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Determinants Containing Powers of Generalized Fibonacci Numbers

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Abstract

We study the determinants of matrices whose entries are powers of the Fibonacci numbers. We then extend the results to include entries that are powers of a second-order linear recurrence relation. These results motivate a fundamental identity of determinants whose entries are powers of linear polynomials. Finally, we discuss the determinants of matrices whose entries are products of the general second-order linear recurrence relations.

1 Introduction

In the first issue of the *Fibonacci Quarterly*, Alfred posed the following problem [1]:

Prove

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1},$$

where F_n is the nth Fibonacci number.

In the second volume of the *Fibonacci Quarterly*, Parker [7] posed a similar problem with the exponent of each entry changed to 4 and the dimension of the matrix changed to 5×5. These two results naturally suggest the following question: what would be the determinant of an analogous form where the dimension of the matrix and the exponent of each entry are arbitrary? Carlitz [4] answered this question by showing that the determinant of the form $D(r,n) = \left| F^r_{n+i+j} \right|$, where $i, j = 0, 1, \ldots, r$, is given by

$$D(r,n) = (-1)^{(n+1)\binom{r+1}{2}} (F_1^r F_2^{r-1} \cdots F_r)^2 \cdot \prod_{i=0}^{r} \binom{r}{i}. \tag{1}$$

In this paper we generalize the entries even further by considering the determinant of the form $D(r,s,k,n) = \left| F^r_{s+k(n+i+j)} \right|$, where $i, j = 0, 1, \ldots, r$. We show that

$$D(r,s,k,n) = (-1)^{(s+kn+1)\binom{r+1}{2}} (F_k^r F_{2k}^{r-1} \cdots F_{rk})^2 \cdot \prod_{i=0}^{r} \binom{r}{i}. \tag{2}$$

Carlitz based his proof of formula (1) on the Binet form of the Fibonacci numbers; whereas we employ the matrix methods, such as the factorization method of Krattenthaler, to prove formula (2). In addition, we require a generalized form of the Catalan identity, proved by Melham and Shannon [6]. In Section 2, we present an alternative proof of the generalized Catalan identity using a matrix representation of the sequence and the properties of the matrix multiplication. In Section 3, we then present the proof of formula (2) as a special case of the determinant with entries involving the powers of the numbers in a second-order linear recurrence with constant coefficients. In the last section, we present the determinant whose entries are the products of the numbers defined as a second-order linear recurrence with constant coefficients using the Desnanot-Jacobi identity. The methodology used for this work relies on a computer programming developed by the second author [8].

2 Generalized Catalan identity

The well-known Catalan identity states that for all nonnegative integers s and i,

$$F_{s+i}^2 - F_s F_{s+2i} = (-1)^s F_i^2.$$

A generalization of this identity useful for this work is given by Melham and Shannon [6]. We shall, however, present an alternative proof of this generalization. For integers a, b, c_1, and c_2 with $c_2 \neq 0$, let $W_n = W_n(a, b; c_1, c_2)$ denote the second-order linear recurrence with constant coefficients, defined by

$$W_0 = a,\, W_1 = b \quad \text{and} \quad W_n = c_1 W_{n-1} + c_2 W_{n-2} \quad \text{for } n \geq 2.$$

With this notation, the Fibonacci sequence (F_n) and the Lucas sequence (U_n) correspond to $F_n = W_n(0, 1; 1, 1)$ and $U_n = W_n(0, 1; c_1, c_2)$, respectively. Moreover, we can use this recurrence to extend the definition of a sequence to the terms with negative indices. Usually, we can explicitly find the relationship between the negative-indexed terms and the positive-indexed terms. For example, for the Fibonacci sequence and the Lucas sequence, we have

$$F_{-n} = (-1)^{n+1} F_n \quad \text{and} \quad U_{-n} = (-1)^{n+1} c_2^{-n} U_n \quad \text{for } n \geq 1.$$

Proposition 1 (Generalized Catalan Identity)**.** *Let* $W_n = W_n(a_0, a_1; c_1, c_2)$ *and* $Y_n = W_n(b_0, b_1; c_1, c_2)$ *be second-order linear recurrences. Then*

$$W_{s+i} Y_{s+j} - W_s Y_{s+i+j} = (-c_2)^s (W_1 Y_j - W_0 Y_{j+1}) \cdot U_i, \tag{3}$$

for all integers $s, j,$ *and* i.

Proof. The proof is by induction on i. For the case when $i = 0$, the identity is trivial. For the case when $i = 1$, we have

$$\begin{pmatrix} W_{s+1} & Y_{s+j+1} \\ W_s & Y_{s+j} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_s & Y_{s+j} \\ W_{s-1} & Y_{s+j-1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^s \begin{pmatrix} W_1 & Y_{j+1} \\ W_0 & Y_j \end{pmatrix} \quad \text{for } s \geq 0,$$

and

$$\begin{pmatrix} W_{s+1} & Y_{s+j+1} \\ W_s & Y_{s+j} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} W_{s+2} & Y_{s+j+2} \\ W_{s+1} & Y_{s+j+1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ 1 & 0 \end{pmatrix}^s \begin{pmatrix} W_1 & Y_{j+1} \\ W_0 & Y_j \end{pmatrix} \quad \text{for } s < 0,$$

where the second equality in both equations follows from repeated application of the matrix representation in the first equality. Taking the determinant of both sides of any one equation yields

$$W_{s+1} Y_{s+j} - W_s Y_{s+j+1} = (-c_2)^s (W_1 Y_j - W_0 Y_{j+1}). \tag{4}$$

Now, consider two cases.

Case $i > 1$**:** Assume that the identity holds for some integers $i-1$ and $i-2$. Then

$$\begin{aligned}
W_{s+i} Y_{s+j} - W_s Y_{s+i+j} &= \begin{vmatrix} W_{s+i} & Y_{s+i+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= \begin{vmatrix} c_1 W_{s+(i-1)} + c_2 W_{s+(i-2)} & c_1 Y_{s+(i-1)+j} + c_2 Y_{s+(i-2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= c_1 \begin{vmatrix} W_{s+(i-1)} & Y_{s+(i-1)+j} \\ W_s & Y_{s+j} \end{vmatrix} + c_2 \begin{vmatrix} W_{s+(i-2)} & Y_{s+(i-2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= (-c_2)^s (W_1 Y_j - W_0 Y_{j+1})(c_1 U_{i-1} + c_2 U_{i-2}) \\
&= (-c_2)^s (W_1 Y_j - W_0 Y_{j+1}) U_i,
\end{aligned}$$

Case $i < 0$**:** Assume that the identity holds for some integers $i+1$ and $i+2$. Then

$$\begin{aligned}
W_{s+i}Y_{s+j} - W_sY_{s+i+j} &= \begin{vmatrix} W_{s+i} & Y_{s+i+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= \begin{vmatrix} \frac{-c_1}{c_2}W_{s+(i+1)} + \frac{1}{c_2}W_{s+(i+2)} & \frac{-c_1}{c_2}Y_{s+(i+1)+j} + \frac{1}{c_2}Y_{s+(i+2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= \frac{-c_1}{c_2}\begin{vmatrix} W_{s+(i+1)} & Y_{s+(i+1)+j} \\ W_s & Y_{s+j} \end{vmatrix} + \frac{1}{c_2}\begin{vmatrix} W_{s+(i+2)} & Y_{s+(i+2)+j} \\ W_s & Y_{s+j} \end{vmatrix} \\
&= (-c_2)^s(W_1Y_j - W_0Y_{j+1})(\frac{-c_1}{c_2}U_{i+1} + \frac{1}{c_2}U_{i+2}) \\
&= (-c_2)^s(W_1Y_j - W_0Y_{j+1})U_i,
\end{aligned}$$

where we apply the induction hypothesis in the penultimate equality in both cases. Hence, the proof is complete. □

We note some special cases of Proposition 1 useful in later sections:

$$U_{s+i}U_{s+j} - U_sU_{s+i+j} = (-c_2)^s \cdot U_iU_j, \tag{5}$$

$$U_{s+i}W_{s+j} - U_sW_{s+i+j} = (-c_2)^s \cdot U_iW_j, \tag{6}$$

and

$$W_{s+i}W_{s+j} - W_sW_{s+i+j} = (-c_2)^s \cdot (W_1W_j - W_0W_{j+1})U_i = (-c_2)^s \cdot \Delta \cdot U_iU_j, \tag{7}$$

where $\Delta = \begin{vmatrix} W_1 & W_2 \\ W_0 & W_1 \end{vmatrix} = a_1^2 - c_1a_0a_1 - c_2a_0^2$.

We justify the second equality of (7) by applying Proposition 1 as follows: In (3), let $Y_n = W_n$, substitute $j = 1$ and $s = 0$, respectively, and rename the index i by j.

The identity (7) can be restated as follows:

Corollary 2. *Let $k, n, r, s,$ and t be integers. Then*

$$W_{s+k(n+t)} = A(t)W_{s+kn} + B(t)U_{s+k(n+r)}, \tag{8}$$

where $A(t) = \dfrac{W_{k(t-r)}}{W_{-kr}}$ *and* $B(t) = \dfrac{-(-c_2)^{-kr} \cdot \Delta \cdot U_{kt}}{W_{-kr}}$.

Proof. Let integers k', n', r', s', and t' be given. Applying (7) with $s = -k'r'$, $i = k't'$, and $j = s' + k'(n' + r')$, we have

$$W_{-k'r'}W_{s'+k'(n'+t')} = W_{k'(t'-r')}W_{s'+k'n'} - (-c_2)^{-k'r'} \cdot \Delta \cdot U_{k't'}U_{s'+k'(n'+r')}.$$

Dividing by $W_{-k'r'}$ on both sides and renaming the variables yield the identity (8), as required. □

3 Determinants involving powers of terms of second-order recurrence

Our goal in this section is to give the closed form of the determinant of the $(r+1)\times(r+1)$ matrix whose entries are $W^r_{s+k(n+i+j)}$, where $i,j=0,1,\ldots,r$, and s and k are any integers. This matrix is

$$\mathbb{A}_n^{s,k}(r)=\begin{pmatrix} W^r_{s+kn} & W^r_{s+k(n+1)} & \cdots & W^r_{s+k(n+r)} \\ W^r_{s+k(n+1)} & W^r_{s+k(n+2)} & \cdots & W^r_{s+k(n+r+1)} \\ \vdots & \vdots & \ddots & \vdots \\ W^r_{s+k(n+r)} & W^r_{s+k(n+r+1)} & \cdots & W^r_{s+k(n+2r)} \end{pmatrix}. \tag{9}$$

We begin with the following proposition on the determinant whose entries are some power of linear polynomials.

Lemma 3. *Let $c_0,\ldots,c_r$ and $x_0,\ldots,x_r$ be real numbers. Then*

$$\det((c_jx_i+1)^r)_{0\le i,j\le r}=\prod_{0\le i<j\le r}(x_i-x_j)\prod_{0\le i<j\le r}(c_i-c_j)\prod_{i=0}^{r}\binom{r}{i}. \tag{10}$$

Proof. We prove Lemma 3 by using the factorization method [5]. The determinant will be 0 if x_0 is replaced by any x_i for $0<i\le r$, since some two rows of the matrix would be equal. This implies that (x_0-x_i) is a factor of the determinant for each $i=1,2,\ldots,r$. Similarly, we have that (x_1-x_i) is a factor of the determinant for each $i=2,\ldots,r$, and so on. In a similar manner, we see that if c_0 is replaced by any c_j for $0<j\le r$, then two columns of the matrix will be the same yielding the zero determinant. This implies that (c_0-c_j) is a factor of the determinant for each $j=1,2,\ldots,r$. Similarly, we have that (c_1-c_j) is a factor of the determinant for each $j=2,\ldots,r$, and so on. Therefore

$$\prod_{0\le i<j\le r}(x_i-x_j)\prod_{0\le i<j\le r}(c_i-c_j) \tag{11}$$

is a factor of this determinant. As a function of x_i for some fixed i or a function of c_j for some fixed j the determinant is a polynomial of degree r. This implies that the factor (11) and the required determinant have the same degree. Therefore, we can write the determinant as

$$\det((c_jx_i+1)^r)_{0\le i,j\le r}=C\prod_{0\le i<j\le r}(x_i-x_j)\prod_{0\le i<j\le r}(c_i-c_j),$$

for some constant C. To find C, we compare on both sides the coefficients of the monomial

$$(c_rx_r)^r(c_{r-1}x_{r-1})^{r-1}\cdots(c_0x_0)^0. \tag{12}$$

On the right-hand side, the monomial (12) appears as

$$\prod_{0<i<j\leq r}(-x_j)(-c_j)=\prod_{0<i<j\leq r}x_jc_j.$$

Hence, the coefficient of the monomial (12) on the right-hand side is just equal to C. We see that for each $0\leq i\leq r$, the term $(c_ix_i)^i$ appears in $(c_ix_i+1)^r$. Hence, on the left-hand side, the monomial (12) appears as

$$\pm\prod_{0\leq i\leq r}(c_ix_i+1)^r.$$

By the definition of the determinant, the sign in front of the expression is determined by the parity of the identity permutation $(0)(1)\cdots(r)$. Since $(0)(1)\cdots(r)=(01)(01)$, it follows that the identity permutation is even. Hence, the sign is determined to be $+$. Since, for each $0\leq i\leq r$, the coefficient of $(c_ix_i)^i$ in $(c_ix_i+1)^r$ is $\binom{r}{i}$, it follows that

$$C=\prod_{i=0}^{r}\binom{r}{i}.$$

This completes the proof of Lemma 3. $\square$

Corollary 4. *Let $A(j),B(j),X_i,Y_i$ be real numbers for $0\leq i,j\leq r$. Then*

$$\begin{aligned}&\det((A(j)X_i+B(j)Y_i)^r)_{0\leq i,j\leq r}\\&\quad=\prod_{0\leq i<j\leq r}(X_iY_j-X_jY_i)\prod_{0\leq i<j\leq r}(A(i)B(j)-A(j)B(i))\prod_{i=0}^{r}\binom{r}{i}.\end{aligned}\tag{13}$$

Proof. We prove in the case of $B(j)\neq 0$ and $Y_i\neq 0$ for all $0\leq i,j\leq r$. Applying Lemma 3 with $c_j=A(j)/B(j)$ and $x_i=X_i/Y_i$ for all $0\leq i,j\leq r$ and clearing the denominators, we obtain (13). The proof of the other case when some of $B(j)$ or Y_i are 0 follows from the fact that the determinant with polynomial entries is a continuous function. $\square$

Thus, this allows us to prove one of the main results of this paper.

Theorem 5. *The determinant of the matrix $\mathbb{A}_n^{s,k}(r)$ is given by*

$$\det\mathbb{A}_n^{s,k}(r)=(-1)^{(s+kn+1)\binom{r+1}{2}}\cdot c_2^{(s+kn)\binom{r+1}{2}+2k\binom{r+1}{3}}\cdot\Delta^{\binom{r+1}{2}}\cdot\prod_{i=0}^{r}\binom{r}{i}U_{(i+1)k}^{2(r-i)}.$$

Proof. By (9), (8), (13), and (6) respectively, we have

$$\det \mathbb{A}_n^{s,k}(r) = \det(W^r_{s+k(n+i+j)})_{0\le i,j\le r} = \det\big(\big(A(j)W_{s+k(n+i)} + B(j)U_{s+k(n+r+i)}\big)^r\big)_{0\le i,j\le r}$$

$$= \prod_{0\le i<j\le r} (W_{s+k(n+i)}U_{s+k(n+r+j)} - W_{s+k(n+j)}U_{s+k(n+r+i)}) \prod_{0\le i<j\le r} \big(A(i)B(j) - A(j)B(i)\big) \prod_{i=0}^{r} \binom{r}{i}$$

$$= \prod_{0\le i<j\le r} ((-c_2)^{s+k(n+r+i)}U_{k(j-i)}W_{-kr}) \prod_{0\le i<j\le r} \left(\frac{-\Delta\cdot(-c_2)^{-kr}}{W^2_{-kr}}(-c_2)^{ki}U_{k(j-i)}W_{-kr}\right) \prod_{i=0}^{r}\binom{r}{i}$$

$$= \prod_{0\le i<j\le r} \Big(-\Delta\cdot(-c_2)^{s+k(n+2i)}U^2_{k(j-i)}\Big) \prod_{i=0}^{r}\binom{r}{i},$$

Rearranging the last expression allows us to obtain the desired identity. □

Remark 6. By letting $W_n = F_n$ in Theorem 5 and noting that for the Fibonacci sequence $c_2 = 1$ and $\Delta = 1$, we then derive (2).

4 Determinants involving products of terms of second-order recurrence

The following lemma was mentioned by Krattenthaler [5] as part of the factorization method. We provide a different proof of this lemma using the Desnanot-Jacobi identity [3].

Lemma 7. *Let $X_0, \ldots, X_r, D_1, \ldots, D_r$, and $E_1, \ldots, E_r$ be indeterminates. Then*

$$\det\left(\prod_{\ell=j+1}^{r}(X_i + D_\ell)\cdot\prod_{m=1}^{j}(X_i+E_m)\right)_{0\le i,j\le r} = \prod_{0\le i<j\le r}(X_j - X_i)\cdot\prod_{1\le i\le j\le r}(D_j - E_i).$$

An alternative way of writing this identity would be

$$\det\left(\prod_{\ell=j+1}^{r}(A(d_\ell)X_i + B(e_\ell)Y_i)\cdot\prod_{m=1}^{j}(A(e_m)X_i + B(e_m)Y_i)\right)_{0\le i,j\le r}$$
$$= \prod_{0\le i<j\le r}(X_iY_j - X_jY_i)\prod_{1\le i\le j\le r}(B(e_i)A(d_j) - A(e_i)B(d_j)). \quad (14)$$

The main result of this section is as follows:

Theorem 8. *Let $d_1, \dots, d_r$ and $e_1, \dots, e_r$ be sequences of integers. Then*

$$\det\left(\prod_{\ell=j+1}^{r} W_{s+k(n+i+d_\ell)} \cdot \prod_{m=1}^{j} W_{s+k(n+i+e_m)}\right)_{0\le i,j\le r}$$
$$= (-\Delta)^{\binom{r+1}{2}} \cdot (-c_2)^{(s+kn)\binom{r+1}{2}+k\binom{r+1}{3}} \cdot \prod_{\ell=1}^{r} U_{k\ell}^{r+1-\ell} \prod_{1\le i\le j\le r} (-c_2)^{kd_j} U_{k(e_i-d_j)}.$$

Proof. We respectively apply the identities (8),(14), and (7), and the details of the proof are similar to those of Theorem 5. □

The following slight variation of the result by Alfred [2] called *basic power determinant* is a special case of this theorem.

Corollary 9. *Let $r, s,$ and k be integers with $r \ge 0$. Then*

$$\begin{vmatrix} F_s^r & F_s^{r-1}F_{s+k} & \cdots & F_{s+k}^r \\ F_{s+k}^r & F_{s+k}^{r-1}F_{s+2k} & \cdots & F_{s+2k}^r \\ \vdots & \vdots & \ddots & \vdots \\ F_{s+rk}^r & F_{s+rk}^{r-1}F_{s+(r+1)k} & \cdots & F_{s+(r+1)k}^r \end{vmatrix} = (-1)^{(s+1)\binom{r+1}{2}+k\binom{r+1}{3}} F_k^{\binom{r+1}{2}} \prod_{\ell=1}^{r} F_{k\ell}^{r+1-\ell}.$$

Proof. This identity follows immediately from Theorem 8 by letting $d_1 = d_2 = \cdots = d_r = 0$, $e_1 = e_2 = \cdots = e_r = 1$, $n = 0$ together with the recurrence and the initial values of the Fibonacci numbers. □

Another interesting case arises when we let (d_j) and (e_j) in Theorem 8 be in some specific forms.

Corollary 10. *Let $s, k, n,$ and p be integers. Let $d_j = p-1+j$ and $e_j = j-1$ for $1 \le j \le r$. Then*

$$\det\left(\prod_{\ell=j+1}^{r} W_{s+k(n+i+d_\ell)} \cdot \prod_{m=1}^{j} W_{s+k(n+i+e_m)}\right)_{0\le i,j\le r}$$
$$= \Delta^{\binom{r+1}{2}} (-c_2)^{(s+kn)\binom{r+1}{2}+2k\binom{r+1}{3}} \cdot \prod_{\ell=1}^{r} U_{\ell k}^{r+1-\ell} \cdot \prod_{\ell=0}^{r-1} U_{k(p+\ell)}^{r-\ell}.$$

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