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Generalizations of the Reciprocal Fibonacci-Lucas Sums of Brousseau

Kunle Adegoke
Department of Physics and Engineering Physics
Obafemi Awolowo University
220005 Ile-Ife
Nigeria
adegoke00@gmail.com

Abstract

We derive closed form expressions for finite and infinite Fibonacci-Lucas sums having products of Fibonacci or Lucas numbers in the denominator of the summand. Our results generalize and extend those obtained by pioneer Brother Alfred Brousseau and later researchers.

1 Introduction

The Fibonacci numbers, F_n, and Lucas numbers, L_n, are defined, for $n \in \mathbb{N}_0$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, with $L_0 = 2$, $L_1 = 1$.

Our main aim in this paper is to derive closed form expressions for the following sums and their corresponding alternating versions, for positive integers m, n and q:

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}L_{nk+2nq}\cdots L_{nk+(m-1)nq}}{F_{nk}F_{nk+nq}\cdots F_{nk+mnq}}, \quad \sum_{k=1}^{\infty} \frac{F_{nk+nq}F_{nk+2nq}\cdots F_{nk+(m-1)nq}}{L_{nk}L_{nk+nq}\cdots L_{nk+mnq}}, \quad m > 1\,,$$

$$\sum_{k=1}^{\infty} \frac{1}{F_{nk}F_{nk+nq}\cdots F_{nk+mnq-nq}F_{nk+mnq+nq}\cdots F_{nk+mnq+2mnq}}\,,$$

$$\sum_{k=1}^{\infty}\frac{1}{L_{nk}L_{nk+nq}\cdots L_{nk+mnq-nq}L_{nk+mnq+nq}\cdots L_{nk+mnq+2mnq}}\,,$$

$$\sum_{k=1}^{\infty}\frac{L_{nk+mnq}}{F_{nk}F_{nk+nq}\cdots F_{nk+2mnq}},\quad \sum_{k=1}^{\infty}\frac{F_{nk+mnq}}{L_{nk}L_{nk+nq}\cdots L_{nk+2mnq}}\,,$$

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq}}{F_{nk}^2F_{nk+nq}^2F_{nk+2nq}^2\cdots F_{nk+mnq}^2},\quad \sum_{k=1}^{\infty}\frac{F_{2nk+mnq}}{L_{nk}^2L_{nk+nq}^2L_{nk+2nq}^2\cdots L_{nk+mnq}^2}\,,$$

$$\sum_{k=1}^{\infty}\frac{L_{nk+mnq}}{F_{nk}^2F_{nk+nq}^2\cdots F_{nk+(m-1)nq}^2F_{nk+mnq}F_{nk+(m+1)q}^2\cdots F_{nk+2mnq}^2}\,,$$

$$\sum_{k=1}^{\infty}\frac{F_{nk+mnq}}{L_{nk}^2L_{nk+nq}^2\cdots L_{nk+(m-1)nq}^2L_{nk+mnq}L_{nk+(m+1)q}^2\cdots L_{nk+2mnq}^2}\,,$$

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq}L_{nk+nq}^2L_{nk+2nq}^2\cdots L_{nk+(m-1)nq}^2}{F_{nk}^2F_{nk+nq}^2F_{nk+2nq}^2\cdots F_{nk+mnq}^2}\,,\quad m>1\,.$$

We require the following telescoping summation identities (see [1])

$$\sum_{k=1}^{N}\left(f(k)-f(k+q)\right)=\sum_{k=1}^{q}f(k)-\sum_{k=1}^{q}f(k+N) \tag{1}$$

and

$$\begin{aligned}&\sum_{k=1}^{N}(-1)^{k-1}\big(f(k)+(-1)^{q-1}f(k+q)\big)\\&\qquad=\sum_{k=1}^{q}(-1)^{k-1}f(k)+(-1)^{N-1}\sum_{k=1}^{q}(-1)^{k-1}f(k+N)\,,\end{aligned} \tag{2}$$

for $N\ge q\in\mathbb{N}_0$.

In general, infinite sums are evaluated using

$$\sum_{k=1}^{\infty}\left(f(k)-f(k+q)\right)=\sum_{k=1}^{q}f(k)-\sum_{k=1}^{q}\lim_{N\to\infty}f(k+N) \tag{3}$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}\big(f(k)+(-1)^{q-1}f(k+q)\big)=\sum_{k=1}^{q}(-1)^{k-1}f(k)\,. \tag{4}$$

If $f(N)$ approaches zero as N approaches infinity, then we have, from (1) and (2), the useful identities

$$\sum_{k=1}^{\infty}\left(f(k)-f(k+q)\right)=\sum_{k=1}^{q}f(k) \tag{5}$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}\left(f(k)+(-1)^{q-1}f(k+q)\right)=\sum_{k=1}^{q}(-1)^{k-1}f(k)\,. \tag{6}$$

The golden ratio, having the numerical value of $(\sqrt{5}+1)/2$, is denoted in this paper by ϕ.

We shall require the following identities (most of which can be found in the book by Vajda [10]):

$$L_vF_u = F_{u+v}+(-1)^vF_{u-v} \tag{7a}$$
$$F_vL_u = F_{u+v}-(-1)^vF_{u-v} \tag{7b}$$

$$2F_{u+v} = L_vF_u+L_uF_v \tag{8a}$$
$$(-1)^v2F_{u-v} = F_uL_v-L_uF_v \tag{8b}$$

$$L_vL_u = L_{u+v}+(-1)^vL_{u-v} \tag{9a}$$
$$5F_vF_u = L_{u+v}-(-1)^vL_{u-v} \tag{9b}$$

$$(-1)^{u-1}(F_{v+u}F_{v-u})=F_u^2(F_{v+1}F_{v-1})-F_v^2(F_{u+1}F_{u-1}) \tag{10}$$

$$(-1)^tF_uF_v=F_{t+u}F_{t+v}-F_tF_{t+u+v} \tag{11a}$$
$$(-1)^{t+1}5F_uF_v=L_{t+u}L_{t+v}-L_tL_{t+u+v} \tag{11b}$$

$$F_{u-v}F_{u+v} = F_u^2+(-1)^{u+v-1}F_v^2 \tag{12a}$$
$$5F_{u-v}F_{u+v} = L_u^2+(-1)^{u+v-1}L_v^2 \tag{12b}$$

$$F_vF_{2u+v+p} = F_{u+v+p}F_{u+v}+(-1)^{v+1}F_{u+p}F_u \tag{13a}$$
$$F_vL_{2u+v+p} = L_{u+v+p}F_{u+v}+(-1)^{v+1}L_{u+p}F_u \tag{13b}$$

The identities (12) and (13) were proved by Howard [6].

The following limiting values are readily established using Binet's formula:

$$\lim_{N\to\infty}\frac{F_{N+m}}{F_{N+n}} = \lim_{N\to\infty}\frac{L_{N+m}}{L_{N+n}}=\phi^{m-n}, \tag{14a}$$
$$\lim_{N\to\infty}\frac{F_{N+m}}{L_{N+n}} = \frac{1}{5}\lim_{N\to\infty}\frac{L_{N+m}}{F_{N+n}}=\frac{\phi^{m-n}}{\sqrt{5}}\,. \tag{14b}$$

We shall adopt the following conventions for empty sums and empty products:

$$\sum_{k=1}^{0}f(k)=0,\quad \prod_{k=1}^{0}f(k)=1\,.$$

2 Results: Generalizations

2.1 Telescoping summation identities

Lemma 1. *If m, n, q and N are positive integers and $f(k)$ is a real sequence, then*

$$\sum_{k=1}^{N}\left\{(f(nk)-f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\right\}$$
$$=\sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}f(nk+jnq)\right\}-\sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}f(nk+nN+jnq)\right\}.$$

If the sequence $f(k)$ is convergent and we denote by f_∞ the limiting value of $f(nN)$ as N approaches infinity, we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\left\{(f(nk)-f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\right\}\\&\quad=\sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}f(nk+jnq)\right\}-{f_\infty}^{m}q\,.\end{aligned}\tag{15}$$

Proof. We have

$$\begin{aligned}&(f(nk)-f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\\&=f(nk)\prod_{j=1}^{m-1}f(nk+jnq)-f(nk+mnq)\prod_{j=1}^{m-1}f(nk+jnq)\\&=\prod_{j=0}^{m-1}f(nk+jnq)-\prod_{j=1}^{m}f(nk+jnq)\end{aligned}$$

$$\begin{aligned}&=\prod_{j=0}^{m-1}f(nk+jnq)-\prod_{j=0}^{m-1}f(nk+jnq+nq)\\&\equiv\prod_{j=0}^{m-1}f(nk+jnq)-\prod_{j=0}^{m-1}f(nk+jnq)\Bigg|_{k\to k+q}.\end{aligned}\tag{16}$$

The result follows by summing both sides of identity (16), using the identity (1) to perform the telescopic summation on the right hand side.

□

Lemma 2. *If $f(k)$ is a real sequence and m, n, q and N are positive integers such that q is even, then*

$$\begin{aligned}&\sum_{k=1}^{N}(-1)^{k-1}\left\{(f(nk)-f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\right\}\\&\quad=\sum_{k=1}^{q}(-1)^{k-1}\left\{\prod_{j=0}^{m-1}f(nk+jnq)\right\}\\&\qquad+(-1)^{N-1}\sum_{k=1}^{q}(-1)^{k-1}\left\{\prod_{j=0}^{m-1}f(nk+nN+jnq)\right\}.\end{aligned}$$

Proof. Multiply through the identity (16) by $(-1)^{k-1}$ and use identity (2). □

Lemma 3. *If $f(k)$ is a real sequence and m, n, q and N are positive integers such that q is odd, then*

$$\begin{aligned}&\sum_{k=1}^{N}(-1)^{k-1}\left\{(f(nk)+f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\right\}\\&\quad=\sum_{k=1}^{q}(-1)^{k-1}\left\{\prod_{j=0}^{m-1}f(nk+jnq)\right\}\\&\qquad+(-1)^{N-1}\sum_{k=1}^{q}(-1)^{k-1}\left\{\prod_{j=0}^{m-1}f(nk+nN+jnq)\right\}.\end{aligned}$$

Proof. We have the identity

$$\begin{aligned}&(f(nk)+f(nk+mnq))\prod_{j=1}^{m-1}f(nk+jnq)\\&\quad=\prod_{j=0}^{m-1}f(nk+jnq)+\prod_{j=0}^{m-1}f(nk+jnq+nq)\qquad(17)\\&\quad\equiv\prod_{j=0}^{m-1}f(nk+jnq)+\left.\prod_{j=0}^{m-1}f(nk+jnq)\right|_{k\to k+q},\end{aligned}$$

from which the result follows after multiplying through by $(-1)^{k-1}$ and summing over k, making use of the identity (2). □

If the sequence $f(k)$ is convergent and $f(2Nn)$ and $f((2N-1)n)$ both have the same limiting value as N approaches infinity, we have

$$\begin{aligned}&\sum_{k=1}^{\infty}(-1)^{k-1}\left\{\big(f(nk)+(-1)^{q-1}f(nk+mnq)\big)\prod_{j=1}^{m-1}f(nk+jnq)\right\}\\&\quad=\sum_{k=1}^{q}(-1)^{k-1}\left\{\prod_{j=0}^{m-1}f(nk+jnq)\right\}.\end{aligned}\tag{18}$$

2.2 Sums with $F_{nk}F_{nk+nq}\cdots F_{nk+mnq}$ or $F_{nk}F_{nk+nq}\cdots F_{nk+(m-1)nq}F_{nk+(m+1)nq}\cdots F_{nk+2mnq}$ in the denominator

Theorem 4. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty}\left((-1)^{nk-1}\frac{\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\right)=\frac{q\sqrt{5^m}}{2F_{mnq}}-\frac{1}{2F_{mnq}}\sum_{k=1}^{q}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}\,,$$

so that

$$\sum_{k=1}^{\infty}\left(\frac{\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\right)=\frac{1}{2F_{mnq}}\sum_{k=1}^{q}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}-\frac{q\sqrt{5^m}}{2F_{mnq}},\quad n\text{ even}\tag{19}$$

and

$$\sum_{k=1}^{\infty}\left((-1)^{k-1}\frac{\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\right)=\frac{q\sqrt{5^m}}{2F_{mnq}}-\frac{1}{2F_{mnq}}\sum_{k=1}^{q}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}},\quad n\text{ odd}.\tag{20}$$

In particular,

$$\sum_{k=1}^{\infty}\frac{(-1)^{nk-1}}{F_{nk}F_{nk+nq}}=\frac{q\sqrt{5}}{2F_{nq}}-\frac{1}{2F_{nq}}\sum_{k=1}^{q}\frac{L_{nk}}{F_{nk}}\,.\tag{21}$$

Brousseau's result ([3], equation (3), also rederived in various equivalent forms by other authors, see for example reference [9]) corresponds to setting $n=1$ in (21), but with a different, but equivalent, form for the right hand side. Bruckman and Good's result ([4], equation (19)) is also a special case of (21), corresponding to setting $q=1$.

Theorem 5. *If m, n and q are integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\right)=\frac{1}{2F_{mnq}}\sum_{k=1}^{q}\left((-1)^{k}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}\right).$$

In particular,

$$\sum_{k=1}^{\infty}\frac{1}{F_{nk}F_{nk+nq}}=\frac{1}{2F_{nq}}\sum_{k=1}^{q}\left((-1)^{k}\frac{L_{nk}}{F_{nk}}\right),\quad n\text{ odd, }q\text{ even}\,,\tag{22}$$

which generalizes the result obtained by Rabinowitz ([9], the second of equation (26)), the latter corresponding to the special case $n=1$ in the identity (22), but with a different, but equivalent, form for the right hand side.

Proof of Theorem 4 and Theorem 5

Dividing through the identity (8b) by $F_u F_v$ and setting $u = nk + mnq$ and $v = nk$, the following identity is established for k, m, n and q positive integers:

$$\frac{(-1)^{nk-1} 2F_{mnq}}{F_{nk}F_{nk+mnq}} = \frac{L_{nk+mnq}}{F_{nk+mnq}} - \frac{L_{nk}}{F_{nk}}\,.$$

Using $f(k) = L_k/F_k$ in Lemma 1 we get the finite summation identity

$$\begin{aligned} &2F_{mnq}\sum_{k=1}^{N}\left((-1)^{nk-1}\frac{\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\right)\\ &\quad=\sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{L_{nk+nN+jnq}}{F_{nk+nN+jnq}}\right)-\sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}\right), \end{aligned} \tag{23}$$

which yields Theorem 4 in the limit N approaches infinity. Theorem 5 is proved by using $f(k) = L_k/F_k$ in identity (18).

Theorem 6. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty}\left(\frac{1}{\prod_{j=0}^{m-1}F_{nk+njq}\prod_{j=m+1}^{2m}F_{nk+njq}}\right) = \frac{1}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{2m-1}F_{nk+njq}}\right).$$

Below are a few explicit examples from Theorem 6:

At $m = 1$:

$$\sum_{k=1}^{\infty}\frac{1}{F_{nk}F_{nk+2nq}} = \frac{1}{L_{nq}}\sum_{k=1}^{q}\frac{1}{F_{nk}F_{nk+nq}},\quad nq \text{ odd}\,. \tag{24}$$

At $(m, n, q) = (1, 1, 1)$ and $(m, n, q) = (1, 1, 3)$:

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}} = 1,\quad \sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}} = \frac{143}{960}\,,$$

corresponding to Formulas (4) and (6) of Brousseau [2].

At $(m, n, q) = (3, 1, 3)$:

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+3}F_{k+6}F_{k+12}F_{k+15}F_{k+18}} = \frac{938359017897442612}{557910472051949235867648\,0}\,.$$

At $(m, n, q) = (5, 3, 1)$:

$$\begin{aligned} &\sum_{k=1}^{\infty}\frac{1}{F_{3k}F_{3k+3}F_{3k+6}F_{3k+9}F_{3k+12}F_{3k+18}F_{3k+21}F_{3k+24}F_{3k+27}F_{3k+30}}\\ &\quad= \frac{1}{1397003209786211551706871087759360\,0}\,. \end{aligned}$$

Theorem 7. *If q, m and n are positive integers such that q is odd and nm is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+njq}\prod_{j=m+1}^{2m}F_{nk+njq}}\right)=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1}F_{nk+njq}}\right).$$

Examples from Theorem 7 include

$$\begin{gathered}\text{At } m=2: \\ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_{nk}F_{nk+nq}F_{nk+3nq}F_{nk+4nq}}=\frac{1}{L_{2nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_{nk}F_{nk+nq}F_{nk+2nq}F_{nk+3nq}},\quad q\text{ odd}.\end{gathered}\tag{25}$$

$$\begin{gathered}\text{At } (m,n,q)=(1,2,1)\text{ and }(m,n,q)=(2,1,1): \\ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_{2k}F_{2k+4}}=\frac{1}{9},\quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_kF_{k+1}F_{k+3}F_{k+4}}=\frac{1}{18}. \\ \text{At } (m,n,q)=(2,6,1): \\ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{F_{6k}F_{6k+6}F_{6k+18}F_{6k+24}}=\frac{1}{44444622716928}.\end{gathered}$$

Proof of Theorem 6 and Theorem 7

With $v=mnq$ and $u=nk+mnq$ in identity (7a), the following identity is established:

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}}=\frac{1}{F_{nk}}+\frac{(-1)^{mnq}}{F_{nk+2mnq}},\tag{26}$$

so that

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}}=\frac{1}{F_{nk}}-\frac{1}{F_{nk+2mnq}},\quad mnq\text{ odd}\tag{27}$$

and

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}}=\frac{1}{F_{nk}}+\frac{1}{F_{nk+2mnq}},\quad mnq\text{ even}.\tag{28}$$

If m, n and q are positive odd integers, then from (27) and using $f(k)=1/F_k$ in Lemma 1 (with $m\to 2m$), we have the following definite summation identity

$$\begin{aligned}&\sum_{k=1}^{N}\left(\frac{1}{\prod_{j=0}^{m-1}F_{nk+jnq}\prod_{j=m+1}^{2m}F_{nk+jnq}}\right)\\&\quad=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{2m-1}F_{nk+jnq}}\right)-\frac{1}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{2m-1}F_{nk+nN+jnq}}\right),\end{aligned}\tag{29}$$

from which Theorem 6 follows as N approaches infinity.

If q is an odd positive integer and either m or n is even, then from (28) and Lemma 3 (with $m \to 2m$) we have the summation identity

$$\begin{aligned}&\sum_{k=1}^{N}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+njq}\prod_{j=m+1}^{2m}F_{nk+njq}}\right)\\&\qquad=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1}F_{nk+njq}}\right)+\frac{(-1)^{N-1}}{L_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1}F_{nk+nN+njq}}\right),\end{aligned} \tag{30}$$

from which Theorem 7 follows in the limit that N approaches infinity.

Theorem 8. *If m, n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+mnq}\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}=\frac{1}{2}\sum_{k=1}^{q}(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+nq}}{F_{nk}F_{nk+nq}}=\frac{1}{2}\sum_{k=1}^{q}(-1)^{k-1}\frac{L_{nk}}{F_{nk}},\quad q\text{ odd}\,. \tag{31}$$

Corollary 9. *If m, n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{nk+mnq}^2\prod_{j=1}^{m-1}L_{nk+jnq}\prod_{j=m+1}^{2m-1}L_{nk+jnq}}{\prod_{j=0}^{m-1}F_{nk+jnq}\prod_{j=m+1}^{2m}F_{nk+jnq}}=\frac{1}{2}\sum_{k=1}^{q}(-1)^{k-1}\prod_{j=0}^{2m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{nk+nq}^2}{F_{nk}F_{nk+2nq}}=\frac{1}{2}\sum_{k=1}^{q}(-1)^{k-1}\frac{L_{nk}L_{nk+nq}}{F_{nk}F_{nk+nq}},\quad q\text{ odd}\,. \tag{32}$$

Proof of Theorem 8 and Corollary 9

Dividing through the identity (8a) by F_uF_v and setting $u=nk+mnq$ and $v=nk$ we have the identity

$$2\frac{F_{2nk+mnq}}{F_{nk}F_{nk+mnq}}=\frac{L_{nk+mnq}}{F_{nk+mnq}}+\frac{L_{nk}}{F_{nk}}\,. \tag{33}$$

If q is odd, then from (33) and with $f(k)=L_k/F_k$ in Lemma 3 we have

$$\begin{aligned}&2\sum_{k=1}^{N}\frac{(-1)^{k-1}F_{2nk+mnq}\prod_{j=1}^{m-1}L_{nk+jnq}}{\prod_{j=0}^{m}F_{nk+jnq}}\\&\qquad=\sum_{k=1}^{q}(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}}{F_{nk+jnq}}+(-1)^{N-1}\sum_{k=1}^{q}(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+nN+jnq}}{F_{nk+nN+jnq}}\,,\end{aligned} \tag{34}$$

from which Theorem (8) follows in the limit as N approaches infinity. Corollary (9) is obtained by specifically requiring m to be even in Theorem (8).

Theorem 10. *If m, n, q and p are positive integers, then*

$$\sum_{k=1}^{\infty}\left\{\frac{(-1)^{nk-1}\prod_{j=1}^{m-1}F_{nk+jnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}}\right\} = \frac{\phi^{mnp}q}{F_{mnq}F_{np}} - \frac{1}{F_{mnq}F_{np}}\sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}\frac{F_{nk+jnq+np}}{F_{nk+jnq}}\right\}.$$

In particular we have

$$\sum_{k=1}^{\infty}\left\{\frac{(-1)^{nk-1}\prod_{j=m+1}^{p+m-1}F_{nk+jn}}{\prod_{j=0}^{p}F_{nk+jn}}\right\} = \frac{\phi^{mnp}}{F_{mn}F_{pn}} - \frac{1}{F_{mn}F_{pn}}\prod_{j=0}^{m-1}\frac{F_{jn+np+n}}{F_{jn+n}} \tag{35}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{nk-1}}{F_{nk}F_{nk+nq}} = \frac{\phi^{n}q}{F_{nq}F_{n}} - \frac{1}{F_{nq}F_{n}}\sum_{k=1}^{q}\frac{F_{nk+n}}{F_{nk}}\,. \tag{36}$$

Observe that identity (36) is equivalent to identity (21) but with a different form for the right hand side. Since $2\phi = \sqrt{5}+1$, $F_{n-1}+F_{n+1}=L_n$ and $\phi^n = \phi F_n + F_{n-1}$, both identities can be combined to yield the following interesting summation identity which is valid for all non-zero integers n and non-negative integers q:

$$\frac{1}{F_n}\sum_{k=1}^{q}\left(\frac{F_{nk+n}}{F_{nk}}\right) - \frac{1}{2}\sum_{k=1}^{q}\left(\frac{L_{nk}}{F_{nk}}\right) = \frac{q}{2}\frac{L_n}{F_n}\,. \tag{37}$$

Theorem 11. *If m, n, q and p are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty}\left\{\frac{\prod_{j=1}^{m-1}F_{nk+jnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}}\right\} = \frac{1}{F_{mnq}F_{np}}\sum_{k=1}^{q}\left\{(-1)^k\prod_{j=0}^{m-1}\frac{F_{nk+jnq+np}}{F_{nk+jnq}}\right\}.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{1}{F_{nk}F_{nk+nq}} = \frac{1}{F_nF_{nq}}\sum_{k=1}^{q}\left((-1)^k\frac{F_{nk+n}}{F_{nk}}\right),\quad n \text{ odd, } q \text{ even}\,. \tag{38}$$

From identity (22) and identity (38) we have the interesting result

$$\frac{1}{2}\sum_{k=1}^{q}\left((-1)^k\frac{L_{nk}}{F_{nk}}\right) = \frac{1}{F_n}\sum_{k=1}^{q}\left((-1)^k\frac{F_{nk+n}}{F_{nk}}\right),\quad q \text{ even}\,. \tag{39}$$

Proof of Theorem 10 and Theorem 11

Dividing through identity (11a) by $F_{t+u}F_t$ and choosing $t = nk$, $u = mnq$ and $v = np$ we obtain the identity:

$$\frac{(-1)^{nk-1}F_{mnq}F_{np}}{F_{nk}F_{nk+mnq}} = \frac{F_{nk+mnq+np}}{F_{nk+mnq}} - \frac{F_{nk+np}}{F_{nk}}\,. \tag{40}$$

With $f(k) = F_{k+np}/F_k$ in Lemma 1 and using the identity (40), we have the finite summation identity

$$\begin{aligned} &F_{mnq}F_{np}\sum_{k=1}^{N}\left\{\frac{(-1)^{nk-1}\prod_{j=1}^{m-1}F_{nk+jnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}}\right\}\\ &= \sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}\frac{F_{nk+nN+jnq+np}}{F_{nk+nN+jnq}}\right\} - \sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}\frac{F_{nk+jnq+np}}{F_{nk+jnq}}\right\}, \end{aligned} \tag{41}$$

from which Theorem 10 follows in the limit as N approaches infinity.

Using $f(k) = F_{k+np}/F_k$ in Lemma 2 gives

$$\begin{aligned} &F_{mnq}F_{np}\sum_{k=1}^{N}\left\{\frac{\prod_{j=1}^{m-1}F_{nk+jnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}}\right\}\\ &= (-1)^N\sum_{k=1}^{q}\left\{(-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+nN+jnq+np}}{F_{nk+nN+jnq}}\right\}\\ &\quad - \sum_{k=1}^{q}\left\{(-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+jnq+np}}{F_{nk+jnq}}\right\}, \end{aligned} \tag{42}$$

from which Theorem 11 follows.

2.3 Sums with $L_{nk}L_{nk+nq}\cdots L_{nk+mnq}$ or $L_{nk}L_{nk+nq}\cdots L_{nk+mnq-nq}L_{nk+mnq+nq}\cdots L_{nk+2mnq}$ in the denominator

The derivations here proceed in the same fashion as in the previous section. The theorems will therefore be stated without proof. The analogous identity to (23) is

$$\begin{aligned} &2F_{mnq}\sum_{k=1}^{N}\left((-1)^{nk-1}\frac{\prod_{j=1}^{m-1}F_{nk+jnq}}{\prod_{j=0}^{m}L_{nk+jnq}}\right)\\ &= \sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}}\right) - \sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{F_{nk+nN+jnq}}{L_{nk+nN+jnq}}\right). \end{aligned} \tag{43}$$

Theorem 12. *If m, n and q are positive integers, then*

$$\sum_{k=1}^\infty \left((-1)^{nk-1}\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^{m} L_{nk+jnq}}\right) = \frac{1}{2F_{mnq}}\sum_{k=1}^q \prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}} - \frac{q}{2F_{mnq}\sqrt{5^m}}\,,$$

so that

$$\sum_{k=1}^\infty \left(\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^{m} L_{nk+jnq}}\right) = \frac{q}{2F_{mnq}\sqrt{5^m}} - \frac{1}{2F_{mnq}}\sum_{k=1}^q \prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}},\quad n\ \text{even} \tag{44}$$

and

$$\sum_{k=1}^\infty \left((-1)^{k-1}\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^{m} L_{nk+jnq}}\right) = \frac{1}{2F_{mnq}}\sum_{k=1}^q \prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}} - \frac{q}{2F_{mnq}\sqrt{5^m}},\quad n\ \text{odd}\,. \tag{45}$$

In particular,

$$\sum_{k=1}^\infty \frac{(-1)^{nk-1}}{L_{nk}L_{nk+nq}} = \frac{1}{2F_{nq}}\sum_{k=1}^q \frac{F_{nk}}{L_{nk}} - \frac{q}{2F_{nq}\sqrt 5},\quad n,q\in\mathbb{Z}^+\,. \tag{46}$$

The case $n=3, q=1$ in (46) was mentioned by Brousseau ([2], equation (14)).

Theorem 13. *If m, n and q are integers such that n is odd and q is even, then*

$$\sum_{k=1}^\infty \left(\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^{m} L_{nk+jnq}}\right) = \frac{1}{2F_{mnq}}\sum_{k=1}^q \left((-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}}\right).$$

In particular,

$$\sum_{k=1}^\infty \frac{1}{L_{nk}L_{nk+nq}} = \frac{1}{2F_{nq}}\sum_{k=1}^q \left((-1)^{k-1}\frac{F_{nk}}{L_{nk}}\right),\quad n\ \text{odd},\ q\ \text{even}\,. \tag{47}$$

Theorem 14. *If m, n and q are positive <u>odd</u> integers, then*

$$\sum_{k=1}^\infty \left(\frac{1}{\prod_{j=0}^{m-1} L_{nk+njq}\prod_{j=m+1}^{2m} L_{nk+njq}}\right) = \frac{1}{L_{mnq}}\sum_{k=1}^q \left(\frac{1}{\prod_{j=0}^{2m-1} L_{nk+njq}}\right).$$

Theorem 15. *If q, m and n are positive integers such that q is odd and nm is even, then*

$$\sum_{k=1}^\infty \left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+njq}\prod_{j=m+1}^{2m} L_{nk+njq}}\right) = \frac{1}{L_{mnq}}\sum_{k=1}^q \left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} L_{nk+njq}}\right).$$

Analogous identity to identity (34) is

$$
\begin{split}
&2\sum_{k=1}^N \frac{(-1)^{k-1}F_{2nk+mnq}\prod_{j=1}^{m-1}F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}}\\
&\qquad=\sum_{k=1}^q (-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}}+(-1)^{N-1}\sum_{k=1}^q (-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+nN+jnq}}{L_{nk+nN+jnq}}\,,
\end{split}
\tag{48}
$$

from which we get the following theorem in the limit as N approaches infinity.

Theorem 16. *If m, n and q are positive integers such that q is odd, then*

$$
\sum_{k=1}^\infty \frac{(-1)^{k-1}F_{2nk+mnq}\prod_{j=1}^{m-1}F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}}=\frac12\sum_{k=1}^q (-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}}\,.
$$

Corollary 17. *If m, n and q are positive integers such that q is odd, then*

$$
\sum_{k=1}^\infty \frac{(-1)^{k-1}F^2_{nk+mnq}\prod_{j=1}^{m-1}F_{nk+jnq}\prod_{j=m+1}^{2m-1}F_{nk+jnq}}{\prod_{j=0}^{m-1} L_{nk+jnq}\prod_{j=m+1}^{2m} L_{nk+jnq}}=\frac12\sum_{k=1}^q (-1)^{k-1}\prod_{j=0}^{2m-1}\frac{F_{nk+jnq}}{L_{nk+jnq}}\,.
$$

Corresponding to identity (40) of section 2.2 we have (from identity (11b))

$$
\frac{(-1)^{nk-1}5F_{mnq}F_{np}}{L_{nk}L_{nk+mnq}}=-\frac{L_{nk+mnq+np}}{L_{nk+mnq}}+\frac{L_{nk+np}}{L_{nk}}\,,
\tag{49}
$$

leading to the summation identities

$$
\begin{split}
&5F_{mnq}F_{np}\sum_{k=1}^N\left\{\frac{(-1)^{nk-1}\prod_{j=1}^{m-1}L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}}\right\}\\
&\qquad=-\sum_{k=1}^q\left\{\prod_{j=0}^{m-1}\frac{L_{nk+nN+jnq+np}}{L_{nk+nN+jnq}}\right\}+\sum_{k=1}^q\left\{\prod_{j=0}^{m-1}\frac{L_{nk+jnq+np}}{L_{nk+jnq}}\right\}
\end{split}
\tag{50}
$$

and

$$
\begin{split}
&5F_{mnq}F_{np}\sum_{k=1}^N\left\{\frac{\prod_{j=1}^{m-1}L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}}\right\}\\
&\qquad=(-1)^{N-1}\sum_{k=1}^q\left\{(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+nN+jnq+np}}{L_{nk+nN+jnq}}\right\}\\
&\qquad\quad+\sum_{k=1}^q\left\{(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+jnq+np}}{L_{nk+jnq}}\right\},
\end{split}
\tag{51}
$$

from which Theorem 18 and Theorem 19 follow.

Theorem 18. *If m, n, q and p are positive integers, then*

$$\sum_{k=1}^{\infty}\left\{\frac{(-1)^{nk-1}\prod_{j=1}^{m-1}L_{nk+jnq+np}}{\prod_{j=0}^{m}L_{nk+jnq}}\right\}$$
$$= -\frac{\phi^{mnp}q}{5F_{mnq}F_{np}} + \frac{1}{5F_{mnq}F_{np}}\sum_{k=1}^{q}\left\{\prod_{j=0}^{m-1}\frac{L_{nk+jnq+np}}{L_{nk+jnq}}\right\}.$$

In particular we have

$$\sum_{k=1}^{\infty}\left\{\frac{(-1)^{nk-1}\prod_{j=m+1}^{p+m-1}L_{nk+jn}}{\prod_{j=0}^{p}L_{nk+jn}}\right\} = -\frac{\phi^{mnp}}{5F_{mn}F_{pn}} + \frac{1}{5F_{mn}F_{pn}}\prod_{j=0}^{m-1}\frac{L_{jn+np+n}}{L_{jn+n}} \tag{52}$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{nk-1}}{L_{nk}L_{nk+nq}} = -\frac{\phi^{n}q}{5F_{nq}F_{n}} + \frac{1}{5F_{nq}F_{n}}\sum_{k=1}^{q}\frac{L_{nk+n}}{L_{nk}}\,. \tag{53}$$

Theorem 19. *If m, n, q and p are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty}\left\{\frac{\prod_{j=1}^{m-1}L_{nk+jnq+np}}{\prod_{j=0}^{m}L_{nk+jnq}}\right\}$$
$$= \frac{1}{5F_{mnq}F_{np}}\sum_{k=1}^{q}\left\{(-1)^{k-1}\prod_{j=0}^{m-1}\frac{L_{nk+jnq+np}}{L_{nk+jnq}}\right\}.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{1}{L_{nk}L_{nk+nq}} = \frac{1}{5F_{n}F_{nq}}\sum_{k=1}^{q}\left((-1)^{k-1}\frac{L_{nk+n}}{L_{nk}}\right),\quad n\text{ odd, } q\text{ even}\,. \tag{54}$$

From identity (47) and identity (54) we have

$$\frac{1}{2}\sum_{k=1}^{q}\left((-1)^{k-1}\frac{F_{nk}}{L_{nk}}\right) = \frac{1}{5F_{n}}\sum_{k=1}^{q}\left((-1)^{k-1}\frac{L_{nk+n}}{L_{nk}}\right),\quad q\text{ even}\,. \tag{55}$$

2.4 Sums with $F_{nk}F_{nk+nq}\cdots F_{nk+2mnq}$ in the denominator

The results in this section are obtained from identity (7b). We have

$$\frac{F_{v}L_{u}}{F_{u-v}F_{u+v}} = \frac{1}{F_{u-v}} - \frac{(-1)^{v}}{F_{u+v}}\,,$$

from which, by setting $v = mnq$ and $u = nk + mnq$, we get

$$\frac{F_{mnq}L_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} - \frac{(-1)^{mnq}}{F_{nk+2mnq}}\,, \tag{56}$$

so that

$$\frac{F_{mnq}L_{nk+mnq}}{F_{nk}F_{nk+2mnq}}=\frac{1}{F_{nk}}-\frac{1}{F_{nk+2mnq}},\qquad mnq\text{ even} \tag{57}$$

and

$$\frac{F_{mnq}L_{nk+mnq}}{F_{nk}F_{nk+2mnq}}=\frac{1}{F_{nk}}+\frac{1}{F_{nk+2mnq}},\quad mnq\text{ odd}\,. \tag{58}$$

The derivations then proceed as in the previous sections.

Theorem 20. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{L_{nk+mnq}}{\prod_{j=0}^{2m}F_{nk+njq}}\right)=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{2m-1}F_{nk+njq}}\right)$$

Examples from Theorem 20 include:

$$\text{At }(m,n,q)=(1,2,1)\text{ and }(m,n,q)=(1,1,2):$$
$$\sum_{k=1}^{\infty}\frac{L_{2k+2}}{F_{2k}F_{2k+2}F_{2k+4}}=\frac{1}{3},\quad \sum_{k=1}^{\infty}\frac{L_{k+2}}{F_kF_{k+2}F_{k+4}}=\frac{5}{6}\,. \tag{59}$$

The first of the identities in (59) was also derived by Melham ([7], equation 3.7).

$$\text{At }(m,n,q)=(2,7,1):$$
$$\sum_{k=1}^{\infty}\frac{L_{7k+14}}{F_{7k}F_{7k+7}F_{7k+14}F_{7k+21}F_{7k+28}}=\frac{1}{6427623373464462}\,.$$

Theorem 21. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}L_{nk+mnq}}{\prod_{j=0}^{2m}F_{nk+njq}}\right)=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1}F_{nk+njq}}\right)$$

Examples from Theorem 21 include:

$$\text{At }(m,n,q)=(1,1,2)\text{ and }(m,n,q)=(1,3,2):$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{k+2}}{F_kF_{k+2}F_{k+4}}=\frac{1}{6}\,,\quad \sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{3k+6}}{F_{3k}F_{3k+6}F_{3k+12}}=\frac{271}{156672}\,.$$

$$\text{At }(m,n,q)=(2,4,2):$$
$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{4k+16}}{F_{4k}F_{4k+8}F_{4k+16}F_{4k+24}F_{4k+32}}=\frac{177072540680427}{166704475185956548320480}\,.$$

2.5 Sums with $L_{nk}L_{nk+nq}\cdots L_{nk+2mnq}$ in the denominator

The results here follow from the identity (9b).

Theorem 22. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{nk+mnq}}{\prod_{j=0}^{2m}L_{nk+jnq}}\right)=\frac{1}{5F_{mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{2m-1}\frac{1}{L_{nk+jnq}}\right).$$

Theorem 23. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{F_{nk+mnq}}{\prod_{j=0}^{2m}L_{nk+jnq}}\right)=\frac{1}{5F_{mnq}}\sum_{k=1}^{q}\left((-1)^{k-1}\prod_{j=0}^{2m-1}\frac{1}{L_{nk+jnq}}\right).$$

2.6 Sums with $F_{nk}F_{nk+2nq}F_{nk+4nq}F_{nk+6nq}\cdots F_{nk+2mnq}$ in the denominator

Theorem 24. *If m, n and q are positive <u>odd</u> integers, then*

$$\sum_{k=1}^{\infty}\frac{(\pm 1)^{k-1}F_{nk+mnq}}{\prod_{j=0}^{m}F_{nk+2jnq}}=\frac{1}{L_{mnq}}\sum_{k=1}^{2q}\frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{F_{nk+nq}}{F_{nk}F_{nk+2nq}}=\frac{1}{L_{nq}}\sum_{k=1}^{2q}\frac{1}{F_{nk}},\quad nq\text{ odd}.\tag{60}$$

Proof. From identity (27) and with $f(k)=1/F_k$ (and $q\to 2q$) in Lemma 1 we have the finite summation identity

$$\begin{aligned}L_{mnq}\sum_{k=1}^{N}\frac{F_{nk+mnq}}{\prod_{j=0}^{m}F_{nk+2jnq}}&=\sum_{k=1}^{2q}\frac{1}{\prod_{j=0}^{m-1}F_{nk+2jnq}} \qquad (mnq\text{ odd})\\&\quad-\sum_{k=1}^{2q}\frac{1}{\prod_{j=0}^{m-1}F_{nk+nN+2jnq}}.\end{aligned}\tag{61}$$

From identity (27) with m, n and q positive odd intergers and with $f(k)=1/F_k$ (and $q\to 2q$) in Lemma 2 we have the alternating finite summation identity

$$\begin{aligned}L_{mnq}\sum_{k=1}^{N}\frac{(-1)^{k-1}F_{nk+mnq}}{\prod_{j=0}^{m}F_{nk+2jnq}}&=\sum_{k=1}^{2q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+2jnq}} \qquad (mnq\text{ odd})\\&\quad+(-1)^{N-1}\sum_{k=1}^{2q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+nN+2jnq}}\end{aligned}\tag{62}$$

Theorem 24 follows from identities (61) and (62) in the limit as N approaches infinity. □

Theorem 25. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^{m} F_{nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}}{F_{nk}F_{nk+2nq}} = \frac{1}{F_{nq}} \sum_{k=1}^{2q} \frac{1}{F_{nk}}, \quad nq \text{ even}\,. \tag{63}$$

Proof. As in Theorem 24, with the identity (57), with mnq even.

The corresponding finite summation identities are

$$\begin{aligned} F_{mnq} \sum_{k=1}^{N} \frac{L_{nk+mnq}}{\prod_{j=0}^{m} F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \qquad (mnq \text{ even}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}} \end{aligned} \tag{64}$$

and

$$\begin{aligned} F_{mnq} \sum_{k=1}^{N} \frac{(-1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^{m} F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \qquad (mnq \text{ even}) \\ &\quad + (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}}\,. \end{aligned} \tag{65}$$

□

2.7 Sums with $L_{nk}L_{nk+2nq}L_{nk+4nq}L_{nk+6nq}\cdots L_{nk+2mnq}$ in the denominator

In this section we state the Lucas versions of the results given in section 2.6.

Here the basic identities (from identities (9)) are:

$$\frac{L_{mnq}L_{nk+mnq}}{L_{nk}L_{nk+2mnq}} = \frac{1}{L_{nk}} - \frac{1}{L_{nk+2mnq}}, \quad mnq \text{ odd} \tag{66}$$

$$\frac{5F_{mnq}F_{nk+mnq}}{L_{nk}L_{nk+2mnq}} = \frac{1}{L_{nk}} - \frac{1}{L_{nk+2mnq}}, \quad mnq \text{ even}\,. \tag{67}$$

Theorem 26. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}}{L_{nk}L_{nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^{2q} \frac{1}{L_{nk}}, \quad nq \text{ odd}\,. \tag{68}$$

Theorem 27. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{nk+nq}}{L_{nk}L_{nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^{2q} \frac{1}{F_{nk}}, \quad nq \text{ even}\,. \tag{69}$$

We have the following finite summation identities:

$$\begin{aligned} L_{mnq} \sum_{k=1}^{N} \frac{L_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \qquad (mnq \text{ odd}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}\,. \end{aligned} \tag{70}$$

$$\begin{aligned} L_{mnq} \sum_{k=1}^{N} \frac{(-1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \qquad (mnq \text{ odd}) \\ &\quad + (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}\,. \end{aligned} \tag{71}$$

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^{N} \frac{F_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \qquad (mnq \text{ even}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}\,. \end{aligned} \tag{72}$$

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^{N} \frac{(-1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^{m} L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \qquad (mnq \text{ even}) \\ &\quad + (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}\,. \end{aligned} \tag{73}$$

2.8 Sums with $F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq}\cdots F_{2nk+2mnq}$ in the denominator

Theorem 28. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq}}{\prod_{j=0}^{m}F_{2nk+2jnq}}=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}F_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{F_{2nk+nq}}{F_{2nk}F_{2nk+2nq}}=\frac{1}{L_{nq}}\sum_{k=1}^{q}\frac{1}{F_{2nk}}\,,\tag{74}$$

$$\sum_{k=1}^{\infty}\frac{F_{2nk+3nq}}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq}}=\frac{1}{L_{3nq}}\sum_{k=1}^{q}\frac{1}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}}\,.\tag{75}$$

The identity derived by Frontczak ([5], identity (18)), corresponds to setting $q=1$ in identity (74).

Proof. From identity (7a) with $v=mnq$ and $u=2nk+mnq$ comes the identity

$$\frac{L_{mnq}F_{2nk+mnq}}{F_{2nk}F_{2nk+2mnq}}=\frac{1}{F_{2nk}}-\frac{1}{F_{2nk+2mnq}},\quad mnq\text{ odd}\,.\tag{76}$$

From identity (76) and Lemma 1 with $f(k)=1/F_{2k}$ we have the finite summation identity:

$$L_{mnq}\sum_{k=1}^{N}\frac{F_{2nk+mnq}}{\prod_{j=0}^{m}F_{2nk+2jnq}}=\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}F_{2nk+2jnq}}-\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}F_{2nk+2nN+2jnq}}\,,\tag{77}$$

from which Theorem 28 follows in the limit as N approaches infinity. □

Theorem 29. *If m, n and q are positive integers such that q is odd and mn is even , then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+mnq}}{\prod_{j=0}^{m}F_{2nk+2jnq}}=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+nq}}{F_{2nk}F_{2nk+2nq}}=\frac{1}{L_{nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_{2nk}}\,.\tag{78}$$

Proof. From identity (7a) with $v = mnq$ and $u = 2nk + mnq$ comes the identity

$$\frac{L_{mnq}F_{2nk+mnq}}{F_{2nk}F_{2nk+2mnq}} = \frac{1}{F_{2nk}} + \frac{1}{F_{2nk+2mnq}}, \quad mnq \text{ even}. \tag{79}$$

From identity (79) and Lemma 3 with $f(k) = 1/F_{2k}$ we have the finite summation identity:

$$\begin{aligned} L_{mnq} \sum_{k=1}^{N} \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^{m} F_{2nk+2jnq}} &= \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} \\ &\quad + (-1)^{N-1} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}, \end{aligned} \tag{80}$$

from which Theorem 29 follows in the limit as N approaches infinity. □

Theorem 30. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{L_{2nk+mnq}}{\prod_{j=0}^{m} F_{2nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{2nk+nq}}{F_{2nk}F_{2nk+2nq}} = \frac{1}{F_{nq}} \sum_{k=1}^{q} \frac{1}{F_{2nk}}, \tag{81}$$

$$\sum_{k=1}^{\infty} \frac{L_{2nk+3nq}}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq}} = \frac{1}{F_{3nq}} \sum_{k=1}^{q} \frac{1}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}}. \tag{82}$$

Proof. From identity (7b) with $v = mnq$ and $u = 2nk + mnq$ comes the identity

$$\frac{F_{mnq}L_{2nk+mnq}}{F_{2nk}F_{2nk+2mnq}} = \frac{1}{F_{2nk}} - \frac{1}{F_{2nk+2mnq}}, \quad mnq \text{ even}. \tag{83}$$

From identity (83) and Lemma 1 with $f(k) = 1/F_{2k}$ we have the finite summation identity:

$$\begin{aligned} F_{mnq} \sum_{k=1}^{N} \frac{L_{2nk+mnq}}{\prod_{j=0}^{m} F_{2nk+2jnq}} &= \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} \\ &\quad - \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}, \end{aligned} \tag{84}$$

from which Theorem 30 follows in the limit as N approaches infinity. □

Theorem 31. *If m, n and q are positive integers such that q is even <u>or</u> mnq is odd , then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^{m} F_{2nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{2nk+nq}}{F_{2nk}F_{2nk+2nq}}=\frac{1}{F_{nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_{2nk}}\,,\quad q\text{ even }\underline{\text{or}}\ nq\text{ odd}\,. \tag{85}$$

The alternating summation identity here, valid for q even or mnq odd, is

$$\begin{aligned}F_{mnq}\sum_{k=1}^{N}\frac{(-1)^{k-1}L_{2nk+mnq}}{\prod_{j=0}^{m}F_{2nk+2jnq}}&=\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{2nk+2jnq}}\\&\quad+(-1)^{N-1}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{2nk+2nN+2jnq}}\,.\end{aligned} \tag{86}$$

2.9 Sums with $L_{2nk}L_{2nk+2nq}L_{2nk+4nq}L_{2nk+6nq}\cdots L_{2nk+2mnq}$ in the denominator

The results in this section are derived from identities (9). The proofs are identical to those in section 2.8 and are therefore omitted.

Theorem 32. *If m, n and q are positive* <u>*odd*</u> *integers, then*

$$\sum_{k=1}^{\infty}\frac{L_{2nk+mnq}}{\prod_{j=0}^{m}L_{2nk+2jnq}}=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}L_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty}\frac{L_{2nk+nq}}{L_{2nk}L_{2nk+2nq}}=\frac{1}{L_{nq}}\sum_{k=1}^{q}\frac{1}{L_{2nk}}\,,\quad nq\text{ odd}\,, \tag{87}$$

$$\sum_{k=1}^{\infty}\frac{L_{2nk+3nq}}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}L_{2nk+6nq}}=\frac{1}{L_{3nq}}\sum_{k=1}^{q}\frac{1}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}}\,. \tag{88}$$

The finite summation identity is

$$\begin{aligned}L_{mnq}\sum_{k=1}^{N}\frac{L_{2nk+mnq}}{\prod_{j=0}^{m}L_{2nk+2jnq}}&=\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}L_{2nk+2jnq}}\qquad mnq\text{ odd}\\&\quad-\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}L_{2nk+2nN+2jnq}}\,.\end{aligned} \tag{89}$$

Theorem 33. *If m, n and q are positive integers such that q is odd and mn is even , then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{2nk+mnq}}{\prod_{j=0}^{m}L_{2nk+2jnq}}=\frac{1}{L_{mnq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}L_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+nq}}{L_{2nk} L_{2nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{L_{2nk}}\,, \qquad q \text{ odd, } n \text{ even}\,. \tag{90}$$

The alternating finite summation identity is

$$\begin{aligned} L_{mnq} \sum_{k=1}^{N} \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^{m} L_{2nk+2jnq}} &= \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \qquad q \text{ odd, } mn \text{ even} \\ &\quad + (-1)^{N-1} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}\,. \end{aligned} \tag{91}$$

Theorem 34. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{\prod_{j=0}^{m} L_{2nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{2nk+nq}}{L_{2nk} L_{2nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^{q} \frac{1}{L_{2nk}}\,, \qquad nq \text{ even}\,, \tag{92}$$

$$\sum_{k=1}^{\infty} \frac{F_{2nk+3nq}}{L_{2nk} L_{2nk+2nq} L_{2nk+4nq} L_{2nk+6nq}} = \frac{1}{5F_{3nq}} \sum_{k=1}^{q} \frac{1}{L_{2nk} L_{2nk+2nq} L_{2nk+4nq}}\,. \tag{93}$$

The finite summation identity is

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^{N} \frac{F_{2nk+mnq}}{\prod_{j=0}^{m} L_{2nk+2jnq}} &= \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \qquad mnq \text{ even} \\ &\quad - \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}\,, \end{aligned} \tag{94}$$

Theorem 35. *If m, n and q are positive integers such that q is even <u>or</u> mnq is odd , then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^{m} L_{2nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}\,.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{L_{2nk} F_{2nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^{q} \frac{(-1)^{k-1}}{L_{2nk}}\,, \quad q \text{ even } \underline{\text{or}}\ nq \text{ odd}\,. \tag{95}$$

The alternating summation identity here, valid for q even or mnq odd, is

$$5F_{mnq}\sum_{k=1}^{N}\frac{(-1)^{k-1}F_{2nk+mnq}}{\prod_{j=0}^{m}L_{2nk+2jnq}}=\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}L_{2nk+2jnq}}+(-1)^{N-1}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}L_{2nk+2nN+2jnq}}\,. \tag{96}$$

2.10 Sums with $F_{nk}^2F_{nk+nq}^2\cdots F_{nk+mnq-nq}^2F_{nk+mnq}F_{nk+mnq+nq}^2\cdots F_{nk+2mnq}^2$ or $F_{nk}^2F_{nk+nq}^2\cdots F_{nk+mnq}^2$ in the denominator

Theorem 36. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{2nk+mnq}}{\prod_{j=0}^{m}F_{nk+jnq}^2}\right)=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}F_{nk+jnq}^2}\,.$$

Explicitly,

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq}}{F_{nk}^2F_{nk+nq}^2\cdots F_{nk+mnq}^2}=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\frac{1}{F_{nk}^2F_{nk+nq}^2\cdots F_{nk+(m-1)nq}^2}\,.$$

Examples include:

At $m=1$

$$\sum_{k=1}^{\infty}\frac{F_{2nk+nq}}{F_{nk}^2F_{nk+nq}^2}=\frac{1}{F_{nq}}\sum_{k=1}^{q}\frac{1}{F_{nk}^2},\quad nq\text{ even}\,. \tag{97}$$

At $(m,n,q)=(1,1,2)$ and $(m,n,q)=(1,2,1)$:

$$\sum_{k=1}^{\infty}\frac{F_{2k+2}}{F_k^2F_{k+2}^2}=2,\quad \sum_{k=1}^{\infty}\frac{F_{4k+2}}{F_{2k}^2F_{2k+2}^2}=1\,. \tag{98}$$

At $(m,n,q)=(3,2,2)$:

$$\sum_{k=1}^{\infty}\frac{F_{4k+12}}{F_{2k}^2F_{2k+4}^2F_{2k+8}^2F_{2k+12}^2}=\frac{1288981}{35850395750400}\,. \tag{99}$$

Corollary 37. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty}\left(\frac{L_{nk+mnq}}{F_{nk+mnq}\prod_{j=0}^{m-1}F_{nk+jnq}^2\prod_{j=m+1}^{2m}F_{nk+jnq}^2}\right)=\frac{1}{F_{2mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{2m-1}\frac{1}{F_{nk+jnq}^2}\right).$$

Explicitly, Corollary 37 is

$$\sum_{k=1}^{\infty} \frac{L_{nk+mnq}}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(m-1)nq}^2 F_{nk+mnq} F_{nk+(m+1)nq}^2 \cdots F_{nk+2mnq}^2} = \frac{1}{F_{2mnq}} \sum_{k=1}^{q} \frac{1}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(2m-1)nq}^2}\,.$$

Below are a couple of examples:

At $(m,n,q)=(1,1,1)$ and $(m,n,q)=(2,1,1)$:

$$\sum_{k=1}^{\infty} \frac{L_{k+1}}{F_k^2 F_{k+1} F_{k+2}^2} = 1, \qquad \sum_{k=1}^{\infty} \frac{L_{k+2}}{F_k^2 F_{k+1}^2 F_{k+2} F_{k+3}^2 F_{k+4}^2} = \frac{1}{108}\,. \tag{100}$$

At $(m,n,q)=(3,2,2)$:

$$\sum_{k=1}^{\infty} \frac{L_{2k+12}}{F_{2k}^2 F_{2k+4}^2 F_{2k+8}^2 F_{2k+12} F_{2k+16}^2 F_{2k+20}^2 F_{2k+24}^2} = \frac{636693716175181614930457}{1701394375843622618689225675379000792710492054565683200}\,. \tag{101}$$

Proof of Theorem 36

By making appropriate choices for the indices u and v in the identity (12a), it is straightforward to establish the following identity:

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} - \frac{(-1)^{mnq}}{F_{nk+mnq}^2}\,, \tag{102}$$

so that

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} - \frac{1}{F_{nk+mnq}^2}, \quad mnq \text{ even} \tag{103}$$

and

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} + \frac{1}{F_{nk+mnq}^2}, \quad mnq \text{ odd}\,. \tag{104}$$

From (103), with $f(k)=1/F_k^2$ in Lemma 1, we have the finite summation identity

$$\sum_{k=1}^{N} \frac{F_{2nk+mnq}}{\prod_{j=0}^{m} F_{nk+jnq}^2} = \frac{1}{F_{mnq}} \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq}^2} - \frac{1}{F_{mnq}} \sum_{k=1}^{q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+jnq}^2}, \quad mnq \text{ even}\,. \tag{105}$$

As N approaches infinity, we have Theorem 36, while specifically requiring m to be even gives Corollary 37.

Theorem 38. *If m, n and q are integers such that q is even* <u>*or*</u> *mnq is odd, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}F_{2nk+mnq}}{\prod_{j=0}^{m}F_{nk+jnq}^2}\right)=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+jnq}^2}\right).$$

Proof. The statement of the theorem follows from (102) and identity (18). □

In particular, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+nq}}{F_{nk}^2F_{nk+nq}^2}=\frac{1}{F_{nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_{nk}^2},\quad q \text{ even or } nq \text{ odd}\,, \tag{106}$$

which generalizes Brousseau's result ([2], Formula (6)), also derived by Melham [8] as a special case of a more general result. Brousseau's formula (6) corresponds to $n=1$ in (106). Another Brousseau's result ([2], Formula (15)) is also contained in the identity (106) above at $n=3$, $q=1$.

More examples from Theorem 38:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+3nq}}{F_{nk}^2F_{nk+nq}^2F_{nk+2nq}^2F_{nk+3nq}^2}=\frac{1}{F_{3nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_{nk}^2F_{nk+nq}^2F_{nk+2nq}^2},\quad q \text{ even or } nq \text{ odd}\,. \tag{107}$$

Corollary 39. *If q is a positive* <u>*even*</u> *integer, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}L_{nk+mnq}}{F_{nk+mnq}\prod_{j=0}^{m-1}F_{nk+jnq}^2\prod_{j=m+1}^{2m}F_{nk+jnq}^2}\right)=\frac{1}{F_{2mnq}}\sum_{k=1}^{q}(-1)^{k-1}\left(\prod_{j=0}^{2m-1}\frac{1}{F_{nk+jnq}^2}\right).$$

In particular:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{k+q}}{F_k^2F_{k+q}F_{k+2q}^2}=\frac{1}{F_{2q}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{F_k^2F_{k+q}^2},\quad q \text{ even}\,. \tag{108}$$

We note that Theorem 36, Theorem 38, Corollary 37 and Corollary 39 correspond to setting $p=0$ in Theorem 48 and Theorem 49 of section 2.12.

Theorem 40. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{nk-1}F_{2nk+mnq}\prod_{j=1}^{m-1}L_{nk+jnq}^2}{\prod_{j=0}^{m}F_{nk+jnq}^2}\right)=\frac{5^mq}{4F_{mnq}}-\frac{1}{4F_{mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{L_{nk+jnq}^2}{F_{nk+jnq}^2}\right).$$

In particular

$$\sum_{k=1}^{\infty}\frac{(-1)^{nk-1}F_{2nk+nq}}{F_{nk}^2F_{nk+nq}^2}=\frac{5q}{4F_{nq}}-\frac{1}{4F_{nq}}\sum_{k=1}^{q}\frac{L_{nk}^2}{F_{nk}^2}\,. \tag{109}$$

Theorem 41. *If n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{2nk+mnq}\prod_{j=1}^{m-1}L_{nk+jnq}^2}{\prod_{j=0}^{m}F_{nk+jnq}^2}\right)=\frac{1}{4F_{mnq}}\sum_{k=1}^{q}\left((-1)^k\prod_{j=0}^{m-1}\frac{L_{nk+jnq}^2}{F_{nk+jnq}^2}\right).$$

Proof of Theorem 40 and Theorem 41

Multiplying identity (8a) and identity (8b) and choosing u and v judiciously, it is easy to establish that the following identity holds for positive integers m, n, q and k:

$$(-1)^{nk-1}\frac{4F_{mnq}F_{2nk+mnq}}{F_{nk}^2F_{nk+mnq}^2}=\frac{L_{nk+mnq}^2}{F_{nk+mnq}^2}-\frac{L_{nk}^2}{F_{nk}^2}\,. \tag{110}$$

From identity (110) and $f(k)=L_k^2/F_k^2$ in Lemma 1 we have the finite summation formula:

$$\begin{aligned}&4F_{mnq}\sum_{k=1}^{N}\frac{(-1)^{nk-1}F_{2nk+mnq}\prod_{j=1}^{m-1}L_{nk+jnq}^2}{\prod_{j=0}^{m}F_{nk+jnq}^2}\\&=\sum_{k=1}^{q}\prod_{j=0}^{m-1}\frac{L_{nk+nN+jnq}^2}{F_{nk+nN+jnq}^2}-\sum_{k=1}^{q}\prod_{j=0}^{m-1}\frac{L_{nk+jnq}^2}{F_{nk+jnq}^2}\,,\end{aligned} \tag{111}$$

from which Theorem 40 follows in the limit N approaches infinity. Theorem 41 follows from identity (18).

2.11 Sums with $L_{nk}^2L_{nk+nq}^2\cdots L_{nk+mnq-nq}^2L_{nk+mnq}L_{nk+mnq+nq}^2\cdots L_{nk+2mnq}^2$ or $L_{nk}^2L_{nk+nq}^2\cdots L_{nk+mnq}^2$ in the denominator

The theorems in this section are the Lucas versions of those of the previous section. We omit their proofs. The basic identity is

$$\frac{5F_{mnq}F_{2nk+mnq}}{L_{nk}^2L_{nk+mnq}^2}=\frac{1}{L_{nk}^2}-\frac{(-1)^{mnq}}{L_{nk+mnq}^2}\,, \tag{112}$$

which follows from the identity (12b).

Theorem 42. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{2nk+mnq}}{\prod_{j=0}^{m}L_{nk+jnq}^2}\right)=\frac{1}{5F_{mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{1}{L_{nk+jnq}^2}\right).$$

Explicitly,

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq}}{L_{nk}^2L_{nk+nq}^2\cdots L_{nk+mnq}^2}=\frac{1}{5F_{mnq}}\sum_{k=1}^{q}\frac{1}{L_{nk}^2L_{nk+nq}^2\cdots L_{nk+(m-1)nq}^2}\,.$$

Corollary 43. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{nk+mnq}}{L_{nk+mnq}\prod_{j=0}^{m-1}L_{nk+jnq}^2\prod_{j=m+1}^{2m}L_{nk+jnq}^2}\right)=\frac{1}{5F_{2mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{2m-1}\frac{1}{L_{nk+jnq}^2}\right).$$

Theorem 44. *If m, n and q are positive integers such that q is even* <u>*or*</u> *mnq is odd, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}F_{2nk+mnq}}{\prod_{j=0}^{m}L_{nk+jnq}^2}\right)=\frac{1}{5F_{mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}L_{nk+jnq}^2}\right).$$

In particular, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+nq}}{L_{nk}^2L_{nk+nq}^2}=\frac{1}{5F_{nq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{L_{nk}^2},\quad q\text{ even or }nq\text{ odd}\,,\tag{113}$$

Corollary 45. *If q is a positive* <u>*even*</u> *integer, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}F_{nk+mnq}}{L_{nk+mnq}\prod_{j=0}^{m-1}L_{nk+jnq}^2\prod_{j=m+1}^{2m}L_{nk+jnq}^2}\right)=\frac{1}{5F_{2mnq}}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1}L_{nk+jnq}^2}\right).$$

In particular:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{k+q}}{L_k^2L_{k+q}L_{k+2q}^2}=\frac{1}{5F_{2q}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{L_k^2L_{k+q}^2},\quad q\text{ even}\,.\tag{114}$$

Theorem 46. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty}\left(\frac{(-1)^{nk-1}F_{2nk+mnq}\prod_{j=1}^{m-1}F_{nk+jnq}^2}{\prod_{j=0}^{m}L_{nk+jnq}^2}\right)=\frac{1}{4F_{mnq}}\sum_{k=1}^{q}\left(\prod_{j=0}^{m-1}\frac{F_{nk+jnq}^2}{L_{nk+jnq}^2}\right)-\frac{1}{4F_{mnq}}\frac{q}{5^m}\,.$$

In particular

$$\sum_{k=1}^{\infty}\frac{(-1)^{nk-1}F_{2nk+nq}}{L_{nk}^2L_{nk+nq}^2}=\frac{1}{4F_{nq}}\sum_{k=1}^{q}\frac{F_{nk}^2}{L_{nk}^2}-\frac{q}{20F_{nq}}\,.\tag{115}$$

Theorem 47. *If n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty}\left(\frac{F_{2nk+mnq}\prod_{j=1}^{m-1}F_{nk+jnq}^2}{\prod_{j=0}^{m}L_{nk+jnq}^2}\right)=\frac{1}{4F_{mnq}}\sum_{k=1}^{q}\left((-1)^{k-1}\prod_{j=0}^{m-1}\frac{F_{nk+jnq}^2}{L_{nk+jnq}^2}\right).$$

In particular

$$\sum_{k=1}^{\infty}\frac{F_{2nk+nq}}{L_{nk}^2L_{nk+nq}^2}=\frac{1}{4F_{nq}}\sum_{k=1}^{q}(-1)^{k-1}\frac{F_{nk}^2}{L_{nk}^2},\quad n\text{ odd, }q\text{ even}\,.\tag{116}$$

2.12 Sums with $F_{nk}F_{nk+np}F_{nk+nq}F_{nk+nq+np}F_{nk+2nq}F_{nk+2nq+np}\cdots F_{nk+mnq}F_{nk+mnq+np}$ in the denominator

Theorem 48. *If m, n, q are positive integers such that mnq is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}F_{nk+jnq+np}}=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1}F_{nk+jnq}F_{nk+jnq+np}}\,.$$

Theorem 49. *If m, n, q are positive integers such that mnq is odd or q is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+mnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}F_{nk+jnq+np}}=\frac{1}{F_{mnq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1}F_{nk+jnq}F_{nk+jnq+np}}\,.$$

Proof of Theorem 48 and Theorem 49

By dividing through the identity (13a) by $F_uF_{u+p}F_{u+v}F_{u+v+p}$ and setting $u=nk$ and $v=mnq$, the following identity is established

$$\frac{F_{mnq}F_{2nk+mnq+np}}{F_{nk}F_{nk+np}F_{nk+mnq}F_{nk+mnq+np}}=\frac{1}{F_{nk}F_{nk+np}}+\frac{(-1)^{mnq+1}}{F_{nk+mnq}F_{nk+mnq+np}}\,,\tag{117}$$

so that

$$\begin{aligned}&\frac{F_{mnq}F_{2nk+mnq+np}}{F_{nk}F_{nk+np}F_{nk+mnq}F_{nk+mnq+np}}\\&\qquad=\frac{1}{F_{nk}F_{nk+np}}-\frac{1}{F_{nk+mnq}F_{nk+mnq+np}},\quad mnq\text{ even}\end{aligned}\tag{118}$$

and

$$\begin{aligned}&\frac{F_{mnq}F_{2nk+mnq+np}}{F_{nk}F_{nk+np}F_{nk+mnq}F_{nk+mnq+np}}\\&\qquad=\frac{1}{F_{nk}F_{nk+np}}+\frac{1}{F_{nk+mnq}F_{nk+mnq+np}},\quad mnq\text{ odd}\,.\end{aligned}\tag{119}$$

From (118) and $f(k)=1/(F_kF_{k+np})$ in Lemma 15 we obtain the finite summation identity

$$\begin{aligned}&F_{mnq}\sum_{k=1}^{N}\left(\frac{F_{2nk+mnq+np}}{\prod_{j=0}^{m}F_{nk+jnq}F_{nk+jnq+np}}\right)\\&\qquad=\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{m-1}F_{nk+jnq}F_{nk+jnq+np}}\right)\\&\qquad\quad-\sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{m-1}F_{nk+nN+jnq}F_{nk+nN+jnq+np}}\right),\end{aligned}\tag{120}$$

from which Theorem 48 follows in the limit as N approaches infinity. The statement of Theorem 49 follows from (117) and identity (18).

2.13 Sums with $L_{nk}L_{nk+np}L_{nk+nq}L_{nk+nq+np}L_{nk+2nq}L_{nk+2nq+np}\cdots L_{nk+mnq}L_{nk+mnq+np}$ in the denominator

The results in this section are the Lucas versions of the results in the preceding section. The basic identities are

$$\frac{5F_{mnq}F_{2nk+mnq+np}}{\prod_{j=0}^{m} L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}} + \frac{(-1)^{mnq+1}}{\prod_{j=1}^{m} L_{nk+jnq}L_{nk+jnq+np}}\,, \qquad (121)$$

and, if mnq is even

$$\begin{aligned} 5F_{mnq}\sum_{k=1}^{N}\left(\frac{F_{2nk+mnq+np}}{\prod_{j=0}^{m} L_{nk+jnq}L_{nk+jnq+np}}\right) &= \sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}\right) \\ &\quad - \sum_{k=1}^{q}\left(\frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+jnq}L_{nk+nN+jnq+np}}\right), \end{aligned} \qquad (122)$$

while if mnq is odd or q is even:

$$\begin{aligned} 5F_{mnq}\sum_{k=1}^{N}\left(\frac{(-1)^{k-1}F_{2nk+mnq+np}}{\prod_{j=0}^{m} L_{nk+jnq}L_{nk+jnq+np}}\right) &= \sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}\right) \\ &\quad + (-1)^{N-1}\sum_{k=1}^{q}\left(\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+jnq}L_{nk+nN+jnq+np}}\right). \end{aligned} \qquad (123)$$

Theorem 50. *If m, n, q are positive integers such that mnq is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty}\frac{F_{2nk+mnq+np}}{\prod_{j=0}^{m} L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{5F_{mnq}}\sum_{k=1}^{q}\frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}\,.$$

Theorem 51. *If m, n, q are positive integers such that mnq is odd or q is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2nk+mnq+np}}{\prod_{j=0}^{m} L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{5F_{mnq}}\sum_{k=1}^{q}\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}\,.$$

2.14 Evaluation of other sums

Theorem 52. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^{m} F_{nk+jnq} F_{nk+jnq+2}} = \frac{q}{F_{mnq}} - \frac{1}{F_{mnq}} \sum_{k=1}^{q} \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}}\,,$$

so that

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^{m} F_{nk+jnq} F_{nk+jnq+2}} = \frac{1}{F_{mnq}} \sum_{k=1}^{q} \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} - \frac{q}{F_{mnq}}, \quad n \text{ even} \quad (124)$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^{m} F_{nk+jnq} F_{nk+jnq+2}} = \frac{q}{F_{mnq}} - \frac{1}{F_{mnq}} \sum_{k=1}^{q} \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}}, \quad n \text{ odd}. \quad (125)$$

Theorem 53. *If m, n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left(\frac{F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^{m} F_{nk+jnq} F_{nk+jnq+2}} \right) = \frac{1}{F_{mnq}} \sum_{k=1}^{q} \left(\prod_{j=0}^{m-1} \frac{(-1)^k F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} \right).$$

Proof of Theorem (52) and Theorem (53)

Dividing through the identity (10) by $F_{v+1}F_{v-1}F_{u+1}F_{u-1}$ and setting $u = nk + 1$ and $v = nk + mnq + 1$ we obtain the identity

$$\frac{(-1)^{nk-1} F_{mnq} F_{2nk+mnq+2}}{F_{nk} F_{nk+2} F_{nk+mnq} F_{nk+mnq+2}} = \frac{F_{nk+mnq+1}^2}{F_{nk+mnq+2} F_{nk+mnq}} - \frac{F_{nk+1}^2}{F_{nk+2} F_{nk}}\,. \quad (126)$$

With $f(k) = F_{k+1}^2/(F_k F_{k+2})$ in Lemma 1 and use of the identity (126) we get the finite summation identity

$$\begin{aligned} F_{mnq} \sum_{k=1}^{N} \left(\frac{(-1)^{nk-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^{m} F_{nk+jnq} F_{nk+jnq+2}} \right) \\ = \sum_{k=1}^{q} \left(\prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq+1}^2}{F_{nk+nN+jnq} F_{nk+nN+jnq+2}} \right) \\ - \sum_{k=1}^{q} \left(\prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} \right), \end{aligned} \quad (127)$$

from which Theorem (52) follows in the limit as N approaches infinity. Theorem (53) follows from the identity (126) and taking $f(k) = F_{k+1}^2/(F_k F_{k+2})$ in identity (18).

Theorem 54.

$$\sum_{k=1}^{\infty}\frac{F_{2k+3}}{F_k^4F_{k+1}^3F_{k+2}^3F_{k+3}^4}=\frac{1}{128},\quad \sum_{k=1}^{\infty}\frac{F_{2k+3}}{L_k^4L_{k+1}^3L_{k+2}^3L_{k+3}^4}=\frac{1}{829440}\,.$$

Theorem 55.

$$\sum_{k=1}^{\infty}\frac{F_{3k+1}F_{3k+2}F_{6k+3}}{F_{3k}^4F_{3k+3}^4}=\frac{1}{128},\quad \sum_{k=1}^{\infty}\frac{L_{3k+1}L_{3k+2}F_{6k+3}}{L_{3k}^4L_{3k+3}^4}=\frac{1}{10240}\,.$$

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