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Quasi-Fibonacci Numbers of the Seventh Order

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Abstract

In this paper we introduce and investigate the so-called quasi-Fibonacci numbers of the seventh order. We discover many surprising relations and identities, and study some applications to polynomials.

1 Introduction

Grzymkowski and Wituła [3] discovered and studied the following two identities:

$$(1+\xi+\xi^4)^n = F_{n+1} + F_n(\xi+\xi^4), \tag{1.1}$$
$$(1+\xi^2+\xi^3)^n = F_{n+1} + F_n(\xi^2+\xi^3), \tag{1.2}$$

where F_n denote the Fibonacci numbers and $\xi \in \mathbb{C}$ is a primitive fifth root of unity (i.e., $\xi^5 = 1$ and $\xi \neq 1$). These identities make it possible to prove many classical relations for Fibonacci numbers as well as to generalize some of them. We may state that these identities make up an independent method of proving such relations, which is an alternative to the methods depending on the application of either Binet formulas or the generating function of Fibonacci and Lucas numbers.

1.1 Example of the application of identities (1.1) and (1.2)

First, we note that

$$(1+\xi+\xi^4)^{u+v} = F_{u+v+1} + F_{u+v}\left(\xi+\xi^4\right) \tag{1.3}$$

and

$$\begin{aligned}(1+\xi+\xi^4)^{u+v} &= (1+\xi+\xi^4)^u\,(1+\xi+\xi^4)^v = \\ &= \left(F_{u+1}+F_u\,(\xi+\xi^4)\right)\left(F_{v+1}+F_v\,(\xi+\xi^4)\right) = \\ &\quad (\text{by the identity } 1+\xi+\xi^2+\xi^3+\xi^4=0) \\ &= F_{u+1}\,F_{v+1}+F_u\,F_v+(F_u\,F_{v+1}+F_{u+1}\,F_v-F_u\,F_v)\,(\xi+\xi^4) = \\ &= F_{u+1}\,F_{v+1}+F_u\,F_v+(F_u\,F_{v+1}+F_{u-1}\,F_v)\,(\xi+\xi^4).\end{aligned} \tag{1.4}$$

Replacing u by $u-r$ and v by $v+r$ in (1.4) we obtain

$$\begin{aligned}(1+\xi+\xi^4)^{u+v} &= F_{u-r+1}\,F_{v+r+1}+F_{u-r}\,F_{v+r}+ \\ &\quad +(F_{u-r}\,F_{v+r+1}+F_{u-r+1}\,F_{v+r}-F_{u-r}\,F_{v+r})(\xi+\xi^4).\end{aligned} \tag{1.5}$$

We note that the numbers 1 and $\xi+\xi^4$ are linearly independent over $\mathbb{Q}$. Hence comparing the parts without $(\xi+\xi^4)$ of (1.3) with (1.4) and (1.4) with (1.5) we get two known identities (see [2, 5])

$$F_{u+v+1} = F_{u+1}\,F_{v+1}+F_u\,F_v$$

and

$$\begin{aligned}F_{u+1}\,F_{v+1}-F_{u-r+1}\,F_{v+r+1} &= F_{u-r}\,F_{v+r}-F_u\,F_v \\ &\quad (\text{after the next } (u-r) \text{ iterations}) \\ &= (-1)^{u-r+1}(F_r\,F_{v-u+r}-F_0\,F_{v-u+2r}) \\ &= (-1)^{u-r+1}F_r\,F_{v-u+r}.\end{aligned}$$

1.2 The aim of the paper

In this paper the relations (1.1) and (1.2) will be generalized in the following way:

$$\begin{aligned}(1+\delta(\xi+\xi^6))^n &= A_n(\delta)+B_n(\delta)(\xi+\xi^6)+C_n(\delta)(\xi^2+\xi^5), \\ (1+\delta(\xi^2+\xi^5))^n &= A_n(\delta)+B_n(\delta)(\xi^2+\xi^5)+C_n(\delta)(\xi^3+\xi^4),\end{aligned}$$

and

$$(1+\delta(\xi^3+\xi^4))^n = A_n(\delta)+B_n(\delta)(\xi^3+\xi^4)+C_n(\delta)(\xi+\xi^6),$$

where $\xi\in\mathbb{C}$ are primitive seventh roots of unity (i.e., $\xi^7=1$ and $\xi\neq 1$), $\delta\in\mathbb{C}$, $\delta\neq 0$. New families of numbers created by these identities

$$\{A_n(\delta)\}_{n=1}^{\infty}, \qquad \{B_n(\delta)\}_{n=1}^{\infty}, \quad \text{and} \quad \{C_n(\delta)\}_{n=1}^{\infty} \tag{1.6}$$

called here "the quasi-Fibonacci numbers of order $(7;\delta)$", $\delta \in \mathbb{C}$, $\delta \neq 0$, are investigated in this paper. The elements of each of the three sequences (1.6) satisfy the same recurrence relation of order three

$$\mathbb{X}_{n+3} + (\delta - 3)\mathbb{X}_{n+2} + (3 - 2\delta - 2\delta^2)\mathbb{X}_{n+1} + (-1 + \delta + 2\delta^2 - \delta^3)\mathbb{X}_n = 0,$$

which enables a direct trigonometrical representation of these numbers (see formulas (3.17)–(3.19)). In consequence, many surprising algebraic and trigonometric identities and summation formulas for these numbers may be generated. Also the polynomials connected with the numbers $\{C_n(\delta)\}_{n=1}^{\infty}$

$$\sum_{k=1}^{n} C_{k+1}(\delta)\, x^{n-k}, \qquad n \in \mathbb{N},$$

are investigated in this paper.

2 Minimal polynomials, linear independence over $\mathbb{Q}$

Let $\Psi_n(x)$ be the minimal polynomial of $\cos(2\pi/n)$ for every $n \in \mathbb{N}$. W. Watkins and J. Zeitlin described [10] (see also [9]) the following identities:

$$T_{s+1}(x) - T_s(x) = 2^s \prod_{d|n} \Psi_d(x)$$

if $n = 2s + 1$ and

$$T_{s+1}(x) - T_{s-1}(x) = 2^s \prod_{d|n} \Psi_d(x)$$

if $n = 2s$, where $T_s(x)$ denotes the s-th Chebyshev polynomial of the first kind. In the sequel, if $n = 2s + 1$ is a prime number, we obtain

$$T_{s+1}(x) - T_s(x) = 2^s \Psi_1(x)\Psi_n(x).$$

For example, we have

$$\begin{aligned}\Psi_7(x) &= \frac{1}{8(x-1)}(T_4(x) - T_3(x)) = \frac{1}{8(x-1)}(8x^4 - 4x^3 - 8x^2 + 3x + 1) \\ &= \frac{1}{8(x-1)} 8(x-1)\left(x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8}\right) = x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{8},\end{aligned}$$

Lemma 2.1. *If $n \geqslant 3$ then the roots of $\Psi_n(x) = 0$ are $\cos(2\pi k/n)$, for $0 < k \leqslant s$ and $(k,n) = 1$ (where $n = 2s$ or $n = 2s + 1$ respectively).*

Below a more elementary proof of this result but only for the polynomial $\Psi_7(x)$, will be presented. Our proof is based only on the following simple fact:

Lemma 2.2. *Let $q \in \mathbb{Q}$ and $\cos(q\pi) \in \mathbb{Q}$. Then $\cos(q\pi) \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}$.*

Proof. For the elementary proof of this Lemma see, for example, the Appendix to the Russian translation of Niven's book [6] (written by I. M. Yaglom). □

Lemma 2.3. *Let* $\xi = \exp(i2\pi/7)$*. Then the polynomial*

$$\begin{aligned} p_7(x) &:= (x-\xi-\xi^6)(x-\xi^2-\xi^5)(x-\xi^3-\xi^4) \\ &= x^3 + x^2(-\xi-\xi^2-\xi^3-\xi^4-\xi^5-\xi^6) \\ &\quad + x((\xi+\xi^6)(\xi^2+\xi^5)+(\xi+\xi^6)(\xi^3+\xi^4)+(\xi^2+\xi^5)(\xi^3+\xi^4)) \\ &\quad - (\xi+\xi^6)(\xi^2+\xi^5)(\xi^3+\xi^4) = x^3+x^2-2x-1 \end{aligned}$$

is a minimal polynomial of the numbers

$$\xi+\xi^6 = 2\cos(2\pi/7), \quad \xi^2+\xi^5 = 2\cos(4\pi/7), \quad \xi^3+\xi^4 = 2\cos(6\pi/7). \tag{2.1}$$

Moreover, we have the identity

$$\Psi_7(x) = \frac{1}{8}p_7(2x). \tag{2.2}$$

Proof. Besides Lemma 2.2, each of the numbers in (2.1) is an irrational number, so the polynomial $p_7(x)$ is irreducible over $\mathbb{Q}$. □

Corollary 2.4. *We have*

$$p_7(x-1) = x^3 - 2x^2 - x + 1.$$

See the identity (3.66) below, where the connection of $p_7(x-1)$ with quasi-Fibonacci numbers of 7-th order is presented.

Corollary 2.5. *The numbers* $\cos(2k\pi/7)$*,* $k = 0, 1, 2$ *are linearly independent over* $\mathbb{Q}$*.*

Proof. If we suppose that

$$a + b\cos(2\pi/7) + c\cos(4\pi/7) = 0$$

for some $a, b, c \in \mathbb{Q}$, then we also have

$$a + b\cos(2\pi/7) + c(2\cos^2(2\pi/7) - 1) = 0,$$

i.e.,

$$a - c + b\cos(2\pi/7) + 2c\cos^2(2\pi/7) = 0$$

so the degree of $\cos(2\pi/7)$ is $\leqslant 2$, which by Lemma 2.3 means that $a = b = c = 0$. □

Corollary 2.6. *Every three numbers which belong to the set* $\{1, \xi+\xi^6, \xi^2+\xi^5, \xi^3+\xi^4\}$ *are linearly independent over* $\mathbb{Q}$*.*

Proof. It follows from the identity $1+\xi+\xi^2+\ldots+\xi^6 = 0$ and from Corollary 2.5. □

Corollary 2.7. *The following decomposition holds:*

$$\begin{aligned} f_n(\mathbb{X}) &:= \left(\mathbb{X} - \left(2\cos\frac{2\pi}{7}\right)^{2^n}\right)\cdot\left(\mathbb{X} - \left(2\cos\frac{4\pi}{7}\right)^{2^n}\right)\cdot\left(\mathbb{X} - \left(2\cos\frac{6\pi}{7}\right)^{2^n}\right) \\ &= \mathbb{X}^3 - \alpha_n\mathbb{X}^2 + \beta_n\mathbb{X} - 1, \end{aligned}$$

where $\alpha_0 = -1$, $\beta_0 = -2$ *and*

$$\begin{cases} \alpha_{n+1} = \alpha_n^2 - 2\beta_n, \\ \beta_{n+1} = \beta_n^2 - 2\alpha_n, \quad n \in \mathbb{N}. \end{cases} \tag{2.3}$$

For example, we have $\alpha_0 = -1$, $\beta_0 = -2$, $\alpha_1 = 5$, $\beta_1 = 6$, $\alpha_2 = 13$, $\beta_2 = 26$, $\alpha_3 = 117$ and $\beta_3 = 650$. (A more general decomposition will be presented later, see Lemma 3.14 a.) We note that $13|\alpha_n$ and $13|\beta_n$ for every $n \in \mathbb{N}$, $n \geq 2$.

Proof. We have

$$\begin{aligned} &\left(\mathbb{Y}^2 - \left(2\cos\tfrac{2\pi}{7}\right)^{2^{n+1}}\right)\cdot\left(\mathbb{Y}^2 - \left(2\cos\tfrac{4\pi}{7}\right)^{2^{n+1}}\right)\cdot\left(\mathbb{Y}^2 - \left(2\cos\tfrac{6\pi}{7}\right)^{2^{n+1}}\right) = \\ &\quad = -f_n(\mathbb{Y})\cdot f_n(-\mathbb{Y}) = (\mathbb{Y}^3 - \alpha_n\mathbb{Y}^2 + \beta_n\mathbb{Y} - 1)(\mathbb{Y}^3 + \alpha_n\mathbb{Y}^2 + \beta_n\mathbb{Y} + 1) = \\ &\quad\quad = (\mathbb{Y}^3 + \beta_n\mathbb{Y})^2 - (\alpha_n\mathbb{Y}^2 + 1)^2 = \mathbb{Y}^6 - (\alpha_n^2 - 2\beta_n)\mathbb{Y}^4 + (\beta_n^2 - 2\alpha_n)\mathbb{Y}^2 - 1. \end{aligned}$$

□

It is obvious that

$$\alpha_n = \left(2\cos\frac{2\pi}{7}\right)^{2^n} + \left(2\cos\frac{4\pi}{7}\right)^{2^n} + \left(2\cos\frac{6\pi}{7}\right)^{2^n}$$

and

$$\beta_n = \left(4\cos\frac{2\pi}{7}\,\cos\frac{4\pi}{7}\right)^{2^n} + \left(4\cos\frac{2\pi}{7}\,\cos\frac{6\pi}{7}\right)^{2^n} + \left(4\cos\frac{4\pi}{7}\,\cos\frac{6\pi}{7}\right)^{2^n}$$

for every $n \in \mathbb{N}$. Moreover, from (2.3), we deduce that

$$\begin{aligned} \alpha_{n+1} + \beta_{n+1} &= \left(\alpha_n - 1\right)^2 + \left(\beta_n - 1\right)^2 - 2, \\ \alpha_{n+1} - \beta_{n+1} &= \left(\alpha_n - 1\right)^2 - \left(\beta_n - 1\right)^2 = \left(\alpha_n - \beta_n\right)\left(\alpha_n + \beta_n + 2\right) = \\ &= \left(\alpha_n - \beta_n\right)\left(\left(\alpha_{n-1} - 1\right)^2 + \left(\beta_{n-1} - 1\right)^2\right), \end{aligned}$$

which yields

$$\beta_{n+1} - \alpha_{n+1} = \prod_{k=0}^{n-1}\left(\left(\alpha_k - 1\right)^2 + \left(\beta_k - 1\right)^2\right) = 13\prod_{k=1}^{n-1}\left(\left(\alpha_k - 1\right)^2 + \left(\beta_k - 1\right)^2\right) =$$

$$= \prod_{k=1}^{n}\left(\alpha_k + \beta_k + 2\right).$$

The following extension lemma is the basic technical tool to generating almost all formulas presented in the next sections.

Lemma 2.8. *Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$ and let $f_k, g_k \in \mathbb{Q}[\delta]$, $k = 1, 2, \ldots, n$. If the identity*

$$\sum_{k=1}^{n} f_k(\delta)\, a_k = \sum_{k=1}^{n} g_k(\delta)\, a_k$$

holds for every $\delta \in \mathbb{Q}$, then $f_k(\delta) = g_k(\delta)$ for every $k = 1, 2, \ldots, n$ and $\delta \in \mathbb{C}$.

3 Quasi-Fibonacci numbers of order 7

Lemma 3.1. *Let $\delta, \xi \in \mathbb{C}$, $\xi^7 = 1$ and $\xi \neq 1$. Then the following identities hold:*

$$\begin{aligned}(1+\delta(\xi+\xi^6))^n &= a_n(\delta) + b_n(\delta)(\xi^{1\cdot 1}+\xi^{1\cdot 6}) + c_n(\delta)(\xi^{2\cdot 1}+\xi^{2\cdot 6}) \\ &\quad + d_n(\delta)(\xi^{3\cdot 1}+\xi^{3\cdot 6}) \\ &= a_n(\delta) + b_n(\delta)(\xi+\xi^6) + c_n(\delta)(\xi^2+\xi^5) + d_n(\delta)(\xi^3+\xi^4) \\ &= A_n(\delta) + B_n(\delta)(\xi+\xi^6) + C_n(\delta)(\xi^2+\xi^5), \end{aligned} \tag{3.1}$$

$$\begin{aligned}(1+\delta(\xi^2+\xi^5))^n &= a_n(\delta) + b_n(\delta)(\xi^{1\cdot 2}+\xi^{1\cdot 5}) + c_n(\delta)(\xi^{2\cdot 2}+\xi^{2\cdot 5}) \\ &\quad + d_n(\delta)(\xi^{3\cdot 2}+\xi^{3\cdot 5}) \\ &= a_n(\delta) + b_n(\delta)(\xi^2+\xi^5) + c_n(\delta)(\xi^3+\xi^4) + d_n(\delta)(\xi+\xi^6) \\ &= A_n(\delta) + B_n(\delta)(\xi^2+\xi^5) + C_n(\delta)(\xi^3+\xi^4), \end{aligned} \tag{3.2}$$

$$\begin{aligned}(1+\delta(\xi^3+\xi^4))^n &= a_n(\delta) + b_n(\delta)(\xi^{1\cdot 3}+\xi^{1\cdot 4}) + c_n(\delta)(\xi^{2\cdot 3}+\xi^{2\cdot 4}) \\ &\quad + d_n(\delta)(\xi^{3\cdot 3}+\xi^{3\cdot 4}) \\ &= a_n(\delta) + b_n(\delta)(\xi^3+\xi^4) + c_n(\delta)(\xi+\xi^6) + d_n(\delta)(\xi^2+\xi^5) \\ &= A_n(\delta) + B_n(\delta)(\xi^3+\xi^4) + C_n(\delta)(\xi+\xi^6), \end{aligned} \tag{3.3}$$

where

$$\begin{cases} a_0(\delta) = 1, \;\; b_0(\delta) = c_0(\delta) = d_0(\delta) = 0, \\ a_{n+1}(\delta) = a_n(\delta) + 2\delta b_n(\delta), \\ b_{n+1}(\delta) = \delta a_n(\delta) + b_n(\delta) + \delta c_n(\delta), \\ c_{n+1}(\delta) = \delta b_n(\delta) + c_n(\delta) + \delta d_n(\delta), \\ d_{n+1}(\delta) = \delta c_n(\delta) + (1+\delta) d_n(\delta) \end{cases} \tag{3.4}$$

for every $n \in \mathbb{N}$ *and*

$$\begin{aligned}
A_{n+1}(\delta) &:= a_{n+1}(\delta) - d_{n+1}(\delta)\\
&= a_n(\delta) + 2\delta b_n(\delta) - \delta c_n(\delta) - (1+\delta)d_n(\delta)\\
&= (a_n(\delta) - d_n(\delta)) + 2\delta(b_n(\delta) - d_n(\delta)) - \delta(c_n(\delta) - d_n(\delta))\\
&= A_n(\delta) + 2\delta B_n(\delta) - \delta C_n(\delta),\\
B_{n+1}(\delta) &:= b_{n+1}(\delta) - d_{n+1}(\delta)\\
&= \delta a_n(\delta) + b_n(\delta) + \delta c_n(\delta) - \delta c_n(\delta) - (1+\delta)d_n(\delta)\\
&= \delta(a_n(\delta) - d_n(\delta)) + b_n(\delta) - d_n(\delta)\\
&= \delta A_n(\delta) + B_n(\delta),\\
C_{n+1}(\delta) &:= c_{n+1}(\delta) - d_{n+1}(\delta)\\
&= \delta b_n(\delta) + c_n(\delta) + \delta d_n(\delta) - \delta c_n(\delta) - (1+\delta)d_n(\delta)\\
&= \delta(b_n(\delta) - d_n(\delta)) + (1-\delta)(c_n(\delta) - d_n(\delta))\\
&= \delta B_n(\delta) + (1-\delta)C_n(\delta),
\end{aligned}$$

i.e.,

$$\begin{cases} A_0(\delta) = 1, \quad B_0(\delta) = C_0(\delta) = 0,\\ A_{n+1}(\delta) = A_n(\delta) + 2\delta B_n(\delta) - \delta C_n(\delta),\\ B_{n+1}(\delta) = \delta A_n(\delta) + B_n(\delta),\\ C_{n+1}(\delta) = \delta B_n(\delta) + (1-\delta)C_n(\delta) \end{cases} \tag{3.5}$$

for every $n \in \mathbb{N}$.

Proof. For example, we have

$$\begin{aligned}
&(1+\delta(\xi+\xi^6))^{n+1} = (1+\delta(\xi+\xi^6))^n(1+\delta(\xi+\xi^6))\\
&\quad = \big(a_n(\delta) + b_n(\delta)(\xi+\xi^6) + c_n(\delta)(\xi^2+\xi^5) + d_n(\delta)(\xi^3+\xi^4)\big)(1+\delta(\xi+\xi^6))\\
&\quad = a_n(\delta) + \delta a_n(\delta)(\xi+\xi^6) + b_n(\delta)(\xi+\xi^6) + \delta b_n(\delta)(\xi^2+2+\xi^5) + c_n(\delta)(\xi^2+\xi^5)\\
&\qquad + \delta c_n(\delta)(\xi^3+\xi+\xi^6+\xi^4) + d_n(\delta)(\xi^3+\xi^4) + \delta d_n(\delta)(\xi^4+\xi^5+\xi^2+\xi^3)\\
&\quad = a_n(\delta) + 2\delta b_n(\delta) + (\xi+\xi^6)(\delta a_n(\delta) + b_n(\delta) + \delta c_n(\delta))\\
&\qquad + (\xi^2+\xi^5)(\delta b_n(\delta) + c_n(\delta) + \delta d_n(\delta)) + (\xi^3+\xi^4)(\delta c_n(\delta) + (1+\delta)d_n(\delta)) =\\
&(\text{by equality } \xi^3+\xi^4 = -1-\xi-\xi^6-\xi^2-\xi^5)\\
&\quad = a_n(\delta) + 2\delta b_n(\delta) - \delta c_n(\delta) - (1+\delta)d_n(\delta)\\
&\qquad + (\xi+\xi^6)(\delta a_n(\delta) + b_n(\delta) - (1+\delta)d_n(\delta))\\
&\qquad + (\xi^2+\xi^5)(\delta b_n(\delta) + (1-\delta)c_n(\delta) - d_n(\delta)).
\end{aligned}$$

□

Definition 3.2. *The numbers (3.5) will be called the quasi-Fibonacci numbers of order* $(7;\delta)$ *(see Table 1 at the end of the paper). To simplify notation we write* a_n, b_n, c_n, d_n, A_n, B_n, C_n *instead of* $a_n(1)$, $b_n(1)$, $c_n(1)$, $d_n(1)$, $A_n(1)$, $B_n(1)$, $C_n(1)$ *respectively, and these numbers will be called the quasi-Fibonacci numbers of the seventh order.*

Remark 3.3. *Note that section III of [8] contains an explicit reference to the sequences A_n, B_n and C_n. This paper is partly based on the properties of $-p_7(-x)$ and $p_7(x-1)$. A_n, B_n and C_n also appear in [4].*

Corollary 3.4. *Adding equalities (3.1–3.3) we obtain the identity*

$$(1+\delta(\xi+\xi^6))^n + (1+\delta(\xi^2+\xi^5))^n + (1+\delta(\xi^3+\xi^4))^n = \\ = 3A_n(\delta) - B_n(\delta) - C_n(\delta). \quad (3.6)$$

Corollary 3.5. *There follows from (3.4) and (3.5) two special systems (for $\delta = 1$)*

$$\begin{cases} a_0 = 1, \ \ b_0 = c_0 = d_0 = 0, \\ a_{n+1} = a_n + 2b_n, \\ b_{n+1} = a_n + b_n + c_n, \\ c_{n+1} = b_n + c_n + d_n, \\ d_{n+1} = c_n + 2d_n \end{cases} \quad (3.7)$$

and

$$\begin{cases} A_0 = 1, \ \ B_0 = C_0 = 0, \\ A_{n+1} = A_n + 2B_n - C_n, \\ B_{n+1} = A_n + B_n, \\ C_{n+1} = B_n \end{cases} \quad (3.8)$$

for every $n \in \mathbb{N}$ (see Table 2, A006356 and A006054 in [7]).

Corollary 3.6. *Another system may also be derived from (3.5) for $\delta = -1$*

$$\begin{cases} A_0(-1) = 1, \ \ B_0(-1) = C_0(-1) = 0, \\ A_{n+1}(-1) = A_n(-1) - 2B_n(-1) + C_n(-1), \\ B_{n+1}(-1) = -A_n(-1) + B_n(-1), \\ C_{n+1}(-1) = -B_n(-1) + 2C_n(-1) \end{cases}$$

for every $n \in \mathbb{N}$ (see Table 3 and A085810 in [7]).

Corollary 3.7. *If $\delta \neq 0$ then from (3.5) the following identities can be generated:*

$$\delta B_n(\delta) = C_{n+1}(\delta) - (1-\delta)C_n(\delta), \quad (3.9)$$

$$\delta A_n(\delta) = B_{n+1}(\delta) - B_n(\delta), \quad (3.10)$$

$$\begin{aligned} \delta^2 A_n(\delta) &= \delta B_{n+1}(\delta) - \delta B_n(\delta) \quad (3.11) \\ &= C_{n+2}(\delta) - (1-\delta)C_{n+1}(\delta) - \big(C_{n+1}(\delta) - (1-\delta)C_n(\delta)\big) \\ &= C_{n+2}(\delta) - (2-\delta)C_{n+1}(\delta) + (1-\delta)C_n(\delta), \end{aligned}$$

$$\begin{aligned} A_{n+1}(\delta) - A_n(\delta) &= 2\delta B_n(\delta) - \delta C_n(\delta) \quad (3.12) \\ &= 2C_{n+1}(\delta) - 2(1-\delta)C_n(\delta) - \delta C_n(\delta) \\ &= 2C_{n+1}(\delta) - (2-\delta)C_n(\delta), \end{aligned}$$

hence, we obtain

$$\delta^2 A_{n+1}(\delta) - \delta^2 A_n(\delta) = 2\delta^2 C_{n+1}(\delta) - (2\delta^2 - \delta^3)C_n(\delta). \quad (3.13)$$

On the other hand, by (3.11) we obtain

$$\delta^2 A_{n+1}(\delta) - \delta^2 A_n(\delta) = C_{n+3}(\delta) - (2-\delta)C_{n+2}(\delta) + (1-\delta)C_{n+1}(\delta) - \\ - C_{n+2}(\delta) + (2-\delta)C_{n+1}(\delta) - (1-\delta)C_n(\delta) = \\ = C_{n+3}(\delta) - (3-\delta)C_{n+2}(\delta) + (3-2\delta)C_{n+1}(\delta) + (\delta-1)C_n(\delta). \quad (3.14)$$

From (3.13) and (3.14) we obtain the final recurrence identities for numbers $C_n(\delta)$

$$C_{n+3}(\delta) + (\delta-3)C_{n+2}(\delta) + (3-2\delta-2\delta^2)C_{n+1}(\delta) + (-1+\delta+2\delta^2-\delta^3)C_n(\delta) = 0. \quad (3.15)$$

Remark 3.8. *We note that the elements of each of the sequences $\{A_n(\delta)\}$ and $\{B_n(\delta)\}$ also satisfy this recurrence relation (which follows from (3.9) and (3.10)).*

Lemma 3.9. *The characteristic polynomial $p_7(\mathbb{X};\delta)$ of (3.15) has the following decomposition:*

$$p_7(\mathbb{X};\delta) := \mathbb{X}^3 + (\delta-3)\mathbb{X}^2 + (3-2\delta-2\delta^2)\mathbb{X} + (-1+\delta+2\delta^2-\delta^3) = \\ = (\mathbb{X}-1-\delta(\xi+\xi^6))(\mathbb{X}-1-\delta(\xi^2+\xi^5))(\mathbb{X}-1-\delta(\xi^3+\xi^4)) \quad (3.16)$$

where $\xi = \exp(i2\pi/7)$. So the relation

$$C_n(\delta) = \alpha(1+\delta(\xi+\xi^6))^n + \beta(1+\delta(\xi^2+\xi^5))^n + \gamma(1+\delta(\xi^3+\xi^4))^n$$

holds for every $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Remark 3.10. *To find α, β, γ, from Lemma 3.9, it is sufficient to solve the following linear system:*

$$\begin{cases} C_1(\delta) = 0 = \alpha(1+\delta(\xi+\xi^6)) + \beta(1+\delta(\xi^2+\xi^5)) + \gamma(1+\delta(\xi^3+\xi^4)), \\ C_2(\delta) = \delta^2 = \alpha(1+\delta(\xi+\xi^6))^2 + \beta(1+\delta(\xi^2+\xi^5))^2 + \gamma(1+\delta(\xi^3+\xi^4))^2, \\ C_3(\delta) = 3\delta^2 - \delta^3 = \alpha(1+\delta(\xi+\xi^6))^3 + \beta(1+\delta(\xi^2+\xi^5))^3 + \gamma(1+\delta(\xi^3+\xi^4))^3. \end{cases}$$

After some calculations we obtain the identities

$$\begin{aligned} 7C_n(\delta) &= (\xi^{2\cdot1}+\xi^{2\cdot6}-\xi^{3\cdot1}-\xi^{3\cdot6})(1+\delta(\xi+\xi^6))^n \\ &\quad + (\xi^{2\cdot2}+\xi^{2\cdot5}-\xi^{3\cdot2}-\xi^{3\cdot5})(1+\delta(\xi^2+\xi^5))^n \\ &\quad + (\xi^{2\cdot3}+\xi^{2\cdot4}-\xi^{3\cdot3}-\xi^{3\cdot4})(1+\delta(\xi^3+\xi^4))^n \\ &= (\xi^2+\xi^5-\xi^3-\xi^4)(1+\delta(\xi+\xi^6))^n + (\xi^3+\xi^4-\xi-\xi^6) \\ &\quad \times (1+\delta(\xi^2+\xi^5))^n + (\xi+\xi^6-\xi^2-\xi^5)(1+\delta(\xi^3+\xi^4))^n \\ &= 2\Big(\cos\frac{4}{7}\pi + \cos\frac{\pi}{7}\Big)\Big(1+2\delta\cos\frac{2\pi}{7}\Big)^n - 2\Big(\cos\frac{\pi}{7}+\cos\frac{2}{7}\pi\Big) \\ &\quad \times \Big(1+2\delta\cos\frac{4\pi}{7}\Big)^n + 2\Big(\cos\frac{2\pi}{7}-\cos\frac{4\pi}{7}\Big)\Big(1+2\delta\cos\frac{6\pi}{7}\Big)^n. \end{aligned} \quad (3.17)$$

Hence, by (3.9) we get

$$
\begin{aligned}
7B_n(\delta) &= \frac{1}{\delta}(7C_{n+1}(\delta) - 7(1-\delta)C_n(\delta)) \qquad (3.18)\\
&= (\xi^{1\cdot 1} + \xi^{1\cdot 6} - \xi^{3\cdot 1} - \xi^{3\cdot 6})(1+\delta(\xi+\xi^6))^n\\
&\quad + (\xi^{1\cdot 2} + \xi^{1\cdot 5} - \xi^{3\cdot 2} - \xi^{3\cdot 5})(1+\delta(\xi^2+\xi^5))^n\\
&\quad + (\xi^{1\cdot 3} + \xi^{1\cdot 4} - \xi^{3\cdot 3} - \xi^{3\cdot 4})(1+\delta(\xi^3+\xi^4))^n\\
&= (\xi+\xi^6-\xi^3-\xi^4)(1+\delta(\xi+\xi^6))^n + (\xi^2+\xi^5-\xi-\xi^6)\\
&\quad \times (1+\delta(\xi^2+\xi^5))^n + (\xi^3+\xi^4-\xi^2-\xi^5)(1+\delta(\xi^3+\xi^4))^n\\
&= 2\left(\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}\right)\left(1+2\delta\cos\frac{2\pi}{7}\right)^n + 2\left(\cos\frac{4\pi}{7}-\cos\frac{2\pi}{7}\right)\\
&\quad \times \left(1+2\delta\cos\frac{4\pi}{7}\right)^n - 2\left(\cos\frac{\pi}{7}+\cos\frac{4\pi}{7}\right)\left(1+2\delta\cos\frac{6\pi}{7}\right)^n
\end{aligned}
$$

and by (3.10)

$$
\begin{aligned}
7A_n(\delta) = \frac{1}{\delta}(7B_{n+1}(\delta) - 7B_n(\delta)) &= (\xi^{0\cdot 1} + \xi^{0\cdot 6} - \xi^{3\cdot 1} - \xi^{3\cdot 6})(1+\delta(\xi+\xi^6))^n +\\
&+ (\xi^{0\cdot 2} + \xi^{0\cdot 5} - \xi^{3\cdot 2} - \xi^{3\cdot 5})(1+\delta(\xi^2+\xi^5))^n +\\
&+ (\xi^{0\cdot 3} + \xi^{0\cdot 4} - \xi^{3\cdot 3} - \xi^{3\cdot 4})(1+\delta(\xi^3+\xi^4))^n =\\
&= (2-\xi^3-\xi^4)(1+\delta(\xi+\xi^6))^n + (2-\xi-\xi^6)(1+\delta(\xi^2+\xi^5))^n +\\
&+ (2-\xi^2-\xi^5)(1+\delta(\xi^3+\xi^4))^n =\\
&= 2\Big(1+\cos\frac{\pi}{7}\Big)\Big(1+2\delta\cos\frac{2\pi}{7}\Big)^n + 2\Big(1-\cos\frac{2\pi}{7}\Big)\Big(1+2\delta\cos\frac{4\pi}{7}\Big)^n +\\
&+ 2\Big(1-\cos\frac{4\pi}{7}\Big)\Big(1+2\delta\cos\frac{6\pi}{7}\Big)^n. \qquad (3.19)
\end{aligned}
$$

Remark 3.11. *In the sequel, for $\delta = 1$, we obtain from (3.15)*

$$C_{n+3} - 2C_{n+2} - C_{n+1} + C_n = 0 \tag{3.20}$$

for every $n \in \mathbb{N}$. Of course, the elements of the sequences $\{B_n\}$ and $\{A_n\}$ satisfy the identity (3.20). The characteristic polynomial of (3.20) has the following decomposition: (see Corollary 2.4)

$$\mathbb{X}^3 - 2\mathbb{X}^2 - \mathbb{X} + 1 = (\mathbb{X} - 1 - \xi - \xi^6)(\mathbb{X} - 1 - \xi^2 - \xi^5)(\mathbb{X} - 1 - \xi^3 - \xi^4),$$

where $\xi = \exp(i2\pi/7)$. Moreover, the decompositions of C_n (there are cyclic transformations of the sums $\xi + \xi^6$, $\xi^2 + \xi^5$ and $\xi^3 + \xi^4$ in elements of the first equality) follows immediately from (3.17)

$$
\begin{aligned}
7C_n &= (\xi^2+\xi^5-\xi^3-\xi^4)(1+\xi+\xi^6)^n +\\
&\quad + (\xi^3+\xi^4-\xi-\xi^6)(1+\xi^2+\xi^5)^n + (\xi+\xi^6-\xi^2-\xi^5)(1+\xi^3+\xi^4)^n =\\
&= (\xi+\xi^6-\xi^3-\xi^4)(1+\xi+\xi^6)^{n-1} + (\xi^2+\xi^5-\xi-\xi^6)(1+\xi^2+\xi^5)^{n-1} +\\
&\quad + (\xi^3+\xi^4-\xi^2-\xi^5)(1+\xi^3+\xi^4)^{n-1} \qquad (3.21)
\end{aligned}
$$

for every $n = 1, 2, 3, \ldots$

Lemma 3.12. *The following summation formulas for the elements of the sequences* $\{C_n(\delta)\}$, $\{B_n(\delta)\}$ *and* $\{A_n(\delta)\}$ *hold:*

$$\delta \sum_{n=1}^{N} A_n(\delta) = B_{N+1}(\delta) - \delta; \tag{3.22}$$

$$\delta \sum_{n=1}^{N} B_n(\delta) = C_{N+1}(\delta) + \delta \sum_{n=1}^{N} C_n(\delta); \tag{3.23}$$

hence, from (3.12) we obtain

$$A_{N+1}(\delta) - A_1(\delta) = 2\delta \sum_{n=1}^{N} B_n(\delta) - \delta \sum_{n=1}^{N} C_n(\delta) = 2C_{N+1}(\delta) + \delta \sum_{n=1}^{N} C_n(\delta), \tag{3.24}$$

i.e.,

$$\delta \sum_{n=1}^{N} C_n(\delta) = -1 + A_{N+1}(\delta) - 2C_{N+1}(\delta); \tag{3.25}$$

$$\begin{aligned} 7\delta \sum_{n=1}^{N} C_n(\delta) = &-7 - 7C_{N+1}(\delta) - 7B_{N+1}(\delta) \\ &+ \Big(3 - \frac{2}{\delta}\Big) A_{N+1}(\delta) + \frac{2}{\delta} A_{N+2}(\delta); \end{aligned} \tag{3.26}$$

$$\begin{aligned} \delta^3 \sum_{n=1}^{N} C_n(\delta) &= C_{N+3}(\delta) + (\delta - 2)C_{N+2}(\delta) + (1 - \delta - 2\delta^2)C_{N+1}(\delta) - \delta^2 = \\ &= C_{N+2}(\delta) + (\delta - 2)C_{N+1}(\delta) + (\delta^3 - 2\delta^2 - \delta + 1)C_N(\delta) - \delta^2; \end{aligned} \tag{3.27}$$

$$\delta \sum_{n=1}^{N} B_n(\delta) = -1 + A_{N+1}(\delta) - C_{N+1}(\delta) \tag{3.28}$$

and

$$\delta^3 \sum_{n=1}^{N} B_n(\delta) = C_{N+3}(\delta) + (\delta - 2)C_{N+2}(\delta) + (1 - \delta - \delta^2)C_{N+1}(\delta) - \delta^2. \tag{3.29}$$

Definition 3.13. *We define* $\mathcal{A}_n(\delta) := 3A_n(\delta) - B_n(\delta) - C_n(\delta)$, $\delta \in \mathbb{C}$, $n \in \mathbb{N}$. *Moreover, we set* $\mathcal{A}_n := \mathcal{A}_n(1)$ *(see Table 7 and A033304 in [7]).*

Lemma 3.14. *From identities (3.17–3.19) the following special identities can be deduced:*
a)

$$\begin{aligned} \mathcal{A}_n^2 &= \mathcal{A}_{2n} + 2(\xi + \xi^6)^n + 2(\xi^2 + \xi^5)^n + 2(\xi^3 + \xi^4)^n \\ &= \mathcal{A}_{2n} + 2\Big(2\cos\frac{2\pi}{7}\Big)^n + 2\Big(2\cos\frac{4\pi}{7}\Big)^n + 2\Big(2\cos\frac{6\pi}{7}\Big)^n \end{aligned}$$

or

$$\Big(2\cos\frac{2\pi}{7}\Big)^n+\Big(2\cos\frac{4\pi}{7}\Big)^n+\Big(2\cos\frac{6\pi}{7}\Big)^n=\frac{1}{2}(\mathcal{A}_n^2-\mathcal{A}_{2n}),$$

and

$$\Big(4\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\Big)^n+\Big(4\cos\frac{2\pi}{7}\cos\frac{6\pi}{7}\Big)^n+\Big(4\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}\Big)^n=$$
$$=(-1)^n\big[(1+\xi+\xi^6)^n+(1+\xi^2+\xi^5)^n+(1+\xi^3+\xi^4)^n\big]=(-1)^n\mathcal{A}_n.$$

Consequently, we get the following decomposition:

$$\Big(\mathbb{X}-\Big(2\cos\frac{2\pi}{7}\Big)^n\Big)\Big(\mathbb{X}-\Big(2\cos\frac{4\pi}{7}\Big)^n\Big)\Big(\mathbb{X}-\Big(2\cos\frac{6\pi}{7}\Big)^n\Big)=$$
$$=\mathbb{X}^3-\frac{1}{2}(\mathcal{A}_n^2-\mathcal{A}_{2n})\mathbb{X}^2+(-1)^n\mathcal{A}_n\mathbb{X}-1;$$

b)

$$2^{-n}\mathcal{A}_n\Big(\frac{1}{2}\Big)=\Big(\cos\frac{\pi}{7}\Big)^{2n}+\Big(\cos\frac{2\pi}{7}\Big)^{2n}+\Big(\cos\frac{3\pi}{7}\Big)^{2n};$$

c)

$$\mathcal{A}_n^3=\mathcal{A}_{3n}+6(-1)^n+3\mathcal{A}_n(-1)+3(\xi+\xi^6-\xi^2-\xi^5-1)^n$$
$$+3(\xi^3+\xi^4-\xi-\xi^6-1)^n+3(\xi^2+\xi^5-\xi^3-\xi^4-1)^n;$$

d)

$$7C_{n+1}\mathcal{A}_n-7C_n\mathcal{A}_{n+1}=(-1+\xi^3+\xi^4-4(\xi^2+\xi^5))(\xi^3+\xi^4)^n+$$
$$+(-1+\xi^2+\xi^5-4(\xi+\xi^6))(\xi^2+\xi^5)^n+(-1+\xi+\xi^6-4(\xi^3+\xi^4))(\xi+\xi^6)^n=$$
$$=\Big(-1+2\cos\frac{6\pi}{7}-8\cos\frac{4\pi}{7}\Big)\Big(2\cos\frac{6\pi}{7}\Big)^n+$$
$$+\Big(-1+2\cos\frac{4\pi}{7}-8\cos\frac{2\pi}{7}\Big)\Big(2\cos\frac{4\pi}{7}\Big)^n+$$
$$+\Big(-1+2\cos\frac{2\pi}{7}-8\cos\frac{6\pi}{7}\Big)\Big(2\cos\frac{2\pi}{7}\Big)^n=\begin{cases}7; & for\quad n=2,4;\\ -7; & for\quad n=6;\\ -70; & for\quad n=8;\\ 14; & for\quad n=1;\\ 7\cdot A_{(n+1)/2}; & for\quad n=3,5,7;\end{cases}$$

e)

$$7(B_{n+1}+C_{n+1}-2A_{n+1})(A_n+B_n-C_n)-7(B_n+C_n-2A_n)(A_{n+1}+B_{n+1}-C_{n+1})=$$
$$=(8+3(\xi^2+\xi^5))(\xi+\xi^6)^n+(8+3(\xi+\xi^6))(\xi^3+\xi^4)^n+(8+3(\xi^3+\xi^4))(\xi^2+\xi^5)^n=$$
$$=\Big(8+6\cos\frac{4\pi}{7}\Big)\Big(2\cos\frac{2\pi}{7}\Big)^n+\Big(8+6\cos\frac{2\pi}{7}\Big)\Big(2\cos\frac{6\pi}{7}\Big)^n+$$
$$+\Big(8+6\cos\frac{6\pi}{7}\Big)\Big(2\cos\frac{4\pi}{7}\Big)^n=\begin{cases}-14; & for\quad n=1;\\ 49; & for\quad n=2;\\ -56; & for\quad n=3,\end{cases}$$

and a more general identity
f)

$$
\begin{aligned}
&(\alpha 7A_{n+1} + \beta 7B_{n+1} + \gamma 7C_{n+1})(\varepsilon 7A_n + \omega 7B_n + \varphi 7C_n) - \\
&\quad - (\alpha 7A_n + \beta 7B_n + \gamma 7C_n)(\varepsilon 7A_{n+1} + \omega 7B_{n+1} + \varphi 7C_{n+1}) = \\
&= \big((\alpha_0 + \beta_0(\xi + \xi^6) + \gamma_0(\xi^2 + \xi^5))(1 + \xi + \xi^6)^{n+1} + \\
&\quad + (\alpha_0 + \beta_0(\xi^2 + \xi^5) + \gamma_0(\xi^3 + \xi^4))(1 + \xi^2 + \xi^5)^{n+1} + \\
&\quad + (\alpha_0 + \beta_0(\xi^3 + \xi^4) + \gamma_0(\xi + \xi^6))(1 + \xi^3 + \xi^4)^{n+1}\big) \times \\
&\quad \times \big((\varepsilon_0 + \omega_0(\xi + \xi^6) + \varphi_0(\xi^2 + \xi^5))(1 + \xi + \xi^6)^n +
\end{aligned}
$$

$$
\begin{aligned}
&+ (\varepsilon_0 + \omega_0(\xi^2 + \xi^5) + \varphi_0(\xi^3 + \xi^4))(1 + \xi^2 + \xi^5)^n + \\
&+ (\varepsilon_0 + \omega_0(\xi^3 + \xi^4) + \varphi_0(\xi + \xi^6))(1 + \xi^3 + \xi^4)^n\big) - \\
&- \big((\alpha_0 + \beta_0(\xi + \xi^6) + \gamma_0(\xi^2 + \xi^5))(1 + \xi + \xi^6)^n + \\
&+ (\alpha_0 + \beta_0(\xi^2 + \xi^5) + \gamma_0(\xi^3 + \xi^4))(1 + \xi^2 + \xi^5)^n + \\
&+ (\alpha_0 + \beta_0(\xi^3 + \xi^4) + \gamma_0(\xi + \xi^6))(1 + \xi^3 + \xi^4)^n\big) \times \\
&\times \big((\varepsilon_0 + \omega_0(\xi + \xi^6) + \varphi_0(\xi^2 + \xi^5))(1 + \xi + \xi^6)^{n+1} + \\
&+ (\varepsilon_0 + \omega_0(\xi^2 + \xi^5) + \varphi_0(\xi^3 + \xi^4))(1 + \xi^2 + \xi^5)^{n+1} + \\
&+ (\varepsilon_0 + \omega_0(\xi^3 + \xi^4) + \varphi_0(\xi + \xi^6))(1 + \xi^3 + \xi^4)^{n+1}\big) = \\
&= \big(\mathcal{A} + (\xi + \xi^6)\mathcal{B} + (\xi^2 + \xi^5)\mathcal{C}\big)(\xi + \xi^6)^n + \\
&\quad + \big(\mathcal{A} + (\xi + \xi^6)\mathcal{C} + (\xi^3 + \xi^4)\mathcal{B}\big)(\xi^3 + \xi^4)^n + \\
&\quad + \big(\mathcal{A} + (\xi^2 + \xi^5)\mathcal{B} + (\xi^3 + \xi^4)\mathcal{C}\big)(\xi^3 + \xi^5)^n = \\
&= \Big(\mathcal{A} + 2\cos\frac{2\pi}{7}\mathcal{B} + 2\cos\frac{4\pi}{7}\mathcal{C}\Big)\Big(2\cos\frac{2\pi}{7}\Big)^n + \\
&\quad + \Big(\mathcal{A} - 2\cos\frac{\pi}{7}\mathcal{B} + 2\cos\frac{2\pi}{7}\mathcal{C}\Big)\Big(-2\cos\frac{\pi}{7}\Big)^n + \\
&\quad + \Big(\mathcal{A} + 2\cos\frac{4\pi}{7}\mathcal{B} - 2\cos\frac{\pi}{7}\mathcal{C}\Big)\Big(2\cos\frac{4\pi}{7}\Big)^n,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} &:= 5\varepsilon_0\beta_0 - 5\alpha_0\omega_0 + \alpha_0\varphi_0 - \varepsilon_0\gamma_0 + 2\omega_0\gamma_0 - 2\beta_0\varphi_0, \\
\mathcal{B} &:= 5\alpha_0\omega_0 - \varepsilon_0\beta_0 + 3\varepsilon_0\gamma_0 - 3\alpha_0\varphi_0 + \gamma_0\omega_0 - \beta_0\varphi_0, \\
\mathcal{C} &:= 2\varepsilon_0\beta_0 - 2\alpha_0\omega_0 + 2\beta_0\varphi_0 - 2\gamma_0\omega_0 + \varepsilon_0\gamma_0 - \alpha_0\varphi_0,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_0 &:= 3\alpha + \beta + \gamma, & \beta_0 &:= \alpha + 2\beta + \gamma, & \gamma_0 &:= \alpha + \beta + 2\gamma, \\
\varepsilon_0 &:= 3\varepsilon + \omega + \varphi, & \omega_0 &:= \varepsilon + 2\omega + \varphi, & \varphi_0 &:= \varepsilon + \omega + 2\varphi.
\end{aligned}
$$

Proof. The identity *b)* follows from (3.6). □

Remark 3.15. *The sequence* $\{\frac{1}{2}(\mathcal{A}_n^2 - \mathcal{A}_{2n})\}_{n=1}^{\infty}$ *(see Table 7) is an accelerator sequence for Catalan's constant (see [1]). The generating function of these numbers has the form*

$$\frac{3+2\,x-2\,x^2}{1+x-2\,x^2-x^3}$$

(see also A094648 in [7]).

Some applications of identities (3.1–3.3) will be presented now.

3.1 Reduction formulas for indices

Lemma 3.16. *The following identities hold:*

$$\begin{aligned} A_{m+n}(\delta) &= A_m(\delta)A_n(\delta) + 2B_m(\delta)B_n(\delta) + C_m(\delta)C_n(\delta) \\ &\quad - B_m(\delta)C_n(\delta) - B_n(\delta)C_m(\delta), \end{aligned} \tag{3.30}$$

$$B_{m+n}(\delta) = -C_m(\delta)C_n(\delta) + A_m(\delta)B_n(\delta) + A_n(\delta)B_m(\delta), \tag{3.31}$$

$$\begin{aligned} C_{m+n}(\delta) &= B_m(\delta)B_n(\delta) - C_m(\delta)C_n(\delta) + A_m(\delta)C_n(\delta) \\ &\quad + A_n(\delta)C_m(\delta) - B_m(\delta)C_n(\delta) - B_n(\delta)C_m(\delta). \end{aligned} \tag{3.32}$$

Proof. First, we note that

$$(1+\delta(\xi+\xi^6))^{m+n} = A_{m+n}(\delta) + B_{m+n}(\delta)(\xi+\xi^6) + C_{m+n}(\delta)(\xi^2+\xi^5).$$

On the other hand, we have

$$\begin{aligned} (1+&\delta(\xi+\xi^6))^{m+n} = (1+\delta(\xi+\xi^6))^m(1+\delta(\xi+\xi^6))^n = \\ &= \Big(A_m(\delta) + B_m(\delta)(\xi+\xi^6) + C_m(\delta)(\xi^2+\xi^5)\Big)\times \\ &\quad \times \Big(A_n(\delta) + B_n(\delta)(\xi+\xi^6) + C_n(\delta)(\xi^2+\xi^5)\Big) \end{aligned}$$

$$\begin{aligned} &= A_m(\delta)A_n(\delta) + B_m(\delta)B_n(\delta)(\xi^2+\xi^5+2) + C_m(\delta)C_n(\delta)(\xi^4+\xi^3+2) \\ &\quad + A_m(\delta)B_n(\delta)(\xi+\xi^6) + A_m(\delta)C_n(\delta)(\xi^2+\xi^5) + A_n(\delta)B_m(\delta)(\xi+\xi^6) \\ &\quad + A_n(\delta)C_m(\delta)(\xi^2+\xi^5) + B_m(\delta)C_n(\delta)(\xi+\xi^6)(\xi^2+\xi^5) \\ &\quad + B_n(\delta)C_m(\delta)(\xi+\xi^6)(\xi^2+\xi^5) \\ &\text{(we have } (\xi+\xi^6)(\xi^2+\xi^5) = \xi^3+\xi^6+\xi+\xi^4 = -1-\xi^2-\xi^5) \\ &= A_m(\delta)A_n(\delta) + 2B_m(\delta)B_n(\delta) + C_m(\delta)C_n(\delta) - B_m(\delta)C_n(\delta) - B_n(\delta)C_m(\delta) \\ &\quad + (\xi+\xi^6)(A_m(\delta)B_n(\delta) + A_n(\delta)B_m(\delta) - C_m(\delta)C_n(\delta)) \\ &\quad + (\xi^2+\xi^5)(B_m(\delta)B_n(\delta) - C_m(\delta)C_n(\delta) + A_m(\delta)C_n(\delta) \\ &\quad + A_n(\delta)C_m(\delta) - B_n(\delta)C_m(\delta) - B_m(\delta)C_n(\delta)) \end{aligned}$$

hence by linear independence of 1, $\xi+\xi^6 = 2\cos\frac{2\pi}{7}$, $\xi^2+\xi^5 = 2\cos\frac{4\pi}{7}$ over $\mathbb{Q}$ and by Lemma 2.8 the reduction formulas follow. □

Corollary 3.17. *We have*

$$A_{2n}(\delta) = A_n^2(\delta) + B_n^2(\delta) + (B_n(\delta) - C_n(\delta))^2, \tag{3.33}$$
$$B_{2n}(\delta) = 2A_n(\delta)B_n(\delta) - C_n^2(\delta), \tag{3.34}$$
$$C_{2n}(\delta) = B_n^2(\delta) - C_n^2(\delta) + 2C_n(\delta)(A_n(\delta) - B_n(\delta)). \tag{3.35}$$

Remark 3.18. *(Which comes from [7], sequence A006356) Let $u(k)$, $v(k)$, $w(k)$ be defined by $u(1) = 1$, $v(1) = 0$, $w(1) = 0$ and*

$$\begin{aligned} u(k+1) &= u(k) + v(k) + w(k), \\ v(k+1) &= u(k) + v(k), \\ w(k+1) &= u(k). \end{aligned}$$

Then we have

$$u(n+1) = A_n, \quad v(n+1) = B_n, \quad w(n+1) = B_n - C_n;$$

for every $n \in \mathbb{N} \cup \{0\}$. So, according to Table 2 we get

$$\{u(n)\} = 1, 1, 3, 6, 14, 31, \dots,$$
$$\{v(n)\} = 0, 1, 2, 5, 11, 25, \dots,$$

and

$$\{w(n)\} = \{u(n)\}$$

(Benoit Cloitre (abcloitre@wanadoo.fr), Apr 05 2002). Moreover

$$\big(u(k)\big)^2 + \big(v(k)\big)^2 + \big(w(k)\big)^2 = u(2k) \quad \text{(Gary Adamson, Dec 23 2003).}$$

The n-th term of the sequence $\{u(n)\}$ is the number of paths for a ray of light that enters two layers of glass, and then is reflected exactly n times before leaving the layers of glass. One such path (with 2 plates of glass and 3 reflections) might be

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...\........./...
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Corollary 3.19. *We have*

$$\begin{aligned} A_{3n}(\delta) &= A_n(\delta)A_{2n}(\delta) - B_n(\delta)C_{2n}(\delta) - B_{2n}(\delta)C_n(\delta) \\ &\quad + 2B_n(\delta)B_{2n}(\delta) + C_n(\delta)C_{2n}(\delta) \\ &= A_n^3(\delta) - B_n^3(\delta) + 6A_n(\delta)B_n^2(\delta) + 3A_n(\delta)C_n^2(\delta) - 3B_n(\delta)C_n^2(\delta) \\ &\quad + 3B_n^2(\delta)C_n(\delta) - 6A_n(\delta)B_n(\delta)C_n(\delta), \end{aligned} \tag{3.36}$$

$$\begin{aligned} B_{3n}(\delta) &= A_{2n}(\delta)B_n(\delta) + A_n(\delta)B_{2n}(\delta) - C_n(\delta)C_{2n}(\delta) \\ &= 2B_n^3(\delta) + C_n^3(\delta) + 3A_n^2(\delta)B_n(\delta) + 3A_n(\delta)C_n^2(\delta) \\ &\quad + 3B_n(\delta)C_n^2(\delta) - 3B_n^2(\delta)C_n(\delta), \end{aligned} \tag{3.37}$$

$$\begin{aligned} C_{3n}(\delta) &= B_n v B_{2n}(\delta) + A_n(\delta)C_{2n}(\delta) + A_{2n}(\delta)C_n(\delta) - C_n(\delta)C_{2n}(\delta) \\ &\quad - B_n(\delta)C_{2n}(\delta) - B_{2n}(\delta)C_n(\delta) \\ &= 3C_n^3(\delta) - B_n^3(\delta) + 3A_n^2(\delta)C_n(\delta) + 3A_n(\delta)B_n^2(\delta) + 3B_n^2(\delta)C_n(\delta) \\ &\quad - 6A_n(\delta)B_n(\delta)C_n(\delta). \end{aligned} \tag{3.38}$$

Lemma 3.20. *The following identities are satisfied:*

$$A_{n-k}(\delta) = \frac{1}{\Delta}\begin{vmatrix} A_n(\delta) & 2B_k(\delta) - C_k(\delta) & C_k(\delta) - B_k(\delta) \\ B_n(\delta) & A_k(\delta) & -C_k(\delta) \\ C_n(\delta) & B_k(\delta) - C_k(\delta) & A_k(\delta) - B_k(\delta) - C_k(\delta) \end{vmatrix}, \tag{3.39}$$

$$B_{n-k}(\delta) = \frac{1}{\Delta}\begin{vmatrix} A_k(\delta) & A_n(\delta) & C_k(\delta) - B_k(\delta) \\ B_k(\delta) & B_n(\delta) & -C_k(\delta) \\ C_k(\delta) & C_n(\delta) & A_k(\delta) - B_k(\delta) - C_k(\delta) \end{vmatrix}, \tag{3.40}$$

and

$$C_{n-k}(\delta) = \frac{1}{\Delta}\begin{vmatrix} A_k(\delta) & 2B_k(\delta) - C_k(\delta) & A_n(\delta) \\ B_k(\delta) & A_k(\delta) & B_n(\delta) \\ C_k(\delta) & B_k(\delta) - C_k(\delta) & C_n(\delta) \end{vmatrix}, \tag{3.41}$$

where

$$\Delta := \begin{vmatrix} A_k(\delta) & 2B_k(\delta) - C_k(\delta) & C_k(\delta) - B_k(\delta) \\ B_k(\delta) & A_k(\delta) & -C_k(\delta) \\ C_k(\delta) & B_k(\delta) - C_k(\delta) & A_k(\delta) - B_k(\delta) - C_k(\delta) \end{vmatrix}.$$

Proof. First we note that

$$\big(1 + \delta(\xi + \xi^6)\big)^{n-k} = A_{n-k}(\delta) + B_{n-k}(\delta)(\xi + \xi^6) + C_{n-k}(\delta)(\xi^2 + \xi^5).$$

On the other hand we obtain

$$\big(1 + \delta(\xi + \xi^6)\big)^{n-k} = \frac{\big(1 + \delta(\xi + \xi^6)\big)^n}{\big(1 + \delta(\xi + \xi^6)\big)^k} = \frac{A_n(\delta) + B_n(\delta)(\xi + \xi^6) + C_n(\delta)(\xi^2 + \xi^5)}{A_k(\delta) + B_k(\delta)(\xi + \xi^6) + C_k(\delta)(\xi^2 + \xi^5)}.$$

The final form of formulas (3.39), (3.40) and (3.41) from the following identity could be derived:

$$\frac{a + b(\xi + \xi^6) + c(\xi^2 + \xi^5)}{d + e(\xi + \xi^6) + f(\xi^2 + \xi^5)} = \alpha + \beta(\xi + \xi^6) + \gamma(\xi^2 + \xi^5),$$

where

$$\alpha := \frac{1}{D}\begin{vmatrix} a & 2e-f & f-e \\ b & d & -f \\ c & e-f & d-e-f \end{vmatrix}, \qquad \beta := \frac{1}{D}\begin{vmatrix} d & a & f-e \\ e & b & -f \\ f & c & d-e-f \end{vmatrix},$$

$$\gamma := \frac{1}{D}\begin{vmatrix} d & 2e-f & a \\ e & d & b \\ f & e-f & c \end{vmatrix} \qquad \text{and} \qquad D := \begin{vmatrix} d & 2e-f & f-e \\ e & d & -f \\ f & e-f & d-e-f \end{vmatrix}.$$

□

Lemma 3.21. *The following identity for the determinant* Δ *from Lemma 3.20 holds:*

$$\Delta = \begin{vmatrix} A_k(\delta) & 2B_k(\delta)-C_k(\delta) & C_k(\delta)-B_k(\delta) \\ B_k(\delta) & A_k(\delta) & -C_k(\delta) \\ C_k(\delta) & B_k(\delta)-C_k(\delta) & A_k(\delta)-B_k(\delta)-C_k(\delta) \end{vmatrix} =$$
$$= \big(A_k(\delta)\big)^3 + \big(B_k(\delta)\big)^3 + \big(C_k(\delta)\big)^3 + 3\,A_k(\delta)\,B_k(\delta)\,C_k(\delta) - \big(A_k(\delta)\big)^2\big(B_k(\delta)+C_k(\delta)\big) -$$
$$- 2\,A_k(\delta)\Big(\big(B_k(\delta)\big)^2 + \big(C_k(\delta)\big)^2\Big) + 3\,\big(B_k(\delta)\big)^2\,C_k(\delta) - 4\,B_k(\delta)\,\big(C_k(\delta)\big)^2 =$$
$$= \big(\delta^3 - 2\,\delta^2 - \delta + 1\big)^k. \quad (3.42)$$

Proof. By (3.16) we obtain

$$\Big[\big(1+\delta(\xi+\xi^6)\big)\big(1+\delta(\xi^2+\xi^5)\big)\big(1+\delta(\xi^3+\xi^4)\big)\Big]^k = \big(\delta^3 - 2\,\delta^2 - \delta + 1\big)^k. \quad (3.43)$$

On the other hand, by (3.1)–(3.3) we get

$$\Big(\big(1+\delta(\xi+\xi^6)\big)\big(1+\delta(\xi^2+\xi^5)\big)\big(1+\delta(\xi^3+\xi^4)\big)\Big)^k =$$
$$= \big(1+\delta(\xi+\xi^6)\big)^k\big(1+\delta(\xi^2+\xi^5)\big)^k\big(1+\delta(\xi^3+\xi^4)\big)^k =$$
$$= \big(A_k(\delta) + B_k(\delta)(\xi+\xi^6) + C_k(\delta)(\xi^2+\xi^5)\big)\big(A_k(\delta) + B_k(\delta)(\xi^2+\xi^5) + C_k(\delta)(\xi^3+\xi^4)\big)\times$$
$$\times\big(A_k(\delta) + B_k(\delta)(\xi^3+\xi^4) + C_k(\delta)(\xi+\xi^6)\big) =$$
$$= \big(A_k(\delta)\big)^3 + a\,\big(B_k(\delta)\big)^3 + a\,\big(C_k(\delta)\big)^3 + b\;A_k(\delta)\,B_k(\delta)\,C_k(\delta) +$$
$$+ c\,\big(A_k(\delta)\big)^2\big(B_k(\delta)+C_k(\delta)\big) + d\;A_k(\delta)\Big(\big(B_k(\delta)\big)^2 + \big(C_k(\delta)\big)^2\Big) +$$
$$+ e\,\big(B_k(\delta)\big)^2\,C_k(\delta) + f\,B_k(\delta)\,\big(C_k(\delta)\big)^2, \quad (3.44)$$

where (see Lemma 2.3)

$$\begin{aligned}
a &= (\xi+\xi^6)(\xi^2+\xi^5)(\xi^3+\xi^4) = 1,\\
b &= (\xi+\xi^6)(\xi^2+\xi^5) + (\xi+\xi^6)(\xi^3+\xi^4) + (\xi^2+\xi^5)(\xi^3+\xi^4) +\\
&\quad + (\xi+\xi^6)^2 + (\xi^2+\xi^5)^2 + (\xi^3+\xi^4)^2 = 3,\\
c &= \xi+\xi^2+\xi^3+\xi^4+\xi^5+\xi^6 = -1,\\
d &= (\xi+\xi^6)(\xi^2+\xi^5) + (\xi+\xi^6)(\xi^3+\xi^4) + (\xi^2+\xi^5)(\xi^3+\xi^4) = -2,\\
e &= (\xi+\xi^6)^2(\xi^2+\xi^5) + (\xi+\xi^6)(\xi^3+\xi^4)^2 + (\xi^2+\xi^5)^2(\xi^3+\xi^4) = 3,\\
f &= (\xi+\xi^6)(\xi^2+\xi^5)^2 + (\xi+\xi^6)^2(\xi^3+\xi^4) + (\xi^2+\xi^5)(\xi^3+\xi^4)^2 = -4.
\end{aligned}$$

Hence (3.42) follows from (3.43) and (3.44). □

3.2 Reduction formulas for δ-arguments

Next, our aim will be to prove some more general identities connecting the quasi-Fibonacci numbers with different δ's ($\delta \neq 2$). Let us start with the following formula:

$$\begin{aligned}\big(2+\delta(\xi^2+\xi^5)+\delta(\xi^3+\xi^4)\big)^n &= \big((2-\delta)-\delta(\xi+\xi^6)\big)^n = \\ &= (2-\delta)^n\left(1+\frac{\delta}{\delta-2}(\xi+\xi^6)\right)^n = (2-\delta)^n A_n\left(\frac{\delta}{\delta-2}\right)+ \\ &+ (2-\delta)^n B_n\left(\frac{\delta}{\delta-2}\right)(\xi+\xi^6)+(2-\delta)^n C_n\left(\frac{\delta}{\delta-2}\right)(\xi^2+\xi^5). \quad (3.45)\end{aligned}$$

But, we can deduce another equivalent formula

$$\begin{aligned}&\Big(2+\delta(\xi^2+\xi^5)+\delta(\xi^3+\xi^4)\Big)^n = \Big(\big(1+\delta(\xi^2+\xi^5)\big)+\big(1+\delta(\xi^3+\xi^4)\big)\Big)^n \quad (3.46)\\ &=\sum_{k=0}^n\binom{n}{k}(1+\delta(\xi^2+\xi^5))^k(1+\delta(\xi^3+\xi^4))^{n-k}\\ &=\sum_{k=0}^n\binom{n}{k}\Big(A_k(\delta)+B_k(\delta)(\xi^2+\xi^5)+C_k(\delta)(\xi^3+\xi^4)\Big)\times\\ &\quad\times\Big(A_{n-k}(\delta)+B_{n-k}(\delta)(\xi^3+\xi^4)+C_{n-k}(\delta)(\xi+\xi^6)\Big)\\ &=\sum_{k=0}^n\binom{n}{k}\Big(A_k(\delta)A_{n-k}(\delta)+A_k(\delta)B_{n-k}(\delta)(\xi^3+\xi^4)+A_k(\delta)C_{n-k}(\delta)(\xi+\xi^6)\\ &\quad+A_{n-k}(\delta)B_k(\delta)(\xi^2+\xi^5)+B_k(\delta)B_{n-k}(\delta)(\xi^5+\xi^6+\xi+\xi^2)\\ &\quad+B_k(\delta)C_{n-k}(\delta)(\xi^3+\xi+\xi^6+\xi^4)+A_{n-k}(\delta)C_k(\delta)(\xi^3+\xi^4)\\ &\quad+B_{n-k}(\delta)C_k(\delta)(\xi+\xi^6+2)+C_k(\delta)C_{n-k}(\delta)(\xi^4+\xi^5+\xi^2+\xi^3)\Big)=\end{aligned}$$

$$\begin{aligned}&=\sum_{k=0}^n\binom{n}{k}(A_k(\delta)A_{n-k}(\delta)-A_k(\delta)B_{n-k}(\delta)-B_k(\delta)C_{n-k}(\delta)-A_{n-k}(\delta)C_k(\delta)\\ &\quad+2B_{n-k}(\delta)C_k(\delta)-C_k(\delta)C_{n-k}(\delta))\\ &\quad+(\xi+\xi^6)\sum_{k=0}^n\binom{n}{k}(-A_k(\delta)B_{n-k}(\delta)+A_k(\delta)C_{n-k}(\delta)+B_k(\delta)B_{n-k}(\delta)\\ &\quad-A_{n-k}(\delta)C_k(\delta)+B_{n-k}(\delta)C_k(\delta)-C_k(\delta)C_{n-k}(\delta))\\ &\quad+(\xi^2+\xi^5)\sum_{k=0}^n\binom{n}{k}(-A_k(\delta)B_{n-k}(\delta)+A_{n-k}(\delta)B_k(\delta)\\ &\quad+B_k(\delta)B_{n-k}(\delta)-B_k(\delta)C_{n-k}(\delta)-A_{n-k}(\delta)C_k(\delta)).\end{aligned}$$

From (3.45) and (3.46), again by linear independence of 1, $\xi+\xi^6=2\cos\frac{2}{7}\pi$ and $\xi^2+\xi^5=2\cos\frac{4}{7}\pi$ over $\mathbb{Q}$ and by Lemma 2.8 the identities below can be generated:

Lemma 3.22. *We have*

$$(2-\delta)^n A_n\Big(\frac{\delta}{\delta-2}\Big) = \sum_{k=0}^{n}\binom{n}{k}\Big(A_k(\delta)A_{n-k}(\delta) - A_k(\delta)B_{n-k}(\delta) \tag{3.47}$$
$$- B_k(\delta)C_{n-k}(\delta) - A_{n-k}(\delta)C_k(\delta) + 2B_{n-k}(\delta)C_k(\delta) - C_k(\delta)C_{n-k}(\delta)\Big),$$

$$(2-\delta)^n B_n\Big(\frac{\delta}{\delta-2}\Big) = \sum_{k=0}^{n}\binom{n}{k}\Big(- A_k(\delta)B_{n-k}(\delta) + A_k(\delta)C_{n-k}(\delta) \tag{3.48}$$
$$+ B_k(\delta)B_{n-k}(\delta) - A_{n-k}(\delta)C_k(\delta) + B_{n-k}(\delta)C_k(\delta) - C_k(\delta)C_{n-k}(\delta)\Big)$$

and

$$(2-\delta)^n C_n\left(\frac{\delta}{\delta-2}\right) = \sum_{k=0}^{n}\binom{n}{k}\Big(- A_k(\delta)B_{n-k}(\delta) \tag{3.49}$$
$$+ A_{n-k}(\delta)B_k(\delta) + B_k(\delta)B_{n-k}(\delta) - B_k(\delta)C_{n-k}(\delta) - A_{n-k}(\delta)C_k(\delta)\Big).$$

In the sequel, from the above formulas the following special identities can be obtained: the case $\delta = 1$

$$A_n(-1) = \sum_{k=0}^{n}\binom{n}{k}\big(A_kA_{n-k} - A_kB_{n-k} - B_kC_{n-k} - A_{n-k}C_k \tag{3.50}$$
$$+ 2B_{n-k}C_k - C_kC_{n-k}\big),$$
$$B_n(-1) = \sum_{k=0}^{n}\binom{n}{k}\big(- A_kB_{n-k} + A_kC_{n-k} + B_kB_{n-k} - A_{n-k}C_k \tag{3.51}$$
$$+ B_{n-k}C_k - C_kC_{n-k}\big),$$

$$C_n(-1) = \sum_{k=0}^{n}\binom{n}{k}\big(- A_kB_{n-k} + A_{n-k}B_k + B_kB_{n-k} \tag{3.52}$$
$$- B_kC_{n-k} - A_{n-k}C_k\big);$$

the case $\delta = 3$

$$(-1)^n A_n(3) = \sum_{k=0}^{n}\binom{n}{k}\big(A_k(3)A_{n-k}(3) - A_k(3)B_{n-k}(3) - \tag{3.53}$$
$$- B_k(3)C_{n-k}(3) - A_{n-k}(3)C_k(3) + 2B_{n-k}(3)C_k(3) - C_k(3)C_{n-k}(3)\big),$$
$$(-1)^n B_n(3) = \sum_{k=0}^{n}\binom{n}{k}\big(- A_k(3)B_{n-k}(3) + A_k(3)C_{n-k}(3) + \tag{3.54}$$
$$+ B_k(3)B_{n-k}(3) - A_{n-k}(3)C_k(3) + B_{n-k}(3)C_k(3) - C_k(3)C_{n-k}(3)\big),$$
$$(-1)^n C_n(3) = \sum_{k=0}^{n}\binom{n}{k}\big(- A_k(3)B_{n-k}(3) + A_{n-k}(3)B_k(3) + \tag{3.55}$$
$$+ B_k(3)B_{n-k}(3) - B_k(3)C_{n-k}(3) - A_{n-k}(3)C_k(3)\big).$$

The case $\delta = 2$ must be treated in another way

$$\begin{aligned}
&\big(2 + 2(\xi^2+\xi^5) + 2(\xi^3+\xi^4)\big)^{2n} = \big(-2(\xi+\xi^6)\big)^{2n} = \\
&\quad = 2^{2n}\big(2 + (\xi^2+\xi^5)\big)^n = 2^{3n}\big(1 + \tfrac{1}{2}(\xi^2+\xi^5)\big)^n = \\
&\quad = 2^{3n}\Big(A_n\big(\tfrac{1}{2}\big) + B_n\big(\tfrac{1}{2}\big)(\xi^2+\xi^5) + C_n\big(\tfrac{1}{2}\big)(\xi^3+\xi^4)\Big) = \\
&\quad = 2^{3n}\Big(A_n\big(\tfrac{1}{2}\big) + B_n\big(\tfrac{1}{2}\big)(\xi^2+\xi^5) + C_n\big(\tfrac{1}{2}\big)(-1-\xi-\xi^6-\xi^2-\xi^5)\Big) = \\
&\quad = 2^{3n}\Big(A_n\big(\tfrac{1}{2}\big) - C_n\big(\tfrac{1}{2}\big) + \big(B_n\big(\tfrac{1}{2}\big) - C_n\big(\tfrac{1}{2}\big)\big)(\xi^2+\xi^5) - C_n\big(\tfrac{1}{2}\big)(\xi+\xi^6)\Big),
\end{aligned}$$

which, by (3.46), implies the next five identities

$$\begin{aligned}
2^{3n}\big(A_n\big(\tfrac{1}{2}\big) - C_n\big(\tfrac{1}{2}\big)\big) = \sum_{k=0}^{2n}\binom{2n}{k}\big(&A_k(2)A_{2n-k}(2) - A_k(2)B_{2n-k}(2) - \\
&- B_k(2)C_{2n-k}(2) - A_{2n-k}(2)C_k(2) + 2B_{2n-k}(2)C_k(2) - C_k(2)C_{2n-k}(2)\big),
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
2^{3n}\big(B_n\big(\tfrac{1}{2}\big) - C_n\big(\tfrac{1}{2}\big)\big) = \sum_{k=0}^{2n}\binom{2n}{k}\big(&-A_k(2)B_{2n-k}(2) - A_{2n-k}(2)B_k(2) + \\
&+ B_k(2)B_{2n-k}(2) - B_k(2)C_{2n-k}(2) - A_{2n-k}(2)C_k(2)\big),
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
-2^{3n}C_n\big(\tfrac{1}{2}\big) = \sum_{k=0}^{2n}\binom{2n}{k}\big(&-A_k(2)B_{2n-k}(2) + A_k(2)C_{2n-k}(2) + \\
&+ B_k(2)B_{2n-k}(2) - A_{2n-k}(2)C_k(2) - B_{2n-k}(2)C_k(2) - C_k(2)C_{2n-k}(k)\big),
\end{aligned} \tag{3.58}$$

hence, after some manipulations, we obtain

$$\begin{aligned}
2^{3n}A_n\big(\tfrac{1}{2}\big) = \sum_{k=0}^{2n}\binom{2n}{k}\big(&A_k(2)A_{2n-k}(2) - A_k(2)C_{2n-k}(2) - \\
&- B_k(2)B_{2n-k}(2) - B_k(2)C_{2n-k}(2) + B_{2n-k}(2)C_k(2)\big),
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
2^{3n}B_n\big(\tfrac{1}{2}\big) = \sum_{k=0}^{2n}\binom{2n}{k}\big(&-A_k(2)C_{2n-k}(2) + A_{2n-k}(2)B_k(2) - \\
&- B_k(2)C_{2n-k}(2) - B_{2n-k}(2)C_k(2) + C_k(2)C_{2n-k}(2)\big).
\end{aligned} \tag{3.60}$$

We have also the identity

$$\big(1+\xi+\xi^6\big)\big(1+\delta\big(\xi^2+\xi^5\big)\big) = \big(1+\xi+\xi^6\big) - \delta,$$

i.e.,

$$\big(1+\xi+\xi^6\big)^n\big(1+\delta\big(\xi^2+\xi^5\big)\big)^n = \Big(\big(1+\xi+\xi^6\big) - \delta\Big)^n,$$

which, by (3.1) and (3.2), yields

$$\Big(A_n + B_n\big(\xi+\xi^6\big) + C_n\big(\xi^2+\xi^5\big)\Big)\Big(A_n(\delta) + B_n(\delta)\big(\xi^2+\xi^5\big) + C_n(\delta)\big(\xi^3+\xi^4\big)\Big) =$$
$$= \sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}\big(1+\xi+\xi^6\big)^k = \sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}A_k +$$
$$+ \big(\xi+\xi^6\big)\sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}B_k + \big(\xi^2+\xi^5\big)\sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}C_k.$$

Hence, by Lemma 2.8, the following three identities can be discovered:

$$A_nA_n(\delta) - A_nC_n(\delta) - B_nB_n(\delta) - B_nC_n(\delta) + C_nB_n(\delta) = \tag{3.61}$$
$$= \sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}A_k,$$
$$-A_nC_n(\delta) + B_nA_n(\delta) - B_nC_n(\delta) - C_nB_n(\delta) + C_nC_n(\delta) = \tag{3.62}$$
$$= \sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}B_k,$$
$$A_nB_n(\delta) - A_nC_n(\delta) - B_nB_n(\delta) + C_nA_n(\delta) - C_nB_n(\delta) + C_nC_n(\delta) = \tag{3.63}$$
$$= \sum_{k=0}^{n}(-\delta)^{n-k}\binom{n}{k}C_k.$$

3.3 Polynomials associated with quasi-Fibonacci numbers of order 7

Directly from equation (3.1) we obtain (for $x_k := 1 + 2\delta\cos\frac{2k\pi}{7}$, $k = 1,2,3$)

$$\begin{aligned} x_k^n &= A_n(\delta) + B_n(\delta)\left(2\cos\frac{2k\pi}{7}\right) + C_n(\delta)\left(2\cos\frac{4k\pi}{7}\right) \\ &= A_n(\delta) + \frac{B_n(\delta)}{\delta}\,x_k - \frac{B_n(\delta)}{\delta} + \frac{C_n(\delta)}{\delta^2}\left(2\delta^2\left(2\cos^2\frac{2k\pi}{7} - 1\right)\right) \\ &= A_n(\delta) + \frac{B_n(\delta)}{\delta}\,x_k - \frac{B_n(\delta)}{\delta} + \frac{C_n(\delta)}{\delta^2}\left(x_k^2 - 2\,x_k + 1 - 2\delta^2\right) \\ &= A_n(\delta) - \frac{B_n(\delta)}{\delta} - \frac{C_n(\delta)}{\delta^2}(2\delta^2-1) + \frac{B_n(\delta)}{\delta}x_k - 2\frac{C_n(\delta)}{\delta^2}x_k + \frac{C_n(\delta)}{\delta^2}x_k^2, \end{aligned}$$

i.e.,

$$W_{n,7}(x;\delta) := \delta^2x^n - C_n(\delta)x^2 + (2C_n(\delta) - \delta B_n(\delta))x +$$
$$+ \delta B_n(\delta) + (2\delta^2-1)C_n(\delta) - \delta^2A_n(\delta) = 0$$

for $x = x_k$, $k = 1,2,3$. So, by identity (3.16) the polynomial $p_7(\mathbb{X};\delta)$ is a divisor of the polynomial $W_{n,7}(\mathbb{X};\delta)$ (for every $n \geqslant 3$). In the sequel we obtain (for $\delta = 1$)

$$(x^3 - 2x^2 - x + 1)\big|(x^n - C_nx^2 + (2C_n - B_n)x + B_n + C_n - A_n) \tag{3.64}$$

for every $n \in \mathbb{N}$, $n \geqslant 3$. More precisely, the following decomposition holds:

$$p_7(\mathbb{X};\delta)\Big(\sum_{k=1}^{n-2} C_{k+1}(\delta)\mathbb{X}^{n-2-k}\Big) = W_{n,7}(\mathbb{X};\delta), \tag{3.65}$$

hence, for $\delta = 1$, we obtain

$$(x^3 - 2x^2 - x + 1)\left(\sum_{k=1}^{n-2} B_k x^{n-2-k}\right) = x^n - C_n x^2 + (2C_n - B_n)x + B_n + C_n - A_n. \tag{3.66}$$

Remark 3.23. *Equation (3.65) after differentiating and some manipulating enables us to generate the sums formulas of the following form:*

$$\Big(p_7(\mathbb{X};\delta)\Big)^{r+1}\left(\sum_{k=r}^{n} k^r C_k(\delta)\mathbb{X}^{n-k-r}\right) = \begin{array}{l}\textit{polynomial depending}\\ \textit{on } W_{n,7}(\mathbb{X};\delta),\ p_7(\mathbb{X};\delta)\\ \textit{and their derivatives}\end{array}$$

For example, the identity (3.65) implies the identity (3.25) (for $\mathbb{X} = 1$).

Remark 3.24. *There exist other ways to obtain the relation (3.64). For example, by (3.5) we have*

$$\begin{aligned}&\alpha A_{n+1}(\delta) + \beta B_{n+1}(\delta) + \gamma C_{n+1}(\delta) = \\ &\qquad = \big(\alpha + \beta\,\delta\big)A_n(\delta) + \big(2\,\alpha\,\delta + \beta + \gamma\,\delta\big)B_n(\delta) + \big(-\delta\,\alpha + (1-\delta)\gamma\big)C_n(\delta).\end{aligned} \tag{3.67}$$

We are interested when the following system of equations holds:

$$\frac{\alpha + \beta\,\delta}{\alpha} = \frac{2\,\alpha\,\delta + \beta + \gamma\,\delta}{\beta} = \frac{-\delta\,\alpha + (1-\delta)\gamma}{\gamma},$$

which is equivalent to the following one:

$$\begin{cases} \gamma = \frac{1}{\alpha}\Big(\beta^2 - 2\,\alpha^2\Big), \\ \Big(\frac{\beta}{\alpha}\Big)^3 + \Big(\frac{\beta}{\alpha}\Big)^2 - 2\Big(\frac{\beta}{\alpha}\Big) - 1 = 0. \end{cases}$$

Hence, by Remark 3.11, it follows that

$$\frac{\beta}{\alpha} \in \Big\{1 + 2\cos\frac{2k\pi}{7} \;:\; k = 1, 2, 3\Big\}.$$

So the equation (3.67) then has the following form:

$$\begin{aligned}&A_{n+1}(\delta) + 2\cos\frac{2k\pi}{7}B_{n+1}(\delta) - \Big(1 + 2\cos\frac{2k\pi}{7}\Big)^{-1}C_{n+1}(\delta) = \\ &= \Big(1 + 2\,\delta\cos\frac{2k\pi}{7}\Big)\Big(A_n(\delta) + 2\cos\frac{2k\pi}{7}B_n(\delta) - \Big(1 + 2\cos\frac{2k\pi}{7}\Big)^{-1}C_n(\delta)\Big) = \\ &= \Big(1 + 2\,\delta\cos\frac{2k\pi}{7}\Big)^n\Big(A_1(\delta) + 2\cos\frac{2k\pi}{7}B_1(\delta) - \Big(1 + 2\cos\frac{2k\pi}{7}\Big)^{-1}C_1(\delta)\Big) = \\ &\textit{(by (3.5))} \\ &= \Big(1 + 2\,\delta\cos\frac{2k\pi}{7}\Big)^{n+1}\end{aligned}$$

or

$$\delta^2\, x^{n+1} = \delta\big(x+\delta-1\big)\, A_n(\delta) + \big(x+\delta-1\big)\big(x-1\big)\, B_n(\delta) - \delta^2 C_n(\delta),$$

i.e.,

$$\delta^2\, x^{n+1} - x^2\, B_n(\delta) + x\big((2-\delta)\, B_n(\delta) - \delta\, A_n(\delta)\big) + \\ + \delta\,(1-\delta)\, A_n(\delta) + (\delta-1)\, B_n(\delta) + \delta^2\, C_n(\delta) = 0$$

for $x := 1 + 2\delta\cos\frac{2k\pi}{7}$, $k = 1, 2, 3$, *which is equivalent to*

$$W_{n,7}(\mathbb{X};\delta) = 0.$$

Lemma 3.25. *Immediately from (3.66) we obtain the following identities:*

$$(x^3-2x^2-x+1)^2 \cdot \Big(\sum_{k=1}^{n-2} B_k x^{n-2-k}\Big)' = \\ = (x^3-2x^2-x+1)^2 \cdot \sum_{k=1}^{n-3}(n-2-k)B_k x^{n-3-k} = \\ = (n-3)x^{n+2} - 2(n-2)x^{n+1} - (n-1)x^n + nx^{n-1} + C_n x^4 + \\ + 2(B_n-2C_n)x^3 + (3A_n-5B_n+2C_n)x^2 + (-4A_n+4B_n+2C_n)x - A_n + 3C_n, \quad (3.68)$$

and

$$(x^3-2x^2-x+1)^3 \cdot \Big(\sum_{k=1}^{n-2} B_k x^{n-2-k}\Big)'' = (n^2-7n+12)x^{n+4} + \\ + (-4n^2+24n-32)x^{n+3} + (2n^2-10n+18)x^{n+2} + (6n^2-24n+6)x^{n+1} + \\ + (-3n^2+9n+6)x^n + (2n-2n^2)x^{n-1} + (n^2-n)x^{n-2} - 2C_n x^6 + \\ + (-6B_n+12C_n)x^5 + (-12A_n+24B_n-18C_n)x^4 + (32A_n-42B_n+6C_n)x^3 + \\ + (-18A_n+30B_n-18C_n)x^2 + (-6A_n-6B_n+30C_n)x - 6A_n + 4B_N + 8C_n. \quad (3.69)$$

Corollary 3.26. *We have*

$$\sum_{k=1}^{n-2} B_k = A_n - 2C_n - 1,$$

$$\sum_{k=1}^{n-2}(-1)^k B_k = -1 + (-1)^n\big(A_n - 2B_n + 2C_n\big),$$

$$\sum_{k=1}^{n-3}(n-2-k)B_k = -2(A_n - 2C_n - 1) - n + B_n,$$

$$\sum_{k=1}^{n-2} kB_k = nA_n - B_n - 2nC_n,$$

$$\sum_{k=1}^{n-3}(n-2-k)(-1)^k B_k = 6 - n + (-1)^{n-1}\big(6A_n - 11B_n + 8C_n\big),$$

and

$$\sum_{k=1}^{n-3}(-1)^k kB_k = -4 + (-1)^n\Big(6A_n - 11B_n + 8C_n +$$
$$+ (n-2)\big(A_n - 2B_n + 2C_n - B_{n-2}\big)\Big).$$

3.4 Numerical remarks about the zeros of polynomials $\mathcal{B}_n(x)$

Let us set

$$\mathcal{B}_n(x) := \sum_{k=1}^{n} B_k x^{n-k}, \qquad n \in \mathbb{N}.$$

Then,

$$\mathcal{B}_1(x) \equiv 1, \quad \mathcal{B}_2(x) = x + 2, \quad \mathcal{B}_3(x) = x^2 + 2x + 5,$$

and, thus, we have the recurrence relations (see (3.20))

$$\mathcal{B}_{n+3}(x) = x^{n+2} + 2\mathcal{B}_{n+2}(x) + \mathcal{B}_{n+1}(x) - \mathcal{B}_n(x), \quad n \geqslant 1.$$

All polynomials $\mathcal{B}_n(x)$ for $n = 2k+1$, $k \in \mathbb{N}$, have an even degree, for example

$$\mathcal{B}_5(x) = x^4 + 2x^3 + 5x^2 + 11x + 25,$$
$$\mathcal{B}_7(x) = x^6 + 2x^5 + 5x^4 + 11x^3 + 25x^2 + 56x + 126.$$

As follows from calculations, these polynomials have no real zeros for $n \leqslant 200$.

However, all polynomials $\mathcal{B}_n(x)$ for $n = 2k$, $k \in \mathbb{N}$, have an odd degree, for example

$$\mathcal{B}_4(x) = x^3 + 2x^2 + 5x + 11,$$
$$\mathcal{B}_6(x) = x^5 + 2x^4 + 5x^3 + 11x^2 + 25x + 56.$$

These polynomials have exactly one real zero, which is less than zero (for $n \leqslant 200$). The zeros s_n of the consecutive polynomials $\mathcal{B}_n$ make a decreasing sequence from $s_2 = -2$ to $s_{200} = -2.2442546$ (by numerical calculations).

Remark 3.27. *All numerical calculations were made in Mathematica.*[1]

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[1] *Mathematica* is registered trademark of Wolfram Research Inc.

Table 1:

	$A_n(\delta)$
$n=0$	1
$n=1$	1
$n=2$	$2\delta^2+1$
$n=3$	$-\delta^3+6\delta^2+1$
$n=4$	$5\delta^4-4\delta^3+12\delta^2+1$
$n=5$	$-5\delta^5+25\delta^4-10\delta^3+20\delta^2+1$
$n=6$	$14\delta^6-30\delta^5+75\delta^4-20\delta^3+30\delta^2+1$
$n=7$	$-19\delta^7+98\delta^6-105\delta^5+175\delta^4-35\delta^3+42\delta^2+1$
	$B_n(\delta)$
$n=0$	0
$n=1$	δ
$n=2$	2δ
$n=3$	$2\delta^3+3\delta$
$n=4$	$-\delta^4+8\delta^3+4\delta$
$n=5$	$5\delta^5-5\delta^4+20\delta^3+5\delta$
$n=6$	$-5\delta^6+30\delta^5-15\delta^4+40\delta^3+6\delta$
$n=7$	$14\delta^7-35\delta^6+105\delta^5-35\delta^4+70\delta^3+7\delta$
	$C_n(\delta)$
$n=0$	0
$n=1$	0
$n=2$	δ^2
$n=3$	$-\delta^3+3\delta^2$
$n=4$	$3\delta^4-4\delta^3+6\delta^2$
$n=5$	$-4\delta^5+15\delta^4-10\delta^3+10\delta^2$
$n=6$	$9\delta^6-24\delta^5+45\delta^4-20\delta^3+15\delta^2$
$n=7$	$-14\delta^7+63\delta^6-84\delta^5+105\delta^4-35\delta^3+21\delta^2$

Table 2:

n	A_n	B_n	C_n
0	1	0	0
1	1	1	0
2	3	2	1
3	6	5	2
4	14	11	5
5	31	25	11
6	70	56	25
7	157	126	56
8	353	283	126
9	793	636	283
10	1782	1429	636
11	4004	3211	1429

Table 3:

n	$A_n(-1)$	$B_n(-1)$	$C_n(-1)$
0	1	0	0
1	1	-1	0
2	3	-2	1
3	8	-5	4
4	22	-13	13
5	61	-35	39
6	170	-96	113
7	475	-266	322
8	1329	-741	910
9	3721	-2070	2561
10	10422	-5791	7192
11	29196	-16213	20175

Table 4:

n	$A_n(\frac{1}{2})$	$B_n(\frac{1}{2})$	$C_n(\frac{1}{2})$
0	1	0	0
1	1	$\frac{1}{2}$	0
2	$\frac{3}{2}$	1	$\frac{1}{4}$
3	$\frac{19}{8}$	$\frac{7}{4}$	$\frac{5}{8}$
4	$\frac{61}{16}$	$\frac{47}{16}$	$\frac{19}{16}$
5	$\frac{197}{32}$	$\frac{155}{32}$	$\frac{33}{16}$
6	$\frac{319}{32}$	$\frac{507}{64}$	$\frac{221}{64}$
7	$\frac{2069}{128}$	$\frac{413}{32}$	$\frac{91}{16}$
8	$\frac{3357}{128}$	$\frac{5373}{256}$	$\frac{595}{64}$
9	$\frac{10897}{256}$	$\frac{4365}{128}$	$\frac{7753}{512}$
10	$\frac{70755}{1024}$	$\frac{28357}{512}$	$\frac{25213}{1024}$
11	$\frac{229725}{2048}$	$\frac{184183}{2048}$	$\frac{81927}{2048}$

Table 5:

n	$A_n(2)$	$B_n(2)$	$C_n(2)$
0	1	0	0
1	1	2	0
2	9	4	4
3	17	22	4
4	97	56	40
5	241	250	72
6	1097	732	428
7	3169	2926	1036
8	12801	9264	4816
9	40225	34866	13712
10	152265	115316	56020
11	501489	419846	174612

Table 6:

n	$A_n(3)$	$B_n(3)$	$C_n(3)$
0	1	0	0
1	1	3	0
2	19	6	9
3	28	63	0
4	406	147	189
5	721	1365	63
6	8722	3528	3969
7	17983	29694	2646
8	188209	83643	83790
9	438697	648270	83349
10	4078270	1964361	1778112
11	10530100	14199171	2336859

Table 7:

n	$\mathcal{A}_n$	$\frac{1}{2}(\mathcal{A}_n^2 - \mathcal{A}_{2n})$	$\mathcal{A}_n(\frac{1}{2})$
0	3	3	3
1	2	-1	$\frac{5}{2}$
2	6	5	$\frac{13}{4}$
3	11	-4	$\frac{19}{4}$
4	26	13	$\frac{117}{16}$
5	57	-16	$\frac{185}{16}$
6	129	38	$\frac{593}{32}$
7	289	-57	$\frac{3827}{128}$
8	650	117	$\frac{12389}{256}$
9	1460	-193	$\frac{40169}{512}$
10	3281	370	$\frac{65169}{512}$
11	7372	-639	$\frac{423065}{2048}$

References

[1] D. M. Bradley, A class of series acceleration formulae for Catalan's constant, *Ramanujan J.* **3** (1999), 149–173.

[2] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics. A Foundation for Computer Science*, Addison-Wesley, 1994.

[3] R. Grzymkowski and R. Wituła, *Calculus Methods in Algebra, Part One*, WPKJS, 2000 (in Polish).

[4] J. Kappraff and G. W. Adamson, A unified theory of proportion, *Vis. Math.* **5** (2003), electronic.

[5] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, 2001.

[6] I. Niven, *Irrational Numbers*, MAA, 1956 (appendix to the russian edition written by I. M. Yaglom).

[7] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2006.

[8] P. Steinbach, Golden fields: a case for the heptagon, *Math. Magazine* **70** (1997), 22–31.

[9] D. Surowski and P. McCombs, Homogeneous polynomials and the minimal polynomial of $\cos(2\pi/n)$, *Missouri J. Math. Sci.* **15** (2003), 4–14.

[10] W. Watkins and J. Zeitlin, The minimal polynomials of $\cos(2\pi/n)$, *Amer. Math. Monthly* **100** (1993), 471–474.

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