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On Sums of Powers of the p-adic Valuation of $n!$

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Abstract

We study sums where the multiplicities of the primes in the prime factorization of $n!$ appear, and obtain a strong connection between these sums and the Riemann zeta function.

1 Introduction

Consider the prime factorization of a positive integer a,

$$a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k},$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of a and $s_1, s_2, \ldots, s_k$ are their multiplicities. We let $\Omega(a)$ denote the total number of prime factors [3], that is,

$$\Omega(a) = s_1 + s_2 + \cdots + s_k.$$

Consider the prime factorization of $n!$. We let $E(p)$ denote the multiplicity of a prime p appearing in this factorization. Consequently, we can write the prime factorization of $n!$ as follows:

$$n! = \prod_{2 \le p \le n} p^{E(p)}.$$

The following asymptotic formula is well-known: (e.g., [3, 5])

$$\Omega(n!) = \sum_{2\le p\le n} E(p) = n\log\log n + An + o\left(\frac{n}{\log n}\right), \tag{1}$$

where the constant A is

$$A = M + \sum_{p} \frac{1}{p(p-1)} \approx 1.034653,$$

and M is Mertens's constant.

Let $k \ge 1$ an arbitrary but fixed positive integer. We study the sequences

$$\sum_{2\le p\le n} \frac{1}{E(p)^k},$$

and prove a strong connection between these sequences and the Riemann zeta function $\zeta(s)$. We also study the sequences

$$\sum_{2\le p\le n} E(p)^k \qquad (k\ge 2).$$

2 Main Results

Theorem 1. *Let $k \ge 1$ be an arbitrary but fixed positive integer. For any sufficiently large integer n we have*

$$\sum_{2\le p\le n} \frac{1}{E(p)^k} = C_k \frac{n}{\log n} + O_k\left(\frac{n}{\log^2 n}\right), \tag{2}$$

where

$$C_k = (-1)^k + \sum_{j=2}^{k+1} (-1)^{k+j-1}\zeta(j). \tag{3}$$

Proof. By the prime number theorem, we have

$$\pi(x) = \sum_{2\le p\le x} 1 = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{4}$$

Consider the prime factorization of $n!$. The multiplicity $E(p)$ of the prime p is, by Legendre's theorem, equal to

$$E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

If p satisfies the inequality

$$\frac{n}{j+1} < p \leq \frac{n}{j},$$

where j is a fixed positive integer, and the inequality

$$p > \sqrt{n},$$

then we obtain

$$E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = j.$$

Now we have

$$\sum_{p \leq n} \frac{1}{E(p)^k} = \sum_{p \leq \sqrt{n}} \frac{1}{E(p)^k} + \sum_{\sqrt{n} < p \leq n} \frac{1}{E(p)^k}, \tag{5}$$

where

$$\sum_{p \leq \sqrt{n}} \frac{1}{E(p)^k} = O\left(\pi\left(\sqrt{n}\right)\right) = O\left(\frac{\sqrt{n}}{\log n}\right) = O\left(\frac{n}{\log^2 n}\right). \tag{6}$$

Let $J = \lfloor \sqrt{n} \rfloor$. If $p > \sqrt{n}$ then $E(p) = \left\lfloor \frac{n}{p} \right\rfloor$. Hence, we have

$$\sum_{\sqrt{n} < p \leq n} \frac{1}{E(p)^k} = \sum_{j=1}^{J-1} \frac{1}{j^k} \sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} 1 = \sum_{j=1}^{J-1} \frac{1}{j^k} \left(\pi\left(\frac{n}{j}\right) - \pi\left(\frac{n}{j+1}\right)\right). \tag{7}$$

Note that, by Eq. (4), we have

$$\pi\left(\frac{n}{j}\right) = \frac{n}{j \log \frac{n}{j}} + O\left(\frac{n}{j \log^2 \frac{n}{j}}\right) = \frac{n}{j \log n} + O\left(\frac{\log j}{j} \frac{n}{\log^2 n}\right) + O\left(\frac{n}{j \log^2 \frac{n}{j}}\right), \tag{8}$$

where we use the formula

$$\frac{1}{1 - f(n)} = 1 + \left(\frac{1}{1 - f(n)}\right) f(n).$$

In the same way, we find that

$$\pi\left(\frac{n}{j+1}\right) = \frac{n}{(j+1) \log n} + O\left(\frac{\log(j+1)}{(j+1)} \frac{n}{\log^2 n}\right) + O\left(\frac{n}{(j+1) \log^2 \frac{n}{(j+1)}}\right) \tag{9}$$

Eqs. (8) and (9) give

$$\frac{1}{j^k}\left(\pi\left(\frac{n}{j}\right)-\left(\pi\left(\frac{n}{j+1}\right)\right)\right)=\frac{1}{j^k}\left(\frac{1}{j}-\frac{1}{j+1}\right)\frac{n}{\log n}+O\left(\frac{\log j}{j^{k+1}}\frac{n}{\log^2 n}\right)$$
$$-\quad O\left(\frac{\log(j+1)}{j^k(j+1)}\frac{n}{\log^2 n}\right)+O\left(\frac{n}{j^{k+1}\log^2\frac{n}{j}}\right)-O\left(\frac{n}{j^k(j+1)\log^2\frac{n}{j+1}}\right). \tag{10}$$

Note that

$$\sum_{j=1}^{J-1}\frac{1}{j^k}\left(\frac{1}{j}-\frac{1}{j+1}\right)\frac{n}{\log n}=C_k\frac{n}{\log n}+O\left(\frac{n}{\log^2 n}\right), \tag{11}$$

where

$$C_k=\sum_{j=1}^{\infty}\frac{1}{j^k}\left(\frac{1}{j}-\frac{1}{j+1}\right). \tag{12}$$

On the other hand, we have

$$\sum_{j=1}^{J-1}O\left(\frac{\log j}{j^{k+1}}\frac{n}{\log^2 n}\right)=O\left(\frac{n}{\log^2 n}\sum_{j=1}^{J-1}\frac{\log j}{j^{k+1}}\right)=O\left(\frac{n}{\log^2 n}\right), \tag{13}$$

Analogously, we find that

$$\sum_{j=1}^{J-1}O\left(\frac{\log(j+1)}{j^k(j+1)}\frac{n}{\log^2 n}\right)=O\left(\frac{n}{\log^2 n}\right). \tag{14}$$

Besides, we have

$$\sum_{j=1}^{J-1}O\left(\frac{n}{j^{k+1}\log^2\frac{n}{j}}\right)=O\left(\sum_{j=1}^{J-1}\frac{n}{j^{k+1}\log^2\frac{n}{j}}\right)=O\left(\frac{n}{\log^2 n}\right), \tag{15}$$

since

$$\sum_{j=1}^{J-1}\frac{n}{j^{k+1}\log^2\frac{n}{j}}=O\left(\sum_{j=1}^{J-1}\frac{n}{j^{k+1}\log^2\frac{n}{J}}\right)=O\left(\frac{n}{\log^2 n}\right).$$

In the same manner, we obtain

$$\sum_{j=1}^{J-1}O\left(\frac{n}{j^k(j+1)\log^2\frac{n}{j+1}}\right)=O\left(\frac{n}{\log^2 n}\right). \tag{16}$$

Substituting equations (11), (13), (14), (15) and (16) into (10) (see (7)) we find that

$$\sum_{\sqrt{n}<p\leq n} \frac{1}{E(p)^k} = C_k \frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right). \tag{17}$$

Equations (5), (6), and (17) give (2).

We have the equation

$$\frac{1}{j^k}\left(\frac{1}{j} - \frac{1}{(j+1)}\right) = \frac{1}{j^{k+1}} - \left(\frac{1}{j^{k-1}}\left(\frac{1}{j} - \frac{1}{j+1}\right)\right).$$

Therefore, by (12)

$$C_k = \zeta(k+1) - C_{k-1}. \tag{18}$$

On the other hand, we have

$$C_1 = \sum_{j=1}^{\infty}\left(\frac{1}{j^2} - \frac{1}{j(j+1)}\right) = \zeta(2) - 1, \tag{19}$$

since

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{j+1}\right) = 1.$$

Equations (18) and (19) give (3). □

Example 2. If $k = 1$ then Theorem 1 is

$$\sum_{2\leq p\leq n} \frac{1}{E(p)} = \left(\frac{\pi^2}{6} - 1\right)\frac{n}{\log n} + O\left(\frac{n}{\log^2 n}\right).$$

Theorem 3. *The sequence of positive numbers C_k is strictly decreasing and*

$$C_k = \frac{1}{2} + O\left(2^{-k}\right).$$

Besides, we have the following limit

$$\lim_{k\to\infty} C_k = \frac{1}{2}.$$

Proof. Clearly the sequence of positive numbers C_k is strictly decreasing (see Eq. (12)). Therefore the limit of this sequence exists and is either positive or zero. Eq. (18) then implies that the limit is $1/2$, since $\lim_{k\to\infty} \zeta(k) = 1$.

To obtain a more precise result we need the following formula [4]:

$$\sum_{j=2}^{\infty}(-1)^j\left(\zeta(j)-1\right)=\frac{1}{2}.$$

Therefore, we have, for $k\geq 1$, that

$$C_k=(-1)^k\left(1-\sum_{j=2}^{k+1}(-1)^j\zeta(j)\right)=(-1)^k\left(1-\sum_{j=2}^{k+1}(-1)^j-\sum_{j=2}^{k+1}(-1)^j\left(\zeta(j)-1\right)\right)$$

$$= \quad (-1)^k\left(1-\frac{1}{2}\left(1-(-1)^k\right)-\sum_{j=2}^{\infty}(-1)^j\left(\zeta(j)-1\right)+\sum_{j>k+1}(-1)^j\left(\zeta(j)-1\right)\right)$$

$$= \quad (-1)^k\left(\frac{1}{2}-\frac{1}{2}\left(1-(-1)^k\right)+\sum_{j>k+1}(-1)^j\left(\zeta(j)-1\right)\right)=\frac{1}{2}+O\left(2^{-k}\right)$$

□

Theorem 4. *Let $k\geq 2$ a fixed but arbitrary positive integer. For every sufficiently large integer we have*

$$\sum_{p\leq n}E(p)^k=n^k\sum_{p}\frac{1}{(p-1)^k}+O_k\left(n^{k-1}\log n\right).$$

Proof. The following equation is well-known [1]. If $p\leq n$ then

$$E(p)=\frac{n}{p-1}+O\left(\frac{\log n}{\log p}\right).$$

Therefore, we have

$$\sum_{p\leq n}E(p)^k=\sum_{p\leq n}\left(\frac{n}{p-1}+O\left(\frac{\log n}{\log p}\right)\right)^k=n^k\sum_{p\leq n}\frac{1}{(p-1)^k}\left(1+O\left(\frac{p\log n}{n\log p}\right)\right)^k$$

$$= \quad n^k\sum_{p}\frac{1}{(p-1)^k}+O\left(n^k\sum_{p>n}\frac{1}{p^k}\right)+O\left(n^{k-1}\log n\sum_{p\leq n}\frac{1}{p^{k-1}\log p}\right)$$

To complete the proof we need the bounds

$$\sum_{p>n}\frac{1}{p^k}\ll_k\frac{1}{n^{k-1}\log n}$$

and

$$\sum_{p\leq n}\frac{1}{p^{k-1}\log p}\ll_k 1,$$

since $k\geq 2$. □

Remark 5.

1. Theorem 4 is a natural generalization of Eq. (1), since we have the well-known formula

$$\sum_{p\le n}\frac{1}{p-1}=\log\log n+A+O\left(\frac{1}{\log n}\right)\qquad (n\ge 2).$$

2. In the case $k=2$, the constant in Theorem 4 is closely related to the constant A in Eq. (1), since

$$\begin{aligned}\sum_{p}\frac{1}{(p-1)^2}&=\sum_{n=2}^{\infty}\frac{J_2(n)-\varphi(n)}{n}\log\zeta(n)\\ &=\sum_{n=2}^{\infty}\frac{J_2(n)}{n}\log\zeta(n)+\gamma-A\approx 1,375064994748635\ldots\end{aligned}$$

   where $J_2(n)=n^2\prod_{p|n}(1-p^{-2})$ is the second Jordan arithmetic function [2].

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