Journal of Integer Sequences, Vol. 21 (2018),
Article 18.4.1

On Two New Classes of B-q-bonacci Polynomials

Suchita Arolkar
Department of Mathematics and Statistics
Dnyanprassarak Mandal's College and Research Centre
Assagao Bardez Goa
403 507, India
Suchita.golatkar@yahoo.com

Yeshwant Shivrai Valaulikar
Department of Mathematics
Goa University
Taleigao Plateau Goa
403 206, India
ysv@unigoa.ac.in

Abstract

In this paper we define two new classes of polynomials associated with generalized Fibonacci polynomials. We call them $h(x)$-B-q-bonacci polynomials and incomplete $h(x)$-B-q-bonacci polynomials. We present some identities for the two classes of polynomials, and the convolution property of $h(x)$-B-q-bonacci polynomials and its applications.

1 Introduction

The Fibonacci sequence, polynomials associated with the Fibonacci sequence, and their extended forms produce interesting and fascinating properties. For details see [8, 14]. Arolkar and Valaulikar introduced the B-tribonacci sequence [2] and B-tribonacci polynomials [1].

The B-tribonacci sequence [2] and B-tribonacci polynomials [1] are further extended to q^{th} order recurrence relations in [5] and [6] respectively. Arolkar and Valaulikar extended and studied the $h(x)$-Fibonacci polynomials [9] to $h(x)$-B-tribonacci polynomials [3]. Filipponi [7] introduced the incomplete Fibonacci and Lucas numbers. Ramírez [11] studied various identities related to the incomplete k-Fibonacci and k-Lucas numbers. Ramírez [13] introduced interesting classes of polynomials, namely, the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Arolkar and Valaulikar [4] extended the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Ramírez and Sirvent [12] defined and studied identities related to the incomplete tribonacci numbers and polynomials. Yilmaz and Taskara [10] obtained identities for the incomplete tribonacci-Lucas numbers and polynomials.

The aim of this paper is to extend two classes of polynomials, namely, the $h(x)$-B-tribonacci polynomials [3] and incomplete $h(x)$-B-tribonacci polynomials of [4] to the q^{th} order relations. We call them the $h(x)$-B-q-bonacci polynomials and incomplete $h(x)$-B-q-bonacci polynomials. We study some properties of these polynomials.

2 $h(x)$-B-q-bonacci polynomials

We first define the class of $h(x)$-B-q-bonacci polynomials.

Definition 1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-B-q-bonacci polynomials, denoted by $({}^{q}B)_{h,n}(x), n \in \mathbb{N} \cup \{0\}, q \geq 2$, are defined by

$$({}^{q}B)_{h,n+q-1}(x) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r}(x)({}^{q}B)_{h,n+q-2-r}(x), \forall n \geq 1, \tag{1}$$

with $({}^{q}B)_{h,i}(x) = 0,\ i = 0, 1, 2, 3, \ldots, q-2$ and $({}^{q}B)_{h,q-1}(x) = 1$, where the coefficients of the terms on the right-hand side are the terms of the binomial expansion of $(h(x)+1)^{q-1}$ and $({}^{q}B)_{h,n}(x)$ is the n^{th} polynomial.

For simplicity, henceforth we denote $({}^{q}B)_{h,n}(x)$ by $({}^{q}B)_{h,n}$ and $h(x)$ by h. We have the following identities for $({}^{q}B)_{h,n}$.

(1) The n^{th} term $({}^{q}B)_{h,n}$ of (1) is given by

$$({}^{q}B)_{h,n} = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1)-r)-r}, \tag{2}$$

$n \geq q-1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. We prove the identity using induction on n.

For $n = q-1$, (2) implies

$$({}^{q}B)_{h,q-1} = \sum_{r=0}^{0} \frac{((q-1)(-r))^{\underline{r}}}{r!} \quad h^{(q-1)(-r)-r} = 1.$$

Hence (2) is true for $n = q-1$. Assume that (2) is true for $n \leq m$. We divide the result into q cases, namely, $m = qk, qk+1, qk+2, \ldots, qk+(q-1)$, for some $k \geq 1$.

Case (i): Let $m = qk$ and $t = \left\lfloor \frac{(q-1)(qk-s-(q-1))}{q} \right\rfloor$. Then

$$\begin{aligned}
&\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \ h^{q-1-s}({}^{q}B)_{h,qk-s} \\
&= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{r=0}^{t} \frac{((q-1)(qk-(q-1)-(r+s)))^{\underline{r}}}{r!} \ h^{(q-1)(qk+1-(q-1)-(r+s))-(r+s)} \\
&= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{p=s}^{(q-1)k-(q-2)} \frac{((q-1)(qk-(q-1)-p))^{\underline{p-s}}}{(p-s)!} \ h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= \Bigg(\frac{(q-1)^{\underline{0}}}{0!} \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)(qk-(q-1)-p))^{\underline{p}}}{p!} \\
&+ \frac{(q-1)^{\underline{1}}}{1!} \sum_{p=1}^{(q-1)k-(q-2)} \frac{((q-1)(qk-(q-1)-p))^{\underline{p-1}}}{(p-1)!} + \cdots \\
&+ \frac{(q-1)^{\underline{(q-1)}}}{(q-1)!} \sum_{p=q-1}^{(q-1)k-(q-2)} \frac{((q-1)(qk-(q-1)-p))^{\underline{p-(q-1)}}}{(p-(q-1))!} \Bigg) \ h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= \Bigg(\frac{((q-1)(qk-(q-1)))^{\underline{0}}}{0!} \\
&+ \left(\frac{((q-1)(qk-(q-1)-1))^{\underline{1}}}{1!} + \frac{(q-1)^{\underline{1}}}{1!} \frac{((q-1)(qk-(q-1)-1))^{\underline{0}}}{0!} \right) + \cdots \\
&+ \sum_{p=q-1}^{(q-1)k-(q-2)} \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \frac{((q-1)(qk-(q-1)-p))^{\underline{p-s}}}{(p-s)!} \Bigg) \ h^{(q-1)(qk+1-(q-1)-p)-p}.
\end{aligned}$$

Therefore, using $\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \frac{n^{p-s}}{(p-s)!} = \frac{(n+(q-1))^{\underline{p}}}{p!}$, we have

$$\begin{aligned}
&\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} ({}^{q}B)_{h,qk-s} \\
&= \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)(qk+1-(q-1)-p))^{\underline{p}}}{p!} h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= ({}^{q}B)_{h,qk+1}.
\end{aligned}$$

Thus, assuming the result for $m = qk$, we have proved it for $m = qk+1$. Similarly, we can prove the other cases.

We conclude that $\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} ({}^{q}B)_{h,m-s} = ({}^{q}B)_{h,m+1}$.

Hence, by induction, the result follows. □

(2) The sum of the first $n+1$ terms of (1) is given by

$$\sum_{r=0}^{n} ({}^{q}B)_{h,r} = \frac{({}^{q}B)_{h,n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \ ({}^{q}B)_{h,n-i} - 1}{(h+1)^{q-1} - 1}, \tag{3}$$

provided $\begin{cases} h \neq 0, & \text{if } q \text{ is even;} \\ h \neq 0, -2, & \text{if } q \text{ is odd.} \end{cases}$

Proof. We obtain the result by induction on n. For $n = q-1$, $\sum_{r=0}^{q-1} ({}^{q}B)_{h,r} = ({}^{q}B)_{h,q-1} = 1$. Also,

$$\begin{aligned}
&\frac{({}^{q}B)_{h,q} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \ ({}^{q}B)_{h,q-1-i} - 1}{(h+1)^{q-1} - 1} \\
&= \frac{h^{q-1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} ({}^{q}B)_{h,q-1} - 1}{(h+1)^{q-1} - 1} = 1.
\end{aligned}$$

Thus, (3) is true for $n = q-1$.

Assume that (3) is true for $n \leq m$. Then

$$\begin{aligned}
\sum_{r=0}^{m+1} ({}^{q}B)_{h,r} &= \sum_{r=0}^{m} ({}^{q}B)_{h,r} + ({}^{q}B)_{h,m+1} \\
&= \frac{({}^{q}B)_{h,m+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} ({}^{q}B)_{h,m-i} - 1}{(h+1)^{q-1} - 1} + ({}^{q}B)_{h,m+1} \\
&= \frac{({}^{q}B)_{h,m+2} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \ ({}^{q}B)_{h,m+1-i} - 1}{(h+1)^{q-1} - 1}.
\end{aligned}$$

Thus, the result is true for $n = m+1$. Hence by induction the result follows. □

(3) The generating function for (1) is given by

$$({}^{q}G_{(B)})_h(z) = \frac{1}{1 - z(h+z)^{q-1}}, \tag{4}$$

provided $|(z(h+z)^{q-1})| < 1$.

Proof. Let $t = \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor$

$$\begin{aligned}
({}^{q}G_{(B)})_h(z) &= \sum_{n=0}^{\infty} ({}^{q}B)_{h,n} \; z^{n-(q-1)} \\
&= \sum_{n=0}^{q-2} ({}^{q}B)_{h,n} \; z^{n-(q-1)} + \sum_{n=q-1}^{\infty} ({}^{q}B)_{h,n} \; z^{n-(q-1)} \\
&= \sum_{n=q-1}^{\infty} ({}^{q}B)_{h,n} \, z^{n-(q-1)}, \text{ since } ({}^{q}B)_{h,i} = 0, \; i = 0, 1, 2, 3, \ldots, q-2 \\
&= \sum_{n=q-1}^{\infty} \sum_{r=0}^{t} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \;\; h^{(q-1)(n-(q-1)-r)-r} \;\; z^{n-(q-1)} \\
&= 1 + h^{q-1} z + \left(h^{2(q-1)} + \frac{(q-1)^{\underline{1}}}{1!} h^{q-2} \right) z^2 + \ldots \\
&= 1 + z(h+z)^{q-1} + z^2 (h+z)^{2(q-1)} + \ldots \\
&= \frac{1}{1 - z(h+z)^{q-1}}, \text{ provided } |(z(h+z)^{q-1})| < 1.
\end{aligned}$$

□

We now obtain the following property.

Theorem 2. (***Convolution property for*** $({}^{q}B)_{h,n}$)
For all $n \geq q-1$, *we have*

$$\frac{d}{dx}\left(({}^{q}B)_{h,n}\right) = (q-1)\frac{dh}{dx}\left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \quad \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i} \; ({}^{q}B)_{h,n+q-2-r-i} \right). \tag{5}$$

Proof. Equation (4) implies

$$\sum_{n=0}^{\infty} ({}^{q}B)_{h,n} \; z^{n-(q-1)} = \frac{1}{1 - z(h+z)^{q-1}}.$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} \frac{d}{dx} \left(({}^qB)_{h,n} \right) z^{n-(q-1)} = z(q-1)(h+z)^{q-2} \; \frac{1}{\left(1 - z(h+z)^{q-1}\right)^2} \; \frac{dh}{dx}$$

$$= \left((q-1) \; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} z^{r+1} \; \left(\sum_{n=0}^{\infty} ({}^qB)_{h,n} \; z^{n-(q-1)} \right)^2 \right) \; \frac{dh}{dx}$$

$$= \left((q-1) \; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; z^{-2(q-1)+r+1} \; \left(\sum_{n=0}^{\infty} ({}^qB)_{h,n} \; z^{n} \right)^2 \right) \; \frac{dh}{dx}$$

$$= (q-1) \frac{dh}{dx} \; \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} ({}^qB)_{h,i} ({}^qB)_{h,n-i} \; z^{n-2(q-1)+r+1} \right).$$

Comparing the coefficients of $z^{n-(q-1)}$, we get

$$\frac{d}{dx} \left(({}^qB)_{h,n} \right) = (q-1) \frac{dh}{dx} \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} ({}^qB)_{h,n+q-2-r-i} \right).$$

□

We now give an application of the convolution property. It is required to prove an identity in the next section.

Theorem 3. *For $n \geq q-1$,*

$$\sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r \; \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr}$$

$$= \frac{(q-1)\,(n-(q-1))}{q} \; ({}^qB)_{h,n}$$

$$-\frac{h}{q} \; (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} \; ({}^qB)_{h,n+q-2-r-i} \right). \tag{6}$$

Proof. Equation (2) implies

$$({}^qB)_{h,n} = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr}.$$

Differentiating both sides with respect to x and simplifying, we get

$$\frac{d}{dx}\big(({}^qB)_{h,n}\big)h = ((q-1)(n-(q-1)))\ ({}^qB)_{h,n}\ \frac{dh}{dx} - q\ \frac{dh}{dx} \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}.$$

Thus,

$$\frac{dh}{dx} \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr} = \frac{(q-1)(n-(q-1))}{q}\ ({}^qB)_{h,n}\ \frac{dh}{dx} - \frac{h}{q}\ \frac{d}{dx}\left(({}^qB)_{h,n}\right).$$

Hence (5) implies

$$\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr} = \frac{(q-1)(n-(q-1))}{q}\ ({}^qB)_{h,n} - \frac{h}{q}\ (q-1)\left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i}\ ({}^qB)_{h,n+q-2-r-i}\right).$$

□

3 Incomplete $h(x)$-B-q-bonacci Polynomials

In this section we define the class of incomplete $h(x)$-B-q-bonacci polynomials and discuss some of its properties.

Definition 4. The incomplete $h(x)$-B-q-bonacci polynomials are defined by

$$({}^qB)^{l}_{h,n}(x) = \sum_{r=0}^{l} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1)-r)-r}(x), \tag{7}$$

$$\forall\ 0 \le l \le \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor \text{ and } n \ge q-1.$$

Note that $({}^{q}B)_{h,n}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor}(x) = ({}^{q}B)_{h,n}(x)$.

For simplicity, we use $({}^{q}B)_{h,n}^{l}(x) = ({}^{q}B)_{h,n}^{l}$, $({}^{q}B)_{h,n}(x) = ({}^{q}B)_{h,n}$ and $h(x) = h$. We prove identities related to the recurrence relation for $({}^{q}B)_{h,n}^{l}$.

Theorem 5. *The recurrence relation for* $({}^{q}B)_{h,n}^{l}$ *is given by*

$$({}^{q}B)_{h,n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \;\; h^{q-1-r} ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r}, \; 0 \leq l \leq \left\lfloor \frac{(q-1)(n-q)}{q} \right\rfloor, \forall n \geq q. \tag{8}$$

Proof. Consider,

$$\begin{aligned}
&\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r} \; ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r} \\
&\sum_{i=0}^{l+q-1-r} \frac{((q-1)(n+q-1-r-(q-1)-i))^{\underline{i}}}{i!} \;\; h^{(q-1)(n+q-1-r-(q-1)-i)-i} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r} \;\; \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^{\underline{i}}}{i!} \; h^{(q-1)(n-r)-qi} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^{\underline{i}}}{i!} \; h^{(q-1)(n+1)-qr-qi} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-(r+i)))^{\underline{i}}}{i!} \; h^{(q-1)(n+1)-q(r+i)}.
\end{aligned}$$

Taking $j = i + r$, we get

$$\begin{aligned}
&\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r} \; h^{q-1-r} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \sum_{j=r}^{l+q-1} \frac{((q-1)(n-j))^{\underline{j-r}}}{(j-r)!} \; h^{(q-1)(n+1)-qj} \\
&= \sum_{j=0}^{l+q-1} \frac{((q-1)(n+1-j))^{\underline{j}}}{j!} \; h^{(q-1)(n+1)-qj} \\
&= ({}^{q}B)_{h,n+q}^{l+q-1}.
\end{aligned}$$

□

Theorem 6. *For $s \geq 1$,*

$$({}^{q}B)_{h,n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \, ({}^{q}B)_{h,n+i}^{l+i} \, h^{i}, \tag{9}$$

$0 \leq l \leq \left\lfloor \frac{(q-1)(n-s-(q-1))}{q} \right\rfloor$.

Proof. Follows using induction. □

Theorem 7. *For $n \geq \left\lfloor \frac{ql+2(q-1)}{q-1} \right\rfloor$, $({}^{q}B)_{h,n+(q-1)+s}^{l+(q-1)} - h^{(q-1)s}({}^{q}B)_{h,n+q-1}^{l+q-1}$*

$$= \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{(q-1)s-(q-1)i-r} \, ({}^{q}B)_{h,n+(q-1)+i-r}^{l+(q-1)-r}. \tag{10}$$

Proof. Follows using induction. □

The next theorem is related to the sum of incomplete $h(x)$-B-q-bonacci polynomials $({}^{q}B)_{h,n}^{l}$.

Theorem 8. *For all $n \geq q-1$,*

$$\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^{q}B)_{h,n}^{l} = \left(\left\lfloor \frac{(q-1)\,(n-(q-1))}{q} \right\rfloor + \frac{q-(q-1)\,(n-(q-1))}{q} \right) ({}^{q}B)_{h,n}$$

$$+ \frac{h}{q} \, (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \, h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i} \, ({}^{q}B)_{h,n+q-2-r-i} \right). \tag{11}$$

Proof. $\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^{q}B)_{h,n}^{l}$

$$= ({}^{q}B)_{h,n}^{0} + ({}^{q}B)_{h,n}^{1} + \cdots + ({}^{q}B)_{h,n}^{r} + \cdots + ({}^{q}B)_{h,n}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor}$$

$$= \frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} \, h^{(q-1)(n-(q-1))}$$

$$+ \left(\frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} h^{(q-1)(n-(q-1))} + \frac{(q-1)(n-(q-1)-1))^{\underline{1}}}{1!} \, h^{(q-1)(n-(q-1))-q} \right)$$

$$+ \cdots +$$

$$\left(\frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} \, h^{(q-1)(n-(q-1))} + \cdots + \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \, h^{(q-1)(n-(q-1))-qr} \right)$$

$$+ \cdots$$

$$+\left(\frac{((q-1)(n-(q-1)))^{0}}{0!}\ h^{(q-1)(n-(q-1))}+\cdots\right.$$
$$+\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}+\cdots$$
$$\left.+\frac{\left((q-1)\left(n-(q-1)-\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor\right)\right)^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}}{(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor)!}\ h^{(q-1)(n-(q-1))-q\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\right)$$
$$=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1-r\right)\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}$$
$$=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\left\lfloor\frac{(q-1)(n-(q-1))}{q}\right\rfloor+1\right)\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}$$
$$-\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor} r\frac{((q-1)(n-(q-1)-r))^{r}}{r!}\ h^{(q-1)(n-(q-1))-qr}$$
$$=\left(\left\lfloor\frac{(q-1)(n-(q-1))}{q}\right\rfloor+1-\frac{(q-1)(n-(q-1))}{q}\right)({}^{q}B)_{h,n}$$
$$+\frac{h}{q}(q-1)\left(\sum_{r=0}^{q-2}\frac{(q-2)^{r}}{r!}\ h^{(q-2)-r}\sum_{i=0}^{n+q-2-r}({}^{q}B)_{h,i}\ ({}^{q}B)_{h,n+q-2-r-i}\right).$$

Using (6) of Theorem 3 in Section 2, the result follows. □

4 Acknowledgments

The authors thank the reviewers/referees for their comments and suggestions that helped to improve the article.

References

[1] S. Arolkar and Y. S. Valaulikar, Generalized bivariate B-tribonacci and B-tri-Lucas polynomials, *Proc. CPMSED-2015,* Krishi sanskriti publications, 2015, pp. 10–13.

[2] S. Arolkar and Y. S. Valaulikar, On an extension of Fibonacci sequence, *Bull. Marathwada Math. Soc.* **17** (2016), 1–8.

[3] S. Arolkar and Y. S. Valaulikar, $h(x)$-B-tribonacci and $h(x)$-B-tri Lucas polynomials, *Kyungpook Math. J.* **56** (2016), 1125–1133.

[4] S. Arolkar and Y. S. Valaulikar, Incomplete $h(x)$-B-tribonacci polynomials, *Turkish Journal of Analysis and Number Theory* **4** (2016), 155–158.

[5] S. Arolkar and Y. S. Valaulikar, On a B-q bonacci sequence, *International Journal of Advances in Mathematics* **1** (2017), 1–8.

[6] S. Arolkar and Y. S. Valaulikar, Identities involving partial derivatives of bivariate B-q bonacci and B-q Lucas polynomials, preprint.

[7] P. Filipponi, Incomplete Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo* **45** (1996), 37–56.

[8] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, 2001.

[9] A. Nalli and P. Haukkane, On generalized Fibonacci and Lucas polynomials, *Chaos, Solitons & Fractals* **42** (2009), 3179–3186.

[10] N. Yilmaz and N. Taskara, Incomplete Tribonacci-Lucas numbers and polynomials, *Adv. Appl. Clifford Algebr* **25** (2015), 741–753.

[11] J. L. Ramírez, Incomplete k-Fibonacci and k-Lucas numbers, *Chinese J. of Math.* (2013), Article ID 107145.

[12] J. L. Ramírez and V. F. Sirvent, Incomplete tribonacci numbers and polynomials, *J. Integer Sequences* **17** (2014), Article 14.4.2.

[13] J. L. Ramírez, Incomplete generalized Fibonacci and Lucas polynomials, *Hacet. J. Math. Stat.* **42** (2015), 363–373.

[14] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover, 2008.

2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B83.
Keywords: Fibonacci polynomial, $h(x)$-B-q-bonacci polynomial, incomplete $h(x)$-B-q-bonacci polynomial.

Received October 3 2017; revised versions received October 5 2017; March 16 2018; April 4 2018; April 6 2018. Published in *Journal of Integer Sequences*, May 7 2018.

Return to Journal of Integer Sequences home page.