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Poisson liftings of holomorphic automorphic forms on semisimple Lie groups

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Abstract. Let G be a semisimple Lie group of Hermitian type, $K \subset G$ a maximal compact subgroup, and $P \subset G$ a minimal parabolic subgroup associated to K. If σ is a finite-dimensional representation of K in a complex vector space, it determines the associated homogeneous vector bundles on the homogeneous manifolds G/P and G/K. The Poisson transform associates to each section of the bundle over G/P a section of the bundle over G/K, and it generalizes the classical Poisson integral. Given a discrete subgroup Γ of G, we prove that the image of a Γ-invariant section of the bundle over G/P under the Poisson transform is a holomorphic automorphic form on G/K for Γ. We also discuss the special case of symplectic groups in connection with holomorphic forms on families of abelian varieties.

1. Introduction

In classical harmonic function theory, it is well-known that the Poisson integral of a complex-valued integrable function defined on a unit circle in the complex plane determines a harmonic function on the corresponding unit disk. The purpose of this paper is to discuss the Lie-theoretic analogue of the Poisson integral that transforms sections of a certain vector bundle to holomorphic automorphic forms on a semisimple Lie group of Hermitian type.

Let S be the unit circle in the complex plane $\mathbb{C}$ given by $S = \{e^{i\theta} \mid -\pi \leq \theta < \pi\}$, and let $L^1(S)$ denote the space of complex-valued integrable functions on S. Let $f \in L^1(S)$, and set $\widehat{f}(\theta) = f(e^{i\theta})$ for $-\pi \leq \theta < \pi$. Then the classical Poisson integral $\mathcal{P}f$ of f is a function defined on the unit disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$ given by

$$\mathcal{P}f(re^{i\phi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2} \widehat{f}(\theta) d\theta \tag{1}$$

for $0 \leq r < 1$ and $\phi \in \mathbb{R}$, and it is known that $\mathcal{P}f$ is a harmonic function (see e.g. [8]). If we use the normalized measure ds for S, then the expression in (1)

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can now be written in the form

$$(\mathcal{P}f)(z) = \int_S \frac{1-|z|^2}{|z-s|^2} f(s)ds \tag{2}$$

for all $z \in U$. We can interpret (2) in terms of Lie groups as follows. Let G_0 be the Lie group $SU(1,1)$ which can be written in the form

$$G_0 = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \, a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1 \right\}.$$

Then G_0 acts on the unit disk U by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z = \frac{az+b}{\bar{b}z+\bar{a}}$$

for $z \in U$ and $\left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right) \in G_0$. We consider the subgroups A_0, K_0, M_0 and N_0 of G_0 given by

$$A_0 = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}, \qquad K_0 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\},$$

$$M_0 = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \middle| \, \varepsilon^2 = 1 \right\}, \qquad N_0 = \left\{ \begin{pmatrix} 1+ix & -ix \\ ix & 1-ix \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

Then the unit circle S and the unit disk U can be identified with the quotient spaces

$$S = G_0/K_0, \quad U = G_0/M_0A_0N_0$$

(cf. [2, Chapter I]), and therefore we see that the Poisson integral associates a harmonic function on U to each integrable function on S. Such an interpretation of the Poisson integral suggests the possibility of extending (2) to the case of a more general semisimple Lie group.

Let G be a semisimple Lie group of Hermitian type, and let σ be a finite-dimensional representation of a maximal compact subgroup K of G in a complex vector space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form. Let P be a minimal parabolic subgroup of G, and let $\mathfrak{a}$ be a maximal abelian subgroup of $\mathfrak{p}$. If $\mathfrak{a}_{\mathbb{C}}$ denotes the complexification of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, then σ can be extended to a representation $\sigma_{P,\lambda}$ of P. Thus we can consider the homogeneous vector bundle $\mathcal{W}(\sigma_{P,\lambda})$ (resp. $\mathcal{V}(\sigma)$) over G/P (resp. G/K) associated to the representation $\sigma_{P,\lambda}$ (resp. σ). Now as the analogue of the classical Poisson integral we can consider the Poisson transform $\mathcal{P}_{\lambda,P}$ which assigns to each section ϕ of $\mathcal{W}(\sigma_{P,\lambda})$ a section of $\mathcal{V}(\sigma)$ (see Section 4 for details; see also [2, §II.3.4], [5], [11]).

Let Γ be a discrete subgroup of G that acts on the Riemannian symmetric space G/K properly discontinuously without fixed points. Then we can discuss holomorphic automorphic forms on G/K for Γ associated to an automorphy factor of Γ. In this paper, we show that the Γ-invariant sections of $\mathcal{W}(\sigma_{P,\lambda})$ are liftings of such automorphic forms. More precisely, we prove that the Poisson transform $\mathcal{P}_{\lambda,P}\phi$ of a Γ-invariant section ϕ of $\mathcal{W}(\sigma_{P,\lambda})$ is a holomorphic automorphic form on G/K for Γ with respect to the canonical automorphy factor of G associated to σ. As an application we show that for symplectic groups such liftings can be regarded as liftings of some holomorphic forms on a family of abelian varieties.

2. Automorphic forms

In this section we discuss holomorphic automorphic forms on semisimple Lie groups of Hermitian type and their interpretation in terms of sections of certain vector bundles over locally symmetric spaces. We also describe canonical automorphy factors associated to semisimple Lie groups of Hermitian type.

Let G be a connected semisimple Lie group Hermitian type, and let K be a maximal compact subgroup of G. Thus the associated Riemannian symmetric space $D = G/K$ is a Hermitian symmetric domain that has a G-invariant complex structure, and it can be realized as a bounded domain in $\mathbb{C}^k$ with $k \dim_{\mathbb{C}} D$. Let W be a finite-dimensional complex vector space, and let $GL(W)$ denote the group of all invertible endomorphism of W. If G' is a subgroup of G, a map $j : G' \times D \to GL(W)$ is called an *automorphy factor of* G' if it satisfies the following conditions:

(i) For fixed $g \in G'$, the map $z \mapsto j(g, z), D \to GL(W)$ is holomorphic.

(ii) For all $g_1, g_2 \in G'$ and $z \in D$, we have

$$j(g_1 g_2, z) = j(g_1, g_2 z) \cdot j(g_2, z) \tag{3}$$

Let Γ be a torsion-free discrete subgroup of G. Then the complex structure on D induces the structure of a complex manifold on the locally symmetric space $X = \Gamma \backslash D$, and we can define automorphic forms on D as follows (cf. [1]).

Definition 2.1. Let $j : \Gamma \times D \to GL(W)$ be an automorphy factor of Γ. A *holomorphic automorphic form on* D *of type* j *for* Γ is a holomorphic map $f : D \to W$ that satisfies

$$f(\gamma z) = j(\gamma, z) \cdot f(z) \tag{4}$$

for all $z \in D$ and $\gamma \in \Gamma$.

Given an automorphy factor $j : \Gamma \times D \to GL(W)$, we can construct an associated vector bundle on the locally symmetric space $X = \Gamma \backslash D$ as follows. Let the discrete subgroup Γ of G act on $D \times W$ by

$$\gamma \cdot (z, w) = (\gamma z, j(\gamma, z) w)$$

for all $\gamma \in \Gamma$ and $(z, w) \in D \times W$. The fact that this operation indeed defines an action of Γ on $D \times W$ follows from the condition (3). We set

$$\mathcal{A}(\Gamma, j) = \Gamma \backslash D \times W,$$

where the quotient is taken with respect to the above action of Γ on $D \times W$. Then the natural projection $D \to \Gamma \backslash D$ induces the structure of a holomorphic vector bundle on the induced map

$$\varpi : \mathcal{A}(\Gamma, j) \to X = \Gamma \backslash D$$

with fiber W. Let $\mathbf{\Gamma}_0(X, \mathcal{A}(\Gamma, j))$ denote the space of holomorphic sections of $\mathcal{A}(\Gamma, j)$ over X, that is, holomorphic maps $s : X \to \mathcal{A}(\Gamma, j)$ such that $\varpi \circ s = 1_X$.

Lemma 2.2. *Let $j : \Gamma \times D \to GL(W)$ be an automorphy factor. Then each element of $\mathbf{\Gamma}_0(X, \mathcal{A}(\Gamma, j))$ can be identified with an automorphic form on D of type j for Γ.*

Proof. See for example [6]. ∎

We shall now describe the construction of the canonical automorphy factor of G. Let I be a G-invariant complex structure on $D = G/K$. Then it determines a complex structure I_z on the tangent space $T_z(D)$ for each $z \in D$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. If z_0 is the point in D with $Kz_0 = z_0$, then we can identify $\mathfrak{p}$ with the tangent space $T_{z_0}(D)$. Thus we obtain a complex structure I_{z_0} on $\mathfrak{p}$. We set

$$\mathfrak{p}_\pm = \{X \in \mathfrak{p}_\mathbb{C} \mid I_{z_0}(X) = \pm iX\},$$

and denote by P_+, P_- the $\mathbb{C}$-subgroups of $G_\mathbb{C}$ corresponding to $\mathfrak{p}_+$, $\mathfrak{p}_-$, respectively; here $(\cdot)_\mathbb{C}$ denotes the complexification. Then we have

$$P_+ \cap K_\mathbb{C} P_- = \{1\}, \quad G \subset P_+ K_\mathbb{C} P_-, \quad G \cap K_\mathbb{C} P_- = K$$

(cf. [9, Lemma II.4.2], [6]). If $g \in P_+ K_\mathbb{C} P_- \subset G_\mathbb{C}$, we denote by $(g)_+ \in P_+$, $(g)_0 \in K_\mathbb{C}$ and $(g)_- \in P_-$ the components of g such that

$$g = (g)_+ \cdot (g)_0 \cdot (g)_-.$$

Let $(G_\mathbb{C} \times \mathfrak{p}_+)_*$ denote the subset of $G_\mathbb{C} \times \mathfrak{p}_+$ consisting of elements (g, z) such that $g \cdot \exp z \in P_+ K_\mathbb{C} P_-$, and set

$$J(g, z) = (g \cdot \exp z)_0 \in K_\mathbb{C}.$$

for $(g, z) \in (G_\mathbb{C} \times \mathfrak{p}_+)_*$. If we identify the Hermitian symmetric domain D with a subset of $\mathfrak{p}_+$ using the Harish-Chandra embedding $D \hookrightarrow \mathfrak{p}_+$ (cf. [9, §II.4]), then we have

$$G \times D \subset (G_\mathbb{C} \times \mathfrak{p}_+)_*.$$

Thus we obtain a map $J : G \times D \to K_\mathbb{C}$ which satisfies the condition

$$J(g_1 g_2, z) = J(g_1, g_2 z) \cdot J(g_2, z) \tag{5}$$

for $g_1 g_2 \in G$ and $z \in D$. Let $\sigma : K \to GL(W)$ be a representation of K in W, and extend it to a representation of $K_\mathbb{C}$. From (5) we see that $\sigma \circ J : G \times D \to GL(W)$ is an automorphy factor.

Definition 2.3. The automorphy factor $J_\sigma = \sigma \circ J : G \times D \to GL(W)$ is called the *canonical automorphy factor of G associated to σ*.

3. Homogeneous vector bundles

Let G be a Lie group, and let H be a closed subgroup of G. Let G/H denote the set $\{gH \mid g \in G\}$ of left cosets modulo H, and let $p : G \to G/H$ be the natural projection $p(g) = gH$. Then G/H has a unique manifold structure such that p is smooth and for each $gH \in G/H$ there is a neighborhood U of gH and a smooth map $\mu : U \to G$ such that $p \circ \mu = \mathrm{id}_U$. The quotient space G/H with such a manifold structure is called a *homogeneous manifold.*

Definition 3.1. Let $M = G/H$ be a homogeneous manifold. A vector bundle E over M is called a *homogeneous vector bundle* if G acts on E on the left and the G-action satisfies the following conditions:

(i) If E_x denotes the fiber of E over $x \in M$, then $g \cdot E_x = E_{gx}$ for all $x \in M$ and $g \in G$.

(ii) The map $E_x \to E_{gx}$ induced by g is linear for all $x \in M$ and $g \in G$.

We shall now construct a homogeneous vector bundle associated to a representation of H. Let $M = G/H$ be a homogeneous manifold, and let τ be a representation of H in a finite-dimensional complex vector space V. Then H acts on the product $G \times V$ on the right by

$$(g, v) \cdot h = (gh, \tau(h)^{-1} v) \tag{6}$$

for all $g, g' \in G$ and $v \in V$. We set

$$\mathcal{V} = G \times V/H,$$

where the quotient is taken with respect to the action of H on $G \times V$ given by (6). The natural projection $G \times V \to G$ induces the map $\pi : \mathcal{V} \to M$ which has the structure of a vector bundle with fiber V (cf. [10]). It can be shown that $\mathcal{V}$ is a homogeneous vector bundle over M. Let $\mathbf{\Gamma}_0(M, \mathcal{V})$ be the space of sections of $\mathcal{V}$, that is, smooth maps $s : M \to \mathcal{V}$ such that $\pi \circ s = 1_M$.

Lemma 3.2. *A section $s \in \mathbf{\Gamma}_0(M, \mathcal{V})$ of $\mathcal{V}$ can be identified with a smooth function $f : G \to V$ on G satisfying*

$$f(gh) = \tau(h)^{-1} f(g)$$

for all $(g, v) \in G \times V$ and $h \in H$.

Proof. See for example [10]. ∎

4. Poisson transforms

Let G, K, $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be as in Section 2 with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then we obtain the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

of $\mathfrak{g}$, where $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ (see e.g. [3, §V.2]). Let A and N be subgroups of G corresponding to the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively, so that we obtain the Iwasawa decomposition $G = KAN$ of G. Let M be the centralizer of A in K, and set $P = MAN$, which is a minimal parabolic subgroup of G. We shall write any element $g \in G$ in the form $g = \kappa(g) \cdot e^{H(g)} \cdot n$ with $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n \in N$.

Let σ be an irreducible representation of K in a finite-dimensional complex vector space W. Given an element $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the representation $\sigma_{\lambda,P}$ of P in W by

$$\sigma_{\lambda,P}(man) = e^{(-\lambda+\rho)H(a)} \sigma(m) \tag{7}$$

for all $m \in M$, $a \in A$ and $n \in N$, where ρ is the half-sum of $\dim(\mathfrak{g}_\alpha)\alpha$ over the positive roots α of $(\mathfrak{g}, \mathfrak{a})$.

Let $\mathcal{V}(\sigma)$ (resp. $\mathcal{W}(\sigma_{\lambda,P})$) be the homogeneous vector bundle on G/K (resp. G/P) associated to the representations σ and $\sigma_{\lambda,P}$, respectively. Using Lemma 3.2 and (7), we see that each section $\phi \in \mathbf{\Gamma}_0(G/P, \mathcal{W}(\sigma_{\lambda,P}))$ of the bundle $\mathcal{W}(\sigma_{\lambda,P})$ can be regarded as a function $\phi : G \to W$ satisfying

$$\phi(gman) = \sigma_{\lambda,P}(man)^{-1}\phi(g) = e^{(\lambda-\rho)H(a)}\sigma(m)^{-1}\phi(g) \tag{8}$$

for $g \in G$, $m \in M$, $a \in A$ and $n \in N$. Similarly, a section $\psi \in \mathbf{\Gamma}_0(G/K, \mathcal{V}(\sigma))$ of $\mathcal{V}(\sigma)$ can be identified with a function $\psi : G \to W$ such that

$$\psi(gk) = \sigma(k)^{-1}\psi(g) \tag{9}$$

for all $k \in K$ and $g \in G$. We now define the Poisson transform which may be regarded as the Lie-theoretic analogue of the classical Poisson integral (cf. [7], [2, §II.3.4], [5]).

Definition 4.1. Let ϕ be an element of $\mathbf{\Gamma}_0(G/P, \mathcal{W}(\sigma_{\lambda,P}))$, that is, a smooth function $\phi : G \to W$ satisfying (8). The *Poisson transform* $\mathcal{P}_{\lambda,P}\phi$ of ϕ is a W-valued function on G given by

$$(\mathcal{P}_{\lambda,P}\phi)(g) = \int_K \sigma(k)\phi(gk)dk \tag{10}$$

for all $g \in G$.

Lemma 4.2. *Let f is a continuous function on K that is right invariant under $K \cap M$. Then we have*

$$\int_K f(k)dk = \int_K e^{-2\rho H(g^{-1}k)} f(\kappa(g^{-1}k))dk$$

for $g \in G$.

Proof. See for example [3, p. 170]. ∎

Lemma 4.3. *The Poisson transform $\mathcal{P}_{\lambda,P}\phi$ of an element $\phi \in \mathbf{\Gamma}_0(G/P, \mathcal{W}(\sigma_{\lambda,P}))$ can be written in the form*

$$(\mathcal{P}_{\lambda,P}\phi)(g) = \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\sigma(\kappa(g^{-1}k))\phi(k)dk \tag{11}$$

for all $g \in G$.

Proof. If $m \in K \cap M$, then by 8 we have

$$\sigma(km)\phi(gkm) = \sigma(k)\sigma(m)\sigma(m)^{-1}\phi(g)$$

for $g \in G$ and $k \in K$. Hence, applying Lemma 4.2 to the function $k \mapsto \sigma(k)\phi(gk)$, we obtain

$$\int_K \sigma(k)\phi(gk)dk = \int_K e^{-2\rho H(g^{-1}k)}\sigma(\kappa(g^{-1}k))\phi(g\kappa(g^{-1}k))dk$$

for $g \in G$. Let $g^{-1}k = \kappa(g^{-1}k) \cdot a_1 \cdot n_1$ with $a_1 \in A$ and $n_1 \in N$. Then we have

$$g\kappa(g^{-1}k) = kn_1^{-1}a_1^{-1} = ka_1^{-1}n'$$

for some $n' \in N$. Hence we see that

$$\begin{aligned}\phi(g\kappa(g^{-1}k)) &= \phi(ka_1^{-1}n') = e^{(\lambda-\rho)H(a_1^{-1})}\phi(k)\\ &= e^{-(\lambda-\rho)H(a_1)}\phi(k) = e^{-(\lambda-\rho)H(g^{-1}k)}\phi(k).\end{aligned}$$

Thus we have

$$\begin{aligned}\int_K \sigma(k)\phi(gk)dk &= \int_K e^{-2\rho H(g^{-1}k)}\sigma(\kappa(g^{-1}k))e^{-(\lambda-\rho)H(g^{-1}k)}\phi(k)dk\\ &= \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\sigma(\kappa(g^{-1}k))\phi(k)dk,\end{aligned}$$

and the lemma follows. ■

Let Γ be a torsion-free discrete subgroup of G as in Section 2, and consider the left-action of Γ on $G \times W$ defined by

$$\gamma \cdot (g, w) = (\gamma g, w) \tag{12}$$

for $\gamma \in \Gamma$ and $(g, w) \in G \times W$. Since this action commutes with the action given by (6) for $\tau = \sigma$ and $V = W$ that was used for the construction of a homogeneous vector bundle, the homogeneous vector bundle $\mathcal{V}(\sigma) \to D = G/K$ associated to σ induces the vector bundle $\mathcal{V}(\Gamma, \sigma) = \Gamma\backslash\mathcal{V}(\sigma)$ over the locally symmetric space $X = \Gamma\backslash D = \Gamma\backslash G/K$ with fiber W. Similarly, we obtain the vector bundle $\mathcal{W}(\Gamma, \sigma_{\lambda,P}) = \Gamma\backslash\mathcal{W}(\sigma_{\lambda,P})$ over the space $\Gamma\backslash G/P$ whose fiber is again W. Thus each section $\phi \in \mathbf{\Gamma}_0(\Gamma\backslash G/P, \mathcal{W}(\Gamma, \sigma_{\lambda,P}))$ of $\mathcal{W}(\Gamma, \sigma_{\lambda,P})$ is a Γ-invariant section of $\mathcal{W}(\Gamma, \sigma_{\lambda,P})$, and it can be identified with a smooth function $\phi : G \to W$ on G satisfying (8) and

$$\phi(\gamma g) = \phi(g) \tag{13}$$

for all $\gamma \in \Gamma$ and $g \in G$. In the same way, a section of $\mathcal{V}(\Gamma, \sigma)$ can be regarded as a smooth function $\psi : G \to W$ satisfying (9) and $\psi(\gamma g) = \psi(g)$ for $\gamma \in \Gamma$ and $g \in G$.

Lemma 4.4. *Let $J_\sigma : G \times D \to GL(W)$ be the canonical automorphy factor given in Definition 2.3, and let $z_0 \in D = G/K$ be the point with $Kz_0 = z_0$. Then the map $[(g, w)] \mapsto [(gz_0, w)]$ determines a canonical isomorphism of the vector bundles*

$$\mathcal{V}(\Gamma, \sigma) = \Gamma\backslash G \times W/K \cong \mathcal{A}(\Gamma, J_\sigma) = \Gamma\backslash D \times W.$$

Proof. See [6, Theorem II.4.1]. ■

Now we state our main theorem in this paper, which implies that each section of the homogeneous vector bundle $\mathcal{W}(\Gamma, \sigma_{\lambda,P})$ can be regarded as a lifting, via the Poisson transform map $\mathcal{P}_{\lambda,P}$, of a holomorphic automorphic form on D for Γ of type J_σ, the canonical automorphy factor of G associated to σ.

Theorem 4.5. *Let ϕ be an element of $\mathbf{\Gamma}_0(\Gamma\backslash G/P, \mathcal{W}(\Gamma, \sigma_{\lambda,P}))$ considered as a W-valued smooth function on G satisfying* (8) *and* (13)*, and let J_σ be the canonical automorphy factor associated to σ described in Definition 2.3. Then the Poisson transform $\mathcal{P}_{\lambda,P}\phi$ of ϕ is a holomorphic automorphic form on D of type J_σ for Γ.*

Proof. Let ϕ be an element of $\mathbf{\Gamma}_0(\Gamma\backslash G/P, \mathcal{W}(\Gamma, \sigma_{\lambda,P}))$. Using Lemma 4.3, for each $g \in G$ and $k_1 \in K$ we obtain

$$\begin{aligned}(\mathcal{P}_{\lambda,P}\phi)(gk_1) &= \int_K e^{-(\lambda+\rho)H((gk_1)^{-1}k)}\sigma(\kappa((gk_1)^{-1}k))\phi(k)dk \\ &= \int_K e^{-(\lambda+\rho)H(k_1^{-1}g^{-1}k)}\sigma(\kappa(k_1^{-1}g^{-1}k))\phi(k)dk \\ &= \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\sigma(k_1^{-1}\kappa(g^{-1}k))\phi(k)dk \\ &= \sigma(k_1^{-1}) \cdot \int_K e^{-(\lambda+\rho)H(g^{-1}k)}\sigma(\kappa(g^{-1}k))\phi(k)dk \\ &= \sigma(k_1)^{-1} \cdot (\mathcal{P}_{\lambda,P}\phi)(g),\end{aligned}$$

which implies that $\mathcal{P}_{\lambda,P}\phi$ is a smooth section of $\mathcal{V}(\sigma)$. Since the function $g \mapsto e^{-(\lambda+\rho)H(g^{-1})}\sigma(\kappa(g^{-1}))$ is analytic and K is compact, using (11), we see that $\mathcal{P}_{\lambda,P}\phi$ is also analytic. Furthermore, since D is assumed to have a G-invariant complex structure, it follows that $\mathcal{P}_{\lambda,P}\phi$ is in fact a holomorphic section of the bundle $\mathcal{V}(\sigma)$. On the other hand, using (10), we obtain

$$(\mathcal{P}_{\lambda,P}\phi)(\gamma g) = \int_K \sigma(k)\phi(\gamma gk)dk = (\mathcal{P}_{\lambda,P}\phi)(g)$$

for all $\gamma \in \Gamma$ and $g \in G$. Therefore $\mathcal{P}_{\lambda,P}\phi$ is Γ-invariant, and hence it follows that $\mathcal{P}_{\lambda,P}\phi$ is an element of $\mathbf{\Gamma}_0(X, \mathcal{V}(\Gamma, \sigma))$. Now the theorem follows by using this and the canonical isomorphism described in Lemma 4.4. ■

5. Symplectic groups

It is well-known that an arithmetic quotient of a Siegel upper half space can be considered as the parameter space of a family of polarized abelian varieties. In this section we show that sections of a certain vector bundle associated to a symplectic group can be regarded as liftings of some holomorphic forms on such a family of abelian varieties.

Throughout this section, let G_0 be the symplectic group $Sp(n, \mathbb{R})$, and let K_0 be a maximal compact subgroup of G_0. Then the associated Hermitian symmetric domain G_0/K_0 can be identified with the Siegel upper half space $\mathcal{H}_n$ of degree n. Let Γ_0 be a torsion-free subgroup of $Sp(n, \mathbb{Z})$ of finite index. Consider the semidirect product $\Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)^m$ whose multiplication operation is given by

$$\begin{aligned}&(g, (u_1, v_1), \dots, (u_m, v_m)) \cdot (g', (u_1', v_1'), \dots, (u_m', v_m')) \\ &\qquad = (gg', (u_1, v_1)g' + (u_1', v_1'), \dots, (u_m, v_m)g' + (u_m', v_m'))\end{aligned}$$

for all $g, g' \in \Gamma_0 \subset Sp(n, \mathbb{R})$ and $(u_j, v_j) \in \mathbb{Z}^n \times \mathbb{Z}^n$ for $1 \leq j \leq m$. Then it acts on the space $\mathcal{H}_n \times \mathbb{C}^{mn}$ by

$$\begin{aligned}\big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (u_1, v_1), \dots, (u_m, v_m)\big) \cdot (z, \zeta_1, \dots, \zeta_m) \\ = \big((az+b)(cz+d)^{-1}, (u_1 z + v_1 + \zeta_1)(cz+d)^{-1}, \dots \\ \dots, (u_m z + v_m + \zeta_m)(cz+d)^{-1}\big)\end{aligned}$$

for all $\zeta_1, \dots, \zeta_m \in \mathbb{C}^n$, $z \in \mathcal{H}_n$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0$, and $(u_j, v_j) \in \mathbb{Z}^n \times \mathbb{Z}^n$ for $1 \leq j \leq m$. We denote by Y_0^m the associated quotient space, that is,

$$Y_0^m = \Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)^m \backslash \mathcal{H}_n \times \mathbb{C}^{mn}. \tag{14}$$

Then the natural projection $\mathcal{H}_n \times \mathbb{C}^{mn} \to \mathcal{H}_n$ induces a map $\pi_0 : Y_0^m \to X_0 = \Gamma_0 \backslash \mathcal{H}_n$, which gives Y_0^m the structure of a fiber bundle over X_0 whose fiber is isomorphic to the complex torus

$$(\mathbb{Z}^n \times \mathbb{Z}^n)^m \backslash \mathbb{C}^{mn} \cong (\mathbb{Z} \times \mathbb{Z} \backslash \mathbb{C})^{mn}.$$

By (14) a holomorphic differential form on Y_0^m can be regarded as a holomorphic form on $\mathcal{H}_n \times \mathbb{C}^{mn}$ that is invariant under the action of $\Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)^m$.

Given a nonnegative integer k, a holomorphic function $f : \mathcal{H}_n \to \mathbb{C}$ is a *Siegel modular form of weight k for Γ_0* if it satisfies

$$f((az+b)(cz+d)^{-1}) = (cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0 \subset Sp(n, \mathbb{R})$. We shall denote by $\mathcal{M}_k(\Gamma_0)$ the space of Siegel modular forms of weight k for Γ_0.

Proposition 5.1. *Let $n \geq 2$ and $\langle n \rangle = n(n+1)/2$. There is a canonical isomorphism between the space $\mathcal{M}_{m+n+1}(\Gamma_0)$ of Siegel modular forms for Γ_0 of weight $m+n+1$ and the space $H^0(Y_0^m, \Omega^{\langle n \rangle + mn})$ of holomorphic forms on Y_0^m of degree $\langle n \rangle + mn$.*

Proof. The proof can be sketched as follows. Let

$$z = (z_1, \dots, z_{\langle n \rangle}), \quad \zeta = (\zeta_1, \dots, \zeta_m)$$

with $\zeta_j = (\zeta_j^1, \dots, \zeta_j^n)$ for $1 \leq j \leq m$ be the canonical coordinate systems for $\mathcal{H}_n$, $\mathbb{C}^{mn}$, respectively. Let f be a Siegel modular form of weight $m+n+1$ for Γ_0, and define the associated holomorphic form Φ_f on $\mathcal{H}_n \times \mathbb{C}^{mn}$ by

$$\Phi_f(z, \zeta) = f(z) dz \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_m,$$

where $dz = dz_1 \wedge \cdots \wedge dz_{\langle n \rangle}$ and $d\zeta_j = d\zeta_j^1 \wedge \cdots \wedge \zeta_j^n$ for $1 \leq j \leq m$. Then it can be shown that Φ_f is invariant under the action of $\Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)^m$, and therefore Φ_f is a holomorphic form on Y_0^m of degree $\langle n \rangle + mn$. On the other hand, given a $(\Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)^m)$-invariant holomorphic form

$$\Phi(z, \zeta) = h(z, \zeta) dz \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_m$$

on $\mathcal{H}_n \times \mathbb{C}^{mn}$ of degree $\langle n \rangle + mn$, it can be shown that $h(z, \zeta)$ is a function of z only and that it is a Siegel modular form for Γ_0 of degree $m+n+1$ (see e.g. the arguments in the proof of [4, Theorem 4.2]). Thus the map $f \mapsto \Phi_f$ determines the desired isomorphism. ■

Let $G_0 = K_0 A_0 N_0$ be the Iwasawa decomposition of G_0 described in Section 4, and let $P_0 = M_0 A_0 N_0$ be the corresponding minimal parabolic subgroup of G_0, where M_0 is the centralizer of A_0 in K_0. Let $\sigma_0 : K_0 \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ be the complex one-dimensional representation of K_0 given by

$$\sigma_0(k) = \det(k)^{m+n+1}$$

for all $k \in K_0$. Let λ_0 be an element of $(\mathfrak{a}_0)^*_{\mathbb{C}}$ with $\mathfrak{a}_0 = \operatorname{Lie} A_0$, and let

$$\mathcal{P}_{\lambda_0,P_0} : \mathbf{\Gamma}_0(\Gamma_0 \backslash G_0/P_0, \mathcal{W}(\Gamma_0, (\sigma_0)_{\lambda_0,P_0})) \to \mathbf{\Gamma}_0(\Gamma_0 \backslash \mathcal{H}_n, \mathcal{V}(\Gamma_0, \sigma_0))$$

be the associated Poisson transform.

Theorem 5.2. *Let ϕ be an element of $\mathbf{\Gamma}_0(\Gamma_0 \backslash G_0/P_0, \mathcal{W}(\Gamma_0, (\sigma_0)_{\lambda_0,P_0}))$. Then the Poisson transform $\mathcal{P}_{\lambda_0,P_0}\phi$ of ϕ is a Siegel modular form of weight $m+n+1$ for Γ_0, and therefore by Proposition 5.1 it can be identified with a holomorphic form of degree $\langle n \rangle + mn$ on the family Y_0^m of abelian varieties parametrized by the Siegel modular variety $X_0 = \Gamma_0 \backslash \mathcal{H}_n$.*

Proof. By Theorem 4.5 the Poisson transform $\mathcal{P}_{\lambda_0,P_0}\phi$ of ϕ is a holomorphic automorphic form on $\mathcal{H}_n$ of type $\sigma_0 \circ J_0$ for Γ_0, where $J_0 : G_0 \times \mathcal{H}_n \to K_0$ is the canonical automorphy factor of G_0. However, for the symplectic group $G_0 = Sp(n, \mathbb{R})$ the canonical automorphy factor is given by

$$J_0(g, z) = cz + d$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0$ and $z \in \mathcal{H}_n$ (cf. [9]). Thus we have

$$\begin{aligned} (\mathcal{P}_{\lambda_0,P_0}\phi)(\gamma z) &= \sigma_0(J_0(\gamma, z))(\mathcal{P}_{\lambda_0,P_0}\phi)(z) \\ &= \det(cz+d)^{m+n+1}(\mathcal{P}_{\lambda_0,P_0}\phi)(z) \end{aligned}$$

for all $\gamma \in \Gamma_0$ and $z \in \mathcal{H}_n$, and hence $\mathcal{P}_{\lambda_0,P_0}\phi$ is a Siegel modular form for Γ_0 of weight $m+n+1$. Thus the theorem follows from this and Proposition 5.1. ∎

References

[1] Borel, A., *Introduction to automorphic forms*, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, 1966, 199–210.

[2] Helgason, S., "Geometric analysis on symmetric spaces," Amer. Math. Soc., Providence, 1994.

[3] Knapp, A., "Representation theory of semisimple groups: An overview based on examples," Princeton University Press, Princeton, 1986.

[4] Lee, M. H., *Mixed automorphic vector bundles on Shimura varieties*, Pacific J. Math. **173** (1996), 105–126.

[5] Minemura, K., *Invariant differential operators and spherical sections on a homogeneous vector bundle*, Tokyo J. Math. **15** (1992), 231–245.

[6] Murakami, S., "Cohomology of vector-valued forms on symmetric spaces," Univ. of Chicago, Chicago, 1966.

[7] Okamoto, K., *Harmonic analysis on homogeneous vector bundles*, Lecture Notes in Math. **266**, Springer-Verlag, Heidelberg, 1972, 255–271.

[8] Rudin, W., "Real and complex analysis," 3rd ed., McGraw-Hill, New York, 1987.

[9] Satake, I., "Algebraic structures of symmetric domains," Princeton Univ. Press, Princeton, 1980.

[10] Wallach, N., "Harmonic analysis in homogeneous spaces," Marcel Dekker, New York, 1973.

[11] Wallach, N., "Real reductive groups II," Academic Press, New York, 1992.

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