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Maximal semigroups in the divisible hull of lattices in nilpotent Lie groups

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Abstract. We consider subsemigroups in the divisible hull of lattices in nilpotent Lie groups. Our principal result shows that such a semigroup is a group provided it is not contained in a proper semigroup with non-empty interior.

1. Introduction

A subsemigroup S of a group G is said to maximal if it is not a group and there is no semigroup $T \neq S$ containing S properly. When studying subsemigroups of groups many questions have a natural formulation and solution by means of the knowledge of maximal semigroups. This makes the problem of determining the maximal semigroups one of the major problems in the theory of semigroups.

Let N be a nilpotent Lie group. In this paper we look at the subsemigroups of the subgroups of N which are obtained by divisible extension of a lattice in the following sense. As it is well known N is a divisilbe group. If N is simply connected and Γ is a lattice in N we look at the semigroups in the smallest divisible group of N containing Γ. In this setting a simply connected N admits a lattice Γ if and only if there exists a basis, say β, of the Lie algebra $\mathfrak{n}$ of N, whose constants of structure are rational (see Raghunathan [5, Thm. 2.12]). Let $\mathfrak{n}_{\mathbb{Q}}$ denote the rational subspace spanned by β. If we identify N with $\mathfrak{n}$ through the exponential map, the group product in $\mathfrak{n}$ is provided by the Campbell-Hausdorff formula, which has constant coefficients. Hence $\mathfrak{n}_{\mathbb{Q}}$ becomes a subgroup of N which contains Γ. We show that $\mathfrak{n}_{\mathbb{Q}}$ is the smallest divisible subgroup of N that contains Γ.

The purpose of this paper is to characterize the maximal semigroups of $\mathfrak{n}_{\mathbb{Q}}$. The main results show that $S \subset \mathfrak{n}_{\mathbb{Q}}$ is a maximal semigroup if and only if it is the intersection of a maximal semigroup with non-void interior in N with $\mathfrak{n}_{\mathbb{Q}}$. Moreover, any semigroup in $\mathfrak{n}_{\mathbb{Q}}$ which is not a group is contained in a maximal semigroup. These results extend those of [6], [7], where the maximal semigroups

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in the lattices of nilpotent Lie groups where given analogous characterizations. Actually, our basic technique comes from the results for semigroups in lattices together with the fact that a rational subgroup is a union of lattices.

2. The Divisible Hull of a Lattice in a Simply Connected Nilpotent Lie Group

In what follows we identify the simply connected nilpotent Lie group with its Lie algebra $\mathfrak{n}$ through the exponential map

$$\exp : \mathfrak{n} \longrightarrow N,$$

which is a diffeomorphism. Through this identification the group structure in N is carried back to $\mathfrak{n}$. We denote by $*$ the group product in $\mathfrak{n}$. It is given by the well known Campbell-Hausdorff formula, whose first terms are

$$X * Y = X + Y + \frac{1}{2}[X,Y] + \cdots \qquad X, Y \in \mathfrak{n}.$$

Since $X^n = nX$ for all $n \in \mathbb{Z}$ and $X \in \mathfrak{n}$, it follows that N is a divisible and torsion free group. Hence if H is a subgroup of a nilpotent simply connected Lie group N, there exists a smallest divisible subgroup $Div(H)$ of N containing H.

Let Γ be a lattice in a simply connected nilpotent Lie group and let $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ be the $\mathbb{Q}$-span of $\exp^{-1}(\Gamma)$. By theorem 2.12 in [5], $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ is a $\mathbb{Q}$-Lie algebra of $\mathfrak{n}$. If we identify N with $(\mathfrak{n}, *)$ we have:

Lemma 2.1. *$\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ is a divisible subgroup of N.*

Proof. If $X \in \mathfrak{n}_{\mathbb{Q}}(\Gamma)$ and $n \in \mathbb{Z} - \{0\}$ then $X = r_1 \ X_1 + \cdots + r_k X_k$,where $X_1, \cdots X_k \in \Gamma$ and $r_1, \ \cdots, r_k \in \mathbb{Q}$. Let us suppose that $r_i = a_i/b_i$, where $a_i, b_i \in \mathbb{Z}$ for $1 \leq i \leq k$ and put $s_i = a_i/nb_i$. Then $S = s_1 X_1 + \cdots s_k X_k \in \mathfrak{n}_{\mathbb{Q}}(\Gamma)$ and $S^n = nS = X$. ■

Let $\mathfrak{n}^l$ the l-th term of the lower central series of $\mathfrak{n}$, that is, $\mathfrak{n}^0 = \mathfrak{n}$ and $\mathfrak{n}^l = [\mathfrak{n}, \mathfrak{n}^{l-1}]$. Let k such that $\mathfrak{n}^k \neq \{0\}$ and $\mathfrak{n}^{k+1} = \{0\}$.

Proposition 2.2. *Let Γ be a lattice in a nilpotent simply connected Lie group N. Then $\mathfrak{n}_{\mathbb{Q}}(\Gamma) = Div(\Gamma)$.*

Proof. Since $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ is a divisible subgroup we have that $Div(\Gamma) \subset \mathfrak{n}_{\mathbb{Q}}(\Gamma)$. In order to show the other inclusion we will argue by induction on $\dim N$. If $\dim N = 1$ then N is abelian and the result is trivial. If $\pi : N \longrightarrow N/[N,N]$ is the natural map, then $\pi(\Gamma)$ is a lattice in $N/[N,N]$. The group $N/[N,N]$ being a real vector space, $\pi(\Gamma)$ is the $\mathbb{Z}$-span of a basis $e_1, e_2, \cdots, e_l$ of this vector space. Let V be the $\mathbb{R}$ -span of $e_1, e_2, \cdots, e_{l-1}$ and put $N' = \pi^{-1}(V)$. Let $\mathfrak{n}'$ be the Lie algebra of N'. Then $\mathfrak{n}'$ may be identified with a subalgebra of $\mathfrak{n}$ of codimension 1. Now, by induction hypothesis, $Div(\Gamma \cap N') = \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N')$. According to the proof of Theorem 2.12 in [5], $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ is a $\mathbb{Q}$-Lie subalgebra, so there exists $v \in \exp^{-1}(\Gamma)$ such that

$$\mathfrak{n}_{\mathbb{Q}}(\Gamma) = \mathbb{Q}v + \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N')$$

and

$$[v, w] \in \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N') , \qquad \forall w \in \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N').$$

Then we have

$$[\mathbb{Q}v, \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N')] \subset \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N') = Div(\Gamma \cap N').$$

Let $r \in \mathbb{Q}$ and $y \in \mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N')$. Since $\mathfrak{n}'_{\mathbb{Q}}(\Gamma \cap N') = Div(\Gamma \cap N') \subset Div(\Gamma)$ and $\mathbb{Q}v \subset Div(\Gamma)$ then $rv * y \in Div(\Gamma)$. By the Campbell-Hausdorff formula we get:

$$rv * y = rv + y + z_1,$$

where $z_1 \in Div(\Gamma \cap N') \cap \mathfrak{n}^1$.

Since $(rv * y) * z_1^{-1} \in Div(\Gamma)$ we have that

$$(rv * y) * z_1^{-1} = (rv + y + z_1) * z_1^{-1} = rv + y + z_2,$$

where $z_2 \in Div(\Gamma \cap N') \cap \mathfrak{n}^2$.

Repeating the above argument j times we obtain $z_1, z_2, \cdots z_j, z_{j+1}$ such that

$$(rv * y) * z_1^{-1} * z_2^{-1} * \cdots * z_j^{-1} \in Div(\Gamma)$$

and

$$(rv * y) * z_1^{-1} * z_2^{-1} * \cdots * z_i^{-1} = rv + y + z_{i+1},$$

where $z_i \in Div(\Gamma \cap N') \cap \mathfrak{n}^i$ for $1 \leq i \leq j+1$. But $\mathfrak{n}$ is nilpotent, so $\mathfrak{n}^{k+1} = \{0\}$ for some $k \in \mathbb{N}$. Thus

$$(rv * y) * z_1^{-1} * z_2^{-1} * \cdots * z_k^{-1} = rv + y \in Div(\Gamma).$$

This shows that $\mathfrak{n}_{\mathbb{Q}}(\Gamma) \subset Div(\Gamma)$. Therefore $\mathfrak{n}_{\mathbb{Q}}(\Gamma) = Div(\Gamma)$. ■

Corollary 2.3. *Let Γ_1 and Γ be lattices of N such that $\Gamma_1 \subset Div(\Gamma)$. Then $Div(\Gamma_1) = Div(\Gamma)$.*

Proof. Since $\Gamma_1 \subset \Gamma_1 \subset Div(\Gamma)$, by Proposition 2.2 we have that

$$\mathfrak{n}_{\mathbb{Q}}(\Gamma_1) = Div(\Gamma_1) \subset Div(\Gamma) = \mathfrak{n}_{\mathbb{Q}}(\Gamma).$$

By Theorem 2.12 of [5], $\mathfrak{n}_{\mathbb{Q}}(\Gamma_1) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\mathfrak{n}_{\mathbb{Q}}(\Gamma) \otimes_{\mathbb{Q}} \mathbb{R}$. Then $\mathfrak{n}_{\mathbb{Q}}(\Gamma_1) = \mathfrak{n}_{\mathbb{Q}}(\Gamma)$ and $Div(\Gamma_1) = Div(\Gamma)$. ■

Lemma 2.4. *If Γ is a lattice of N then $Div(\Gamma)$ is a union of lattices of N.*

Proof. Suppose that $\beta = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ is a basis of $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$. Then it is clear that

$$\mathfrak{n}_{\mathbb{Q}}(\Gamma) = \bigcup_{(k_1,\dots,k_n)\in\mathbb{N}^n} \frac{1}{k_1}\mathbb{Z}\alpha_1 + \frac{1}{k_2}\mathbb{Z}\alpha_2 + \cdots + \frac{1}{k_n}\mathbb{Z}\alpha_n$$

and for $(k_1, \cdots, k_n) \in \mathbb{N}^n$,

$$\Gamma_{(k_1,\cdots,k_n)} = \frac{1}{k_1}\mathbb{Z}\alpha_1 + \frac{1}{k_2}\mathbb{Z}\alpha_2 + \cdots + \frac{1}{k_n}\mathbb{Z}\alpha_n$$

is a lattice in the vector space $\mathfrak{n}$. But the proof of Theorem 2.12 in [5], shows that if $\Gamma_1 \subset \mathfrak{n}_{\mathbb{Q}}(\Gamma)$ is a lattice in the vector space $\mathfrak{n}$ then the subgroup of N generates by Γ_1 is a lattice of $\dot{N}$. Then $\Gamma_{(k_1,\cdots,k_n)}$ generated a lattice $\langle\Gamma_{(k_1,\cdots,k_n)}\rangle$ of N which is contained in $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$. ■

We prove below that any finitely generated subgroup of a rational group is discrete. By a rational subspace of a vector space V we mean the space spanned over $\mathbb{Q}$ by a discrete subgroup in V.

Lemma 2.5. *Let V be a finite dimensional vector space over $\mathbb{R}$. Then, every finitely generated subgroup of a $\mathbb{Q}$-subspace W in V is a discrete subgroup of V.*

Proof. We may assume that $V = \mathbb{R}^n$ and $W = \mathbb{Q}^n$. Let H be a subgroup of $\mathbb{R}^n$ generated by $X_1, X_2, \dots, X_l$ in $\mathbb{Q}^n$, where l is a non negative integer. In this case there is an integer d such that $dX_i \in \mathbb{Z}^n$ for $1 \le i \le l$. Hence dH is a subgroup of $\mathbb{Z}^n$, showing that H is a discrete subgroup of $\mathbb{R}^n$. ■

Proposition 2.6. *If Γ is a lattice of N then every finitely generated subgroup of $Div(\Gamma)$ is a discrete subgroup.*

Proof. By proposition 2.2 $Div(\Gamma) = \mathfrak{n}_{\mathbb{Q}}(\Gamma)$. Let β be a basis of $\mathfrak{n}_{\mathbb{Q}}(\Gamma)$ and take $x_1, \dots, x_l \in R$, where l is a non negative integer. By Lemma 2.5 the subgroup H of the vector space $\mathfrak{n}$ generated by $x_1, \dots, x_l$ is discrete in $\mathfrak{n}$. By the proof of Theorem 2.12 in [5] this implies that H generates a discrete subgroup of N. ■

3. Semigroups in $Div(\Gamma)$

A standard assumption in the control theory of Lie groups is that semigroups under consideration have non-void interior. A controllability criterion is provided by the fact that a neighborhood of the identity in a topological group generates, as a semigroup, its identity component.

In order to avoid repetition we make the following technical definition:

Definition 3.1. A subset X of a topological group G is said to have *property (C)* if it is not contained in a proper subsemigroup of G with interior points.

This is a kind of controllability condition because it means that the only semigroup with non-void interior containing X is G itself.

For a subsemigroup S of a simply connected nilpotent Lie group N we let

$$\widetilde{S} = \{t_1x_1 * t_2x_2 * \cdots * t_lx_l : t_i \geq 0, x_i \in S, \text{ arbitrary } l\}$$

be the semigroup generated by the real positive powers of the elements in S. It is a ray semigroup in N and S is a set of infinitesimal generators of $\widetilde{S}$.

Lemma 3.2. *Let $S \subset N$ a semigroup. If S has the property (C), then $\widetilde{S} = \mathfrak{n}$.*

Proof. Since $\widetilde{S}$ is arcwise connected the subgroup $G(\widetilde{S})$ of N generated by $\widetilde{S}$ in N is also arcwise connected. It follows that the group $G(\widetilde{S})$ generated by $\widetilde{S}$ is an analytic subgroup of N (see [1], Theorem V.1.1). Therefore the Lie algebra of $G(\widetilde{S})$ is the subalgebra $\langle\langle S\rangle\rangle$ of $\mathfrak{n}$ generated by S. Since S has the property (C) and the maximal semigroups with interior points in N are in one-to-one correspondence with the half-spaces in $\mathfrak{n}$ whose boundary is a subalgebra, (cf. [1], Corollary V.5.41), we have that $\langle\langle S\rangle\rangle = \mathfrak{n}$. Therefore, $G(\widetilde{S}) = N$. This implies that $\widetilde{S}$ has non void dense interior in N, (see [1], Theorem V.1.16). Since $S \subset \widetilde{S}$ we conclude that $\widetilde{S} = N$. ■

The next result is purely combinatorial geometry, but we are unaware of a direct reference in the literature.

Proposition 3.3. *Let Γ a lattice of N and $S \subset Div(\Gamma)$ a semigroup. If S has the property (C), then there exists $I \subset S$ finite with property (C).*

Proof. The maximal semigroups with interior points in N are the closed half spaces in $\mathfrak{n}$ bounded by a hyperplane containing the derived algebra $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$, (see [1], Corollary V.5.41). Since every proper subsemigroup with interior points in a topological group is contained in a maximal subsemigroup with interior points it is sufficient to consider the maximal semigroups with non-empty interior.

Let $\varphi_1, \varphi_2, \ldots, \varphi_l$ be a system of generators of the annihilator $[\mathfrak{n}, \mathfrak{n}]^\perp \subset \mathfrak{n}^*$ of $[\mathfrak{n}, \mathfrak{n}]$. For $1 \leq i \leq l$ and $j \in \{0, 1\}$ let

$$V_j^i = \{x \in \mathfrak{n} : (-1)^j \varphi_i(x) > 0\}$$

be the open half-spaces given by the linear functional φ_i. For $\sigma = \{j_1, j_2, \ldots, j_l\}$ such that $j_k \in \{0, 1\}$ we consider also the open cones

$$V_\sigma = \bigcap_{i=1}^{l} V_{j_i}^i.$$

By the above lemma we have that $\widetilde{S} = \mathfrak{n}$. Hence for each subset σ there are $a_1, a_2, \ldots, a_p \in S$ and non-negative numbers $t_1, t_2, \ldots, t_p$ such that

$$x_\sigma = t_1a_1 * t_2a_2 * \cdots * t_pa_p \in V_\sigma.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can take $t_1, t_2, \dots, t_N \in \mathbb{Q}$. Let $d \in \mathbb{N}$ be such that $dt_i \in \mathbb{N}$, $i = 1, \dots, p$, and consider the element

$$a_\sigma = dt_1 a_1 * dt_2 a_2 * \cdots * dt_p a_p$$

in S. We claim that $a_\sigma \in V_\sigma$. In fact, using the Campbell-Hausdorff formula for $\mathfrak{n}$, we get explicitly that

$$a_\sigma = \sum_{i=1}^{p} dt_i a_i + Y \text{ and } x_\sigma = \sum_{i=1}^{p} t_i a_i + Z$$

with $Y, Z \in [\mathfrak{n}, \mathfrak{n}]$. Therefore, if $1 \leq i \leq l$ we have that

$$0 < (-1)^{j_i} \varphi_{j_i}(x_\sigma) = (-1)^{j_i} \varphi_{j_i}(\sum_{k=1}^{p} t_k a_k),$$

and, then

$$0 < (-1)^{j_i} \varphi_{j_i}(\sum_{k=1}^{p} dt_k a_k) = (-1)^{j_i} \varphi_{j_i}(a_\sigma),$$

Therefore $a_\sigma \in V_{j_i}^i \cap S$ for $i = 1, 2, \dots, l$ and then $a_\sigma \in V_\sigma \cap S$.

Now let M be a maximal semigroup with interior points in N. We can assume without loss of generality that

$$M = \{x \in \mathfrak{n} : \lambda(x) \geq 0\},$$

where λ is a non zero linear functional which is identically zero in $[\mathfrak{n}, \mathfrak{n}]$. Let us suppose that

$$\lambda = b_1 \varphi_1 + b_2 \varphi_2 + \cdots + b_l \varphi_l$$

and let $\sigma = (j_1, j_2, ..., j_l)$ be such that

$$j_i = \begin{cases} 0 & if \quad b_i < 0 \\ 1 & if \quad b_i \geq 0. \end{cases}$$

Put $a_\sigma \in V_\sigma \cap S$. Then $b_i \varphi_i(a_\sigma) \leq 0$ for $1 \leq i \leq l$ and $\lambda(a_\sigma) < 0$ because $\varphi_i(a_\sigma) \neq 0$ and $\lambda \neq o$. This shows that S is not contained in M.

Let I be the subset of elements a_σ where $\sigma = (j_1, j_2, \dots, j_l)$ and $j_i \in \{0, 1\}$. Then I is a finite subset of S that property (C). ■

Theorem 3.4. *Let N be a connected and simply connected nilpotent Lie group, $\Gamma \subset G$ a lattice and $S \subset Div(\Gamma)$ a semigroup with property C. Then S is a group.*

Proof. Pick $a \in S$. By the previous proposition there are $a_1, a_2, \dots, a_l$ in S that have property (C). Let Ω be the semigroup generated by $a_1, a_2, \dots, a_l$ and a. By Proposition 2.6, Ω is a discrete semigroup of N. Therefore Ω satisfies the conditions of Theorem 4.1 in [7]. Hence Ω is a subgroup of N. This implies that $a^{-1} \in S$, showing that S is a group. ■

From this theorem we can get the maximal semigroups in a rational group. The proof of the next result is similar to that of Corollary 4.3 in [6].

Corollary 3.5. *Let Γ a lattice in a connected and simply connected nilpotent Lie group N. If S is a maximal subsemigroup of $Div(\Gamma)$, then $S = Div(\Gamma) \cap T$, where T is a maximal subsemigroup of N with non-void interior. Moreover, any semigroup of $Div(\Gamma)$, which is not a group, is contained in a maximal one.*

4. The non-simply connected case

Let N be a connected nilpotent Lie group, $\pi : \widetilde{N} \longrightarrow N$ be the universal covering of N and put $D = \ker(\pi)$, a discrete central subgroup of $\widetilde{N}$. If S is a semigroup of N with property (C) such that the identity of N is in S and $\overline{S}$ is a proper semigroup of $\widetilde{N}$, with non-void interior, which contains $\widetilde{S} = \pi^{-1}(S)$ then $D \subset \widetilde{S} \subset \overline{S}$ and $\pi(\overline{S})$ is a proper semigroup of N with non-void interior which contains S. Therfore $\pi^{-1}(S)$ also has the property (C).

Now, if Γ is a lattice in N, then $\widetilde{\Gamma} = \pi^{-1}(\Gamma)$ is a lattice in $\widetilde{N}$ and $D \subset \widetilde{\Gamma}$.

Corollary 4.1. *Let N be a connected nilpotent Lie group, $\Gamma \subset N$ a lattice. Let D be a divisible subgroup of N that contains Γ with $D \subset \pi(Div(\widetilde{\Gamma}))$. If $S \subset D$ is a semigroup which has the property (C) then S is a subgroup.*

Proof. Assume without loss of generality that S contains the identity of N. Since $D \subset \pi(Div(\widetilde{\Gamma}))$, $\pi^{-1}(D) \subset Div(\widetilde{\Gamma})$ and thus $\widetilde{S} = \pi^{-1}(S) \subset Div(\widetilde{\Gamma})$. But $\widetilde{S}$ has property (C). By Theorem 3.4, $\widetilde{S}$ is a group. Therefore $S = \pi(\widetilde{S})$ is a group. ∎

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