the time it is placed (we assume elements are arranged one-by-one in ascending order). If (iii) is to be satisfied, the element x at the time it is placed must directly follow $r+1$ within its block. In order for (iv) to hold, the element x must be inserted at the very beginning of the leftmost non-special block (which in this case contains $r+1$). Note that the last two options entail that x be assigned one of m possible colors. Thus, there are $2[r]_p + 2m$ ways to place each x and hence $(2[r]_p + 2m)^{n-k}$ ways in all to position the remaining $n - k$ unchosen members of I.

Now assume that $r+1$ is not minimal and suppose that all elements of $J = [r+1, r+n-i]$ for some i belong to special blocks, with $r + n - i + 1$ being the smallest minimal element of I. Note that $k \leq i \leq n-1$ since $r+1$ is not minimal. There are $2[r]_p$ choices concerning the placement of $r + 1$ since it may directly follow or precede any member of $[r]$ and $2[r]_p + m$ choices for each of the other members of J since they may also directly follow $r + 1$, whence there are $2[r]_p(2[r]_p + m)^{n-i-1}$ possibilities in all for the members of J. Concerning the other $k - 1$ minimal elements, they can be picked arbitrarily from the set $[r + n - i + 2, r + n]$, which can be done in $\binom{i-1}{k-1}$ ways. For each of the remaining $i - k$ elements belonging to this set, there are $2[r]_p + 2m$ possibilities since each one may satisfy conditions (ii), (iii) or (iv), whence there are $(2[r]_p + 2m)^{i-k}$ possibilities concerning their arrangement. Summing over i gives the weight of all members of $\mathcal{L}(n, k)$ having β value zero in which the element $r + 1$ is not minimal. This yields the second term on the right-hand side of (16) and completes the proof. $\square$

Note that (16) does not follow from (12) since the $b_i = [i]_q$ terms are not distinct when $q = 0$. Formula (12) also does not apply in the case when $q = -1$ for the same reason, which we now consider. Note that the sign-balance of the β statistic on $\mathcal{L}_{n,k}$ can be obtained by evaluating $L_{p,q}(n, k)$ at $q = -1$. Let $H(x, y) = \sum_{n,k\geq 0} L_{p,-1}(n, k)x^n y^k$. There is the following explicit formula for $H(x, y)$.

Proposition 18. *We have*

$$H(x, y) = \frac{1 + x(y + c - m) - x^2((y - c)^2 + cm) - x^3(y - c - m)(y^2 - c^2 + cm)}{1 - 2x^2(y^2 + c^2) + x^4(y^2 - c^2 - cm)(y^2 - c^2 + cm)}, \tag{18}$$

where $c = 2[r]_p + m$.

Proof. Note that $H(x, y) = \sum_{n\geq 0} a(n; y)x^n$, where $a(n; y) = \sum_{k=0}^{n} L_{p,-1}(n, k)y^k$. We first write a recurrence for $a(n; y)$. Letting $q = -1$ in (7), and observing $[i]_q \mid_{q=-1} = [i \text{ is odd}]$, yields for $n \geq 1$ the formulas

$$L_{p,-1}(2n, k) = L_{p,-1}(2n - 1, k - 1) + (2[r]_p + m)L_{p,-1}(2n - 1, k), \qquad k \ \text{ even},$$

and

$$L_{p,-1}(2n, k) = L_{p,-1}(2n - 1, k - 1) + (2[r]_p + 2m)L_{p,-1}(2n - 1, k), \qquad k \ \text{ odd}.$$

Multiplying the last two recurrences by y^k, summing over even and odd values of k, respectively, and adding the resulting equations implies

$$\begin{aligned}a(2n;y) &= (y+2[r]_p+m)a(2n-1;y) + m\sum_{\substack{k=0\\k \text{ odd}}}^{n} L_{p,-1}(2n-1,k)y^k \\ &= (y+2[r]_p+m)a(2n-1;y) + \frac{m}{2}(a(2n-1;y)-a(2n-1;-y)) \\ &= \left(y+2[r]_p+\frac{3m}{2}\right)a(2n-1;y) - \frac{m}{2}a(2n-1;-y), \qquad n \geq 1. \end{aligned} \tag{19}$$

By similar reasoning, we also have

$$a(2n-1;y) = \left(y+2[r]_p+\frac{m}{2}\right)a(2n-2;y) - \frac{m}{2}a(2n-2;-y), \qquad n \geq 1. \tag{20}$$

Multiplying (19) and (20) by x^{2n} and x^{2n-1}, respectively, and summing over all $n \geq 1$, implies

$$\begin{aligned}H(x,y)-1 &= \sum_{n\geq 1} a(n;y)x^n \\ &= x\left(y+2[r]_p+\frac{m}{2}\right)H(x,y) - \frac{mx}{2}H(x,-y) + mx\sum_{n\geq 1} a(2n-1;y)x^{2n-1} \\ &= x\left(y+2[r]_p+\frac{m}{2}\right)H(x,y) - \frac{mx}{2}H(x,-y) + \frac{mx}{2}(H(x,y)-H(-x,y)),\end{aligned}$$

which gives the functional equation

$$(1-xy-cx)H(x,y) + \frac{mx}{2}H(-x,y) + \frac{mx}{2}H(x,-y) = 1. \tag{21}$$

Replacing (x,y) by $(-x,y)$, $(x,-y)$, and $(-x,-y)$ in (21), and solving the resulting linear system in four unknowns using Cramer's rule, yields (18). □

Remark 19. Note that $H(x,1)$ is the generating function for $u_n := \sum_{k=0}^{n} L_{p,-1}(n,k)$, which gives the sign balance of the β statistic over all (r,m)-Lah distributions of size $n+r$. From (18) when $y=1$, this quantity is seen to satisfy the fourth-order recurrence

$$u_n = 2(c^2+1)u_{n-2} - (c^2+cm-1)(c^2-cm-1)u_{n-4}, \qquad n \geq 4.$$

Similarly, using (18), one may deduce a more complicated linear recurrence satisfied by the array $L_{p,-1}(n,k)$.

4 Further formulas in the case $q = 1$

We have the following further results for $L_{p,q}(n,k)$ when $q=1$. The first formula in the next theorem generalizes [3, Corollary 4.3] and reduces to this result when $p=1$.

Theorem 20. *If $n, k \geq 0$ and $r \geq s \geq 0$, then*

$$L_{p,1}(n,k;r,m) = \sum_{i=k}^{n} \binom{n}{i} [2p^s[r-s]_p|m]_{n-i} L_{p,1}(i,k;s,m) \tag{22}$$

and

$$L_{p,1}(n,k;r,m) = \sum_{i=k}^{n}\sum_{j=k}^{i} \binom{n}{i}\binom{i}{j} m^{j-k}[2[r-s]_p|m]_{n-i}[2p^{r-s}[s]_p|m]_{i-j} L(j,k;0,1). \tag{23}$$

Proof. We argue that the right-hand side of (22) gives the sum of the p-weights of all members of $\mathcal{L}_{r,m}(n,k)$ according to the number, $n-i$, of elements of I belonging to the *last* $r-s$ special blocks. Once these elements have been chosen in $\binom{n}{i}$ ways, there are $2(p^s + \cdots + p^{r-1}) = 2p^s[r-s]_p$ possibilities concerning the choice of position for the smallest chosen element. After this smallest element is positioned, it is seen that there are $2p^s[r-s]_p+m$ possibilities for the position of the second smallest chosen element. Continuing in this manner, it follows that there are $[2p^s[r-s]_p|m]_{n-i}$ possibilities concerning the positions of the elements in the final $r-s$ special blocks. The remaining unselected i members of I, together with the elements of $[s]$, then constitute a configuration in $\mathcal{L}_{s,m}(i,k)$, whence the factor of $L_{p,1}(i,k;s,m)$. Summing over all i gives (22).

To show (23), we consider instead the number, $n-i$, of members of I belonging to the *first* $r-s$ special blocks. Note that there are $\binom{n}{i}[2[r-s]_p|m]_{n-i}$ ways in which to choose and arrange these elements. Concerning the remaining i members of I, we must specify how many of them belong to the final s special blocks, say $i-j$. There are $\binom{i}{j}[2p^{r-s}[s]_p|m]_{i-j}$ ways in which to choose and arrange these elements. The remaining j members of I then go in the non-special blocks and constitute a Lah distribution having k blocks, wherein the $j-k$ non-minimal elements are each assigned one of m possible colors. Summing over all possible i and j gives (23). □

The $p=1$ case of formula (23) does not seem to have been previously noted. On the other hand, the $p=1$ case of the next result occurs as [3, Theorem 4.2] and was shown by algebraic methods using the Riordan group. Here, we provide a combinatorial proof using the interpretation developed above for $L_{p,q}(n,k)$.

Theorem 21. *If $n, k \geq 0$, then*

$$L_{p,1}(n,k;r,m) = \sum_{i=k}^{n} \binom{n}{i} [2[r]_p(1-m)|m]_{n-i} m^{i-k} L_{p,1}(i,k;r,1). \tag{24}$$

Proof. To give a bijective proof of this result, within each (r,m)-Lah distribution, we distinguish further certain elements, assigning some of these elements different weights. We then consider the sum of the weights of configurations containing three types of elements defined as follows. A *critical element* (ce) is one that satisfies the nearness property and is either underlined (type 1) or overlined (type 2). A *sub-critical element* (sce) is a member of I having a predecessor that is either a ce or another sce. Critical elements of type 1 are assigned no color, whereas ce's of type 2 and sce's are assigned one of m colors. Finally, a *non-critical element* (nce) is a member of I that is neither critical nor sub-critical. By a *distinguished non-critical element* (dcne), we mean an nce that satisfies the nearness property. Note that a dnce is to be neither underlined nor overlined. All nce's except those that are the smallest within their block are assigned one of m colors (including dnce's, which did not receive a color in the definition of $\mathcal{L}(n,k)$ above). Let us refer to members of $\mathcal{L}_{r,m}(n,k)$ marked and colored as described above as *extended* (r,m)*-Lah distributions*.

Given $i \geq 0$, let $\mathcal{R}^{(i)}(n,k) = \mathcal{R}^{(i)}_{r,m}(n,k)$ denote the set of all extended (r,m)-Lah distributions containing exactly i nce's (including dnce's). Note that $\mathcal{R}^{(i)}(n,k)$ is empty unless $k \leq i \leq n$ since ce's and sce's must belong to special blocks, by definition. Let $\mathcal{R}(n,k) = \cup_{i=k}^{n}\mathcal{R}^{(i)}(n,k)$. We define the p-weight of $\lambda \in \mathcal{R}(n,k)$ just as we did above for $\mathcal{L}(n,k)$ (that is, per Definitions 5 and 9), with all elements satisfying the nearness property contributing (namely, the dnce's and ce's). Define the sign of $\lambda \in \mathcal{R}(n,k)$ to be $(-1)^{\nu(\lambda)}$, where $\nu(\lambda)$ denotes the number of ce's of type 2.

We now argue that the i-th term on the right-hand side of (24) gives the (signed) weight of all members of $\mathcal{R}^{(i)}(n,k)$. Let $S = \{s_1 < s_2 < \cdots < s_{n-i}\}$ be a subset of I of size $n-i$. We show that the weight of all members of $\mathcal{R}^{(i)}(n,k)$ whose set of ce's and sce's is precisely S is given by $[2[r]_p(1-m)|m]_{n-i}m^{i-k}L_{p,1}(i,k;r,1)$ for all S. First consider placing the elements of $[s_1-1]$. We arrange these elements in contents-ordered blocks such that the members of $[r]$ lie in different blocks and assign colors as described above for nce's. We next choose the position of the element s_1. Note that s_1 must be a ce (by the definitions), being the smallest element of S, and thus must be positioned directly adjacent to some member of $[r]$ within a special block. Thus, there are $2[r]_p$ possibilities for s_1 if it is of type 1, and $-2m[r]_p$ if it is of type 2, which implies that there are $2[r]_p(1-m)$ possibilities in all.

Having positioned all elements of $[s_1]$, we next arrange members of $[s_1+1, s_2-1]$, doing so one at a time, starting with s_1+1. Since each $j \in [s_1+1, s_2-1]$ is to be an nce, there are four disjoint possibilities concerning the position of each j: (i) j satisfies the nearness property (in which case, j is a dnce), (ii) j has an nce as predecessor, (iii) j is a left-to-right minimum that is not minimal within some non-special block (i.e., j is the first but not the smallest letter in its block at the time it is positioned), or (iv) j is the smallest letter of its block (i.e., j goes in its own block when its position is chosen). Note in all cases except (iv) that j is assigned one of m colors. We next place the element s_2. For this, there are $2[r]_p(1-m)$ possibilities for s_2 as before if s_2 is a ce, and m possibilities if s_2 is to be an sce (note in the latter case that s_2 would directly follow s_1 in its block), whence there are $2[r]_p(1-m)+m$ possibilities for s_2 in all.

We continue to add elements of I to a growing extended Lah distribution as described above until all elements have been added (lastly, choosing the positions of the elements of $[s_{n-i}+1, n+r]$). Reasoning as above, one sees that there are $2[r]_p(1-m)+tm$ possibilities regarding the choice of the position of the $(t+1)$-st element of S for $0 \leq t \leq n-i-1$. As these choices are in essence independent of one another, we get $[2[r]_p(1-m)|m]_{n-i}$ possibilities concerning the placement of the elements of S within a member of $\mathcal{R}^{(i)}(n,k)$. Since members of $I-S$ are never positioned directly after either a ce or an sce when placed and since these elements comprise all of the elements within the non-special blocks (as members of S must all go in special blocks, by definition), choosing the positions for members of $I-S$ is essentially the same as arranging the elements of $[r]$, together with i elements of I, according to some member of $\mathcal{L}_{1,r}(i,k)$ (with all non-minimal elements assigned one of m colors). Thus, there are $m^{i-k}L_{p,1}(n,k;r,1)$ possibilities concerning the placement of the elements of $I-S$. It follows that the weight of all members of $\mathcal{R}^{(i)}(n,k)$ is given by the i-th term of the sum on the right side of (24), as desired. Allowing i to vary, we have that the right side of (24) gives the signed weight of all members of $\mathcal{R}(n,k)$.

To complete the proof, we define an involution of $\mathcal{R}(n,k)$. Given $\lambda \in \mathcal{R}(n,k)$, let ℓ_0 be the largest ℓ (if it exists) satisfying either (i) ℓ is a ce of type 2, or (ii) ℓ is a dnce. If (i) occurs for ℓ_0, then erase the overlining, which makes ℓ_0 a dnce, whereas if (ii) occurs, then designate ℓ_0 a ce and overline it. In either case, note that the element ℓ_0 is among those assigned one of m colors, which remains the same. Let λ' denote the resulting configuration. For example, if $n=21$, $k=2$, $r=3$, $m=1$ and $\lambda \in \mathcal{R}(21,2)$ is given by

$$\lambda = \{20, \underline{16}, 1, \underline{11}, 22\},\ \{21, 24, \overline{18}, 2, \underline{9}, 8, 23, \overline{4}, 13\},\ \{\overline{17}, 3, 10, \underline{5}, 15\},\ \{7, 19, 6\},\ \{12, 14\},$$

then the elements 4, 8 and 18 meet either (i) or (ii) in the second special block, while 10 and 17 do so in the third, whence $\ell_0 = 18$. Then λ' is obtained from λ by replacing $\overline{18}$ with 18, keeping all other elements the same. Observe that we have $\lambda \in \mathcal{R}^{(8)}(21,2)$, whereas $\lambda' \in \mathcal{R}^{(11)}(21,2)$, since the elements 21 and 24 are sub-critical in λ but are non-critical in λ'.

The mapping $\lambda \mapsto \lambda'$ is well-defined since both ce's and dnce's satisfy the nearness property. Furthermore, if ℓ_0 is changed from a ce to a dnce, then all letters within the block between ℓ_0 and the closest element of I that is strictly smaller than ℓ_0 and on the same side of the special element s as ℓ_0 (or all letters beyond ℓ_0 when moving away from s if no such element of I exists) would go from being sce's to nce's, and conversely, if ℓ_0 is changed from a dnce to a ce. Since sce's and nce's are assigned one of m colors, the weight is preserved in both cases. Thus, the mapping $\lambda \mapsto \lambda'$ is seen to be a sign-changing, weight-preserving involution of $\mathcal{R}(n,k)$ where it is defined. It is not defined for those configurations $\rho \in \mathcal{R}(n,k)$ containing no dnce's and in which all ce's are of type 1. Note that all of these ρ have positive sign and that any elements contained within them that satisfy the nearness property are ce's of type 1 and hence are not assigned a color, though they do contribute to the p-weight. It is then seen that the total weight of all such ρ is given by $L_{p,1}(n,k;r,m)$, which completes the proof. $\square$

Our final result in this section generalizes [18, Theorem 3.11], reducing to it when $p = m = 1$.

Theorem 22. *If $n \geq k \geq 0$, then*

$$\sum_{i=k}^{n}(-1)^{i-k}L_{p,1}(n,i;r,m)L_{p,1}(i,k;s,m) = \binom{n}{k}[2[r]_p - 2[s]_p|m]_{n-k}. \qquad (25)$$

Proof. By Corollary 2, we have

$$
\begin{aligned}
[x+2[r]_p-2[s]_p|m]_n &= \sum_{i=0}^{n} L_{p,1}(n,i;r,m)(x-2[s]_p|m)_i \\
&= \sum_{i=0}^{n} L_{p,1}(n,i;r,m)\sum_{j=0}^{i}(-1)^{i-j}L_{p,1}(i,j;s,m)[x|m]_j \\
&= \sum_{j=0}^{n}\sum_{i=j}^{n}(-1)^{i-j}L_{p,1}(n,i;r,m)L_{p,1}(i,j;s,m)[x|m]_j.
\end{aligned}
$$

On the other hand, by the binomial theorem for the rising factorial (see, e.g., [7, p. 245]), which also applies for an arbitrary increment m, we have

$$[x+2[r]_p-2[s]_p|m]_n = \sum_{j=0}^{n}\binom{n}{j}[2[r]_p-2[s]_p|m]_{n-j}[x|m]_j.$$

Equating coefficients of $[x|m]_k$ gives (25). □

Taking $r=s$ in (25) gives the orthogonality relation

$$\sum_{i=k}^{n}(-1)^{i-k}L_{p,1}(n,i)L_{p,1}(i,k)=\delta_{n,k}, \qquad 0\le k\le n. \tag{26}$$

5 (p,q)-Whitney-Lah matrix

In this section, we introduce the (p,q)-Whitney-Lah matrix and determine various properties.

Definition 23. The (p,q)-Whitney-Lah matrix is the $n\times n$ matrix defined by

$$L^{p,q}(n) := [L_{p,q}(i,j)]_{0\le i,j\le n-1}.$$

For example when $n=4$, the matrix $L^{p,q}(n)$ is given by

$$
\begin{bmatrix}
1 & 0 & 0 & 0\\
2[r]_p & 1 & 0 & 0\\
4[r]_p^2+2m[r]_p & 4[r]_p+2m & 1 & 0\\
8[r]_p^3+4m[r]_p^2(q+2)+2m^2[r]_p(q+1) & 12[r]_p^2+4m[r]_p(q+4)+2m^2(q+2) & 6[r]_p+2m(q+2) & 1
\end{bmatrix}.
$$

In particular, note that $L^{p,q}(n)$ reduces to the r-Whitney-Lah matrix [15] when $p=q=1$, the $r=0$, $m=1$ case of which is referred to simply as the Lah matrix (see, e.g., [17]).

Before giving a factorization of $L^{p,q}(n)$, we need to recall some previously studied matrices. The (p,q)-Whitney matrices [20, 12] of the first and second kind are defined as follows:

$$
\begin{aligned}
\mathcal{W}^{(1)}_{p,q}(n) &:= [w_{p,q}(i,j;r,m)]_{0\le i,j\le n-1}\,,\\
\mathcal{W}^{(2)}_{p,q}(n) &:= [W_{p,q}(i,j;r,m)]_{0\le i,j\le n-1}\,.
\end{aligned}
$$

In particular, when $p = q = 1$, these matrices reduce to the r-Whitney matrices [15] of the first and second kind. When $m = p = 1$ and $r = 0$, the $\mathcal{W}^{(1)}_{p,q}(n)$ coincide with the $x = -1$ case of q-Stirling matrices of the first kind defined by $S^{(1)}_{q,n}[x] := [s_q(i,j)x^{i-j}]_{0\le i,j\le n-1}$, whereas the $\mathcal{W}^{(2)}_{p,q}(n)$ coincide with the $x = 1$ case of the q-Stirling matrices of the second kind $S^{(2)}_{q,n}[x] := [S_q(i,j)x^{i-j}]_{0\le i,j\le n-1}$ (see, e.g., [6, 19]).

It follows from the definition of the (p,q)-Whitney-Lah numbers that

$$L^{p,q}(n) = \mathcal{W}^{(1)}_{p,q}(n)\mathcal{W}^{(2)}_{p,q}(n).$$

Recall now the generalized $n \times n$ Pascal matrix $P_n[x]$ (see [2]) defined as

$$P_n[x] := \left[\binom{i}{j}x^{i-j}\right]_{0\le i,j\le n-1},$$

which reduces to the Pascal matrix P_n of order n when $x = 1$.

It was shown [20, 12] for $n \ge 1$ that

$$\mathcal{W}^{(1)}_{p,q}(n) = S^{(1)}_{q,n}[-m]P_n[[r]_p], \tag{27}$$

$$\mathcal{W}^{(2)}_{p,q}(n) = P_n[[r]_p]S^{(2)}_{q,n}[m]. \tag{28}$$

Therefore, we have

$$L^{p,q}(n) = S^{(1)}_{q,n}[-m]P_n[2[r]_p]S^{(2)}_{q,n}[m], \qquad n \ge 1, \tag{29}$$

which implies the following formula:

$$L_{p,q}(i,j) = \sum_{l=j}^{i}\sum_{k=l}^{i}\binom{k}{l}(-1)^{i-k}m^{i+l-(k+j)}s_q(i,k)S_q(l,j)(2[r]_p)^{k-l}, \qquad 0 \le j \le i. \tag{30}$$

Given $n \ge 1$, let $S_n[x]$ be the $n \times n$ matrix defined by $S_n[x] := [x^{i-j}]_{0\le j\le i\le n-1}$. Recall the factorization of the generalized Pascal matrix due to Zhang [22, Theorem 1] given by

$$P_n[x] = G_n[x]G_{n-1}[x]\cdots G_1[x], \qquad n \ge 1, \tag{31}$$

where $G_n[x] = S_n[x]$ and $G_k[x] = I_{n-k} \oplus S_k[x]$ for $1 \le k \le n-1$, with $\oplus$ denoting the matrix direct sum.

Proposition 24. *If $n \ge 2$, then*

$$L^{p,q}(n) = \overline{P}_1[mq^{n-2}]\overline{P}_2[mq^{n-3}]\cdots\overline{P}_{n-1}[m]P_n[2[r]_p]\overline{P}_{n-1}[m]\overline{P}_{n-2}[mq]\cdots\overline{P}_1[mq^{n-2}] \tag{32}$$

$$= \overline{P}_1[mq^{n-2}]\overline{P}_2[mq^{n-3}]\cdots\overline{P}_{n-1}[m]G_n[2[r]_p]\cdots G_1[2[r]_p]\overline{P}_{n-1}[m]\overline{P}_{n-2}[mq]\cdots\overline{P}_1[mq^{n-2}], \tag{33}$$

where

$$\overline{P}_k[x] = I_{n-k} \oplus P_k[x].$$

Proof. The matrix $P_n[[r]_p]$ can be factorized by means of (31). The matrices $S^{(1)}_{q,n}[m]$ and $S^{(2)}_{q,n}[m]$ have the factorizations (see [6, 19]):

$$\begin{aligned} S^{(1)}_{q,n}[m] &= \overline{P}_1[-mq^{n-2}]\cdots\overline{P}_{n-2}[-mq]\overline{P}_{n-1}[-m], \\ S^{(2)}_{q,n}[m] &= \overline{P}_{n-1}[m]\overline{P}_{n-2}[mq]\cdots\overline{P}_1[mq^{n-2}]. \end{aligned}$$

Therefore, formulas (32) and (33) follow from (29). □

When $q = 1$, one may also express $L^{p,q}(n)$ as follows.

Theorem 25. *If $n \geq 1$, then*

$$L^{p,1}(n) = I_n + M + M^2 + \frac{1}{2!}M^2 + \cdots + \frac{1}{(n-1)!}M^{n-1}, \tag{34}$$

where

$$(M)_{i,j} = \begin{cases} im\left(i + \frac{2[r]_p}{m} - 1\right), & \text{if } \ j = i-1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. From Corollary 14, we have

$$\begin{aligned} \left(\frac{1}{k!}M^k\right)_{k+l,l} &= \frac{1}{k}\left(\frac{1}{(k-1)!}M^{k-1}M\right)_{k+l,l} = \frac{1}{k}\left(\frac{1}{(k-1)!}M^{k-1}\right)_{k+l,l+1}(M)_{l+1,l} \\ &= \frac{1}{k}\left(m^k\frac{(k+l)!}{l!}\left(l + 2\frac{[r]_p}{m}\right)\binom{k+l+2\frac{[r]_p}{m}-1}{k-1}\right) \\ &= \left(m^k\frac{(k+l)!}{l!}\binom{k+l+2\frac{[r]_p}{m}-1}{k}\right) = L_{p,1}(k+l,l). \end{aligned}$$

This implies

$$\frac{1}{k!}(M^k)_{i,j} = \begin{cases} L_{p,1}(i, i-k), & \text{if } \ j = i-k; \\ 0, & \text{otherwise,} \end{cases}$$

for $i = k, \ldots, n-1$, which gives (34). □

In particular, taking $m = p = 1$ and $r = 0$ in (34), we obtain [17, Equation 2.4]. Let us define the generalized p-Whitney-Lah matrix $L^p_n[x]$ as

$$L^p_n[x] := \left[L_{p,1}(i,j)x^{i-j}\right]_{0\leq i,j\leq n-1},$$

the $x = 1$ case of which coincides with $L^{p,1}(n)$.

Theorem 26. *If $n \geq 1$, then*

$$L^p_n[x+y] = L^p_n[x]L^p_n[y]. \tag{35}$$

Proof. By multiplication of matrices and Corollary 14, we have

$$\begin{aligned}
(L_n^p[x]L_n^p[y])_{i,j} &= \sum_{k=j}^{i} x^{i-k} L_{p,1}(i,k) y^{k-j} L_{p,1}(k,j) \\
&= \sum_{k=j}^{i} x^{i-k} y^{k-j} m^{i-k} \frac{i!}{k!} \binom{i+\frac{2[r]_p}{m}-1}{i-k} m^{k-j} \frac{k!}{j!} \binom{k+\frac{2[r]_p}{m}-1}{k-j} \\
&= \sum_{k=j}^{i} x^{i-k} y^{k-j} m^{i-j} \frac{i!}{j!} \binom{i+\frac{2[r]_p}{m}-1}{i-j} \binom{i-j}{k-j} \\
&= \sum_{k=0}^{i-j} x^{i-j-k} y^{k} L_{p,1}(i,j) \binom{i-j}{k} \\
&= (x+y)^{i-j} L_{p,1}(i,j) = (L_n^p[x+y])_{i,j}\,.
\end{aligned}$$

□

The $m=p=1, r=0$ case of (35) occurs as [17, Theorem 7]. Taking $y=-x$ in (35), and noting $L_n^p[0]=I_n$, gives the following result.

Corollary 27. *If $n \geq 1$, then*

$$(L_n^p[x])^{-1} = L_n^p[-x]. \tag{36}$$

In particular, $(L^{p,1}(n))^{-1} = L_n^p[-1]$.

We also have the following formula for powers of $L^p(n)$.

Corollary 28. *If $n \geq 1$ and s are integers, then*

$$(L^{p,1}(n))^s = L_n^p[s]. \tag{37}$$

Proof. Applying (35) repeatedly implies

$$L_n^p[x_1+x_2+\cdots+x_m] = L_n^p[x_1]L_n^p[x_2]\cdots L_n^p[x_m],$$

for indeterminates $x_1, x_2, \ldots, x_m$. Taking $m=s$ and $x_1=x_2=\cdots=x_s=1$ gives (37) when s is non-negative. The case when s is negative in (37) follows from the positive case and (36). □

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