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On Penney's Cayley transform of a homogeneous Siegel domain

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Abstract. In this paper we introduce a Cayley transform $\mathcal{C}$ of a homogeneous Siegel domain D as a slight modification of Penney's one. We give an explicit formula to the inverse map of $\mathcal{C}$, and thus clarify the biholomorphic nature of $\mathcal{C}$ in a direct and visible manner. When D is quasisymmetric, our Cayley transform $\mathcal{C}$ is shown to be naturally coincident with Dorfmeister's one. A phenomenon which does not appear in the case of quasisymmetric domains is presented by an example in the last section.

Introduction

This paper deals with a Cayley transform $\mathcal{C}$ which maps a homogeneous Siegel domain D biholomorphically onto a bounded domain. Since Siegel domains are holomorphically equivalent to bounded domains, what actually concerns here is a canonical bounded model of $\mathcal{C}(D)$, by which we mean a realization that reduces, in an appropriate sense, to Harish-Chandra's realization (cf. [18, Chapter II] for example) if $\mathcal{C}(D)$ is symmetric. In this regard, Penney [14] has given a realization, named the Harish-Chandra realization, which is identifiable with the image of an explicitly defined Cayley transform of D. While his description of Cayley transform is quite pretty, we feel a lack of satisfaction on the following two points. First an explicit formula for the inverse Cayley transform is not given. As a result, a direct proof is missing for the fact that the Cayley transform actually maps the Siegel domain biholomorphically onto the Harish-Chandra realization, even though this biholomorphy can be conceptually convinced. The second point is that when D is quasisymmetric, the identification with Dorfmeister's Cayley transform [3] is not so obvious, although this is alluded in the Introduction of [14].

In this paper we modify a little the definition of Penney's Cayley transform to get rid of the above two dissatisfactions. Our Cayley transform still looks like a fractional linear transform, which requires a sort of denominator. Penney's denominator is given by the map $x \mapsto x^*$ considered by Vinberg in [19, §4], whereas ours comes from a function η that is related to the Bergman kernel of D, see section 2.1 of this paper for details. Use of this function η is not new and already found in Dorfmeister's study of Siegel domains [4]. To be precise,

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let Ω be a regular open convex cone in a real vector space V. Then our inverse $\mathcal{I}(x)$ of an element $x \in \Omega$ is $-\nabla \log \eta(x)$, minus of the gradient of $\log \eta(x)$, and we shall call the map $\mathcal{I} : x \mapsto \mathcal{I}(x)$ the *pseudoinverse map*. One knows by [4] that $\mathcal{I}$ is a rational map and gives a diffeomorphism of Ω onto the dual cone Ω^* in V^*. Since η is relatively invariant under the simple transitive solvable Lie group acting on Ω, we can introduce its dual counterpart η^* easily by dualizing the relative invariance. The function η^* induces another diffeomorphic rational pseudoinverse map $\mathcal{I}^* : \Omega^* \to \Omega$, and we have $\mathcal{I}^* = \mathcal{I}^{-1}$, so that $\mathcal{I}$ is birational. Then we continue $\mathcal{I}$ (resp. $\mathcal{I}^*$) analytically to $W := V_{\mathbb{C}}$ (resp. to W^*) and show that $\mathcal{I}$ (resp. $\mathcal{I}^*$) is holomorphic on the tube domain $\Omega + iV$ (resp. $\Omega^* + iV^*$). What is more important to our purpose is to show that the image $\mathcal{I}(\Omega + iV)$ (resp. $\mathcal{I}^*(\Omega^* + iV^*)$) is contained in the holomorphic domain of $\mathcal{I}^*$ (resp. $\mathcal{I}$). This is done in Theorem 2.11. It should be noted that in general we do *not* have $\mathcal{I}(\Omega + iV) \subset \Omega^* + iV^*$ unlike the case of symmetric tube domains. An example of this failure is given in section 5.

Now with a fixed specific element $E \in \Omega$, our Cayley transform C of the tube domain $\Omega + iV$ is defined to be $C(w) = \mathcal{I}(E) - 2\mathcal{I}(w + E)$. We refer the reader to the formula (29) in this paper for the definition of our Cayley transform $\mathcal{C}$ for a general Siegel domain D. Penney's proof [14] that the Cayley image of D is bounded still works for $\mathcal{C}(D)$ with a suitable minor modification. The explicit inverse maps B and $\mathcal{B}$ of C and $\mathcal{C}$ will be given in (33) and (34) respectively, and the biholomorphy of $\mathcal{C} : D \to \mathcal{C}(D)$ is visibly clarified.

In section 4, we compare our $\mathcal{C}$ with Dorfmeister's Cayley transform when D is quasisymmetric. We will describe the relevant Jordan algebra structure of V in terms of the normal j-algebra structure with which we begin this paper. Some constancy of dimensions of root spaces equates Vinberg's x^* with our $\mathcal{I}(x)$ up to a constant multiple. By using a known property of x^* in the case of selfdual cone, $\mathcal{I}(x)$ is naturally identified with the Jordan algebra inverse x^{-1} of x. Another technical point is to show that the linear map $\varphi : w \mapsto \varphi(w)$ appearing in the definition (34) of $\mathcal{B} = \mathcal{C}^{-1}$ is a Jordan algebra representation of W. To carry out this, we shall express $\varphi(x)$ for $x \in V$ in our language of normal j-algebra, and provide some details in order that the reader can trace a quotation from Dorfmeister's works without difficulties. Our Theorem 4.10 shows that the Cayley transform $\mathcal{C}$ is coincident in a canonical way with Dorfmeister's.

The Cayley transform $\mathcal{C}$ defined in this paper will play a fundamental role in the forthcoming paper [13] for a characterization of symmetric Siegel domains.

1. Preliminaries

1.1. Normal j-algebras. As is known by the work of Pjatetskii-Shapiro [15] (see also [16], [17]), homogeneous Siegel domains are described by means of normal j-algebras. Thus we begin this paper with the definition of normal j-algebra. A *normal j-algebra* is a triple $(\mathfrak{g}, J, \omega)$ of a split solvable Lie algebra $\mathfrak{g}$, a linear operator[1] J on $\mathfrak{g}$ such that $J^2 = -I$ and a linear form ω on $\mathfrak{g}$ satisfying

$$[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad (\text{for all } x, y \in \mathfrak{g}), \tag{1}$$

$$\langle\, x \mid y \,\rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines a } J\text{-invariant inner product on } \mathfrak{g}. \tag{2}$$

We summarize here some of the fundamental facts about normal j-algebras following [15] and [17] (see also [16]). Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. Let $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$. Moreover, $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{a})$ consists of semisimple operators on $\mathfrak{g}$. For every $\alpha \in \mathfrak{a}^*$ we set

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n}\,;\ [h, x] = \langle h, \alpha \rangle\, x \quad \text{for all } h \in \mathfrak{a}\}.$$

Take all $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{n}_\alpha \neq \{0\}$ and $J\mathfrak{n}_\alpha \subset \mathfrak{a}$, and number them as $\alpha_1, \ldots, \alpha_r$. We have $\dim \mathfrak{a} = r$ and $\dim \mathfrak{n}_{\alpha_k} = 1$ for every k. The number r is called the *rank* of the normal j-algebra $\mathfrak{g}$. We can reorder $\alpha_1, \ldots, \alpha_r$, if necessary, so that all the α such that $\mathfrak{n}_\alpha \neq \{0\}$ (such an α is called a *root* of the normal j-algebra) are of the following form (not all possibilities need occur):

$$\begin{array}{llll} \tfrac{1}{2}(\alpha_m + \alpha_k) & (1 \leqq k < m \leqq r), & \tfrac{1}{2}(\alpha_m - \alpha_k) & (1 \leqq k < m \leqq r), \\ \tfrac{1}{2}\alpha_k & (1 \leqq k \leqq r), & \alpha_k & (1 \leqq k \leqq r). \end{array} \tag{3}$$

We note that if α, β are distinct roots, then $\mathfrak{n}_\alpha$ is orthogonal to $\mathfrak{n}_\beta$. Put

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}, \qquad \mathfrak{g}(1/2) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i/2},$$
$$\mathfrak{g}(1) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m + \alpha_k)/2}.$$

Understanding $\mathfrak{g}(i) = 0$ for $i > 1$, we have $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i + j)$. Moreover

$$J\mathfrak{n}_{(\alpha_m - \alpha_k)/2} = \mathfrak{n}_{(\alpha_m + \alpha_k)/2} \quad (m > k), \qquad J\mathfrak{n}_{\alpha_i/2} = \mathfrak{n}_{\alpha_i/2} \quad (1 \leqq i \leqq r), \tag{4}$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$ and $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. Taking $E_i \in \mathfrak{n}_{\alpha_i}$ $(i = 1, \ldots, r)$ such that $\alpha_k(JE_i) = \delta_{ki}$, we put $H_i := JE_i \in \mathfrak{a}$. Then $H_1, \ldots, H_r$ form a basis of $\mathfrak{a}$. Set

$$H := H_1 + \cdots + H_r, \qquad E := E_1 + \cdots + E_r. \tag{5}$$

[1] We prefer to write J rather than the common usage j to emphasize that it is an operator. However, we do not call the triple $(\mathfrak{g}, J, \omega)$ a normal J-algebra, because this might cause a serious confusion with a Jordan algebra.

We remark here that the action of J on the elements of $\mathfrak{g}(0)$ is described as

$$JT = -[T, E] \qquad (T \in \mathfrak{g}(0)). \tag{6}$$

We write down here the constants used frequently in this paper:

$$\begin{aligned} n_{mk} &:= \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m - \alpha_k)/2} = \dim_{\mathbb{R}} \mathfrak{n}_{(\alpha_m + \alpha_k)/2} \quad (1 \leqq k < m \leqq r), \\ b_i &:= \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{n}_{\alpha_i/2} \quad (1 \leqq i \leqq r), \\ d_j &:= 1 + \frac{1}{2}\Big(\sum_{k>j} n_{kj} + \sum_{i<j} n_{ji}\Big) \qquad (1 \leqq j \leqq r), \\ \omega_k &:= \langle E_k, \omega \rangle = \|E_k\|_\omega^2 > 0 \qquad (1 \leqq k \leqq r). \end{aligned} \tag{7}$$

1.2. Homogeneous Siegel domains. Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra, and $G = \exp \mathfrak{g}$ the connected and simply connected Lie group corresponding to $\mathfrak{g}$. Since $\mathfrak{g}(0)$ is a Lie subalgebra of $\mathfrak{g}$, we denote by $G(0)$ the corresponding subgroup $\exp \mathfrak{g}(0)$ of G. By 1.1, we know that $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. Recall $E \in V$ in (5) and let Ω be the $G(0)$-orbit through E. By [17, Theorem 4.15] Ω is a regular open convex cone in V, and $G(0)$ acts on Ω simply transitively. By (4) the subspace $\mathfrak{g}(1/2)$ is invariant under J, so that it is considered as a *complex* vector space by means of $-J$. We shall write this complex vector space by U. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation of W relative to the real form V is written as $w \mapsto w^*$. The real bilinear map Q defined by

$$Q(u, u') := \frac{1}{2}\big([Ju, u'] - i[u, u']\big) \qquad (u, u' \in \mathfrak{g}(1/2)) \tag{8}$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map $U \times U \to W$ which is Ω-positive. This means that

$$Q(u', u) = Q(u, u')^* \ \ (u, u' \in U), \quad Q(u, u) \in \overline{\Omega} \setminus \{0\} \ \text{ for all } u \in U \setminus \{0\}.$$

With these data we define the *Siegel domain* corresponding to the normal j-algebra $(\mathfrak{g}, J, \omega)$ to be

$$D := \{(u, w) \in U \times W \ ; \ w + w^* - Q(u, u) \in \Omega\}. \tag{9}$$

Note that we take a generalized *right* half plane rather than a more familiar upper half plane.

Consider the Lie subalgebra $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is at most 2-step nilpotent by 1.1. Let $N_D = \exp \mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in G. Writing the elements of N_D by $n(a, b)$ $(a \in \mathfrak{g}(1),\ b \in \mathfrak{g}(1/2))$, we see by the Campbell-Hausdorff formula that the group operation is described as (with Q as in (8))

$$n(a, b)n(a', b') = n(a + a' - Q_I(b, b'),\ b + b'), \tag{10}$$

where Q_I denotes the real alternating bilinear form on $\mathfrak{g}(1/2) \times \mathfrak{g}(1/2)$ defined by $Q_I(b, b') := \operatorname{Im} Q(b, b')$. The group N_D acts on D by

$$n(a, b) \cdot (u, w) = \big(u + b,\ w + ia + \tfrac{1}{2}Q(b, b) + Q(u, b)\big) \qquad ((u, w) \in D). \tag{11}$$

Concerning the action of $G(0)$ on D, we first note that by a simple observation using (1) the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$ commutes with J. This implies that $G(0)$ acts on U complex-linearly. On the other hand the adjoint action of $G(0)$ on $V = \mathfrak{g}(1)$ extends complex-linearly to W. Thus $G(0)$ acts on D complex-linearly, so that $G = N_D \rtimes G(0)$ acts on D simply transitively. Put $\mathsf{e} := (0, E) \in D$. Given $z = (u, w) \in D$, we can find a unique $h \in G(0)$ satisfying $hE = \operatorname{Re} w - Q(u,u)/2$. Then taking $n = n(\operatorname{Im} w, u) \in N_D$, we see by (11) that $z = nh \cdot \mathsf{e}$. This makes explicit the simple transitive action of G on D.

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ we put

$$\chi_{\mathbf{s}}\Big(\exp \sum_k t_k H_k\Big) = \exp\Big(\sum_k s_k t_k\Big) \qquad (t_1, \ldots, t_r \in \mathbb{R}). \tag{12}$$

Then $\chi_{\mathbf{s}}$ is a one-dimensional representation of $A := \exp \mathfrak{a}$. On the other hand, it is clear that

$$\mathfrak{n}_0 := \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}$$

is a nilpotent Lie subalgebra of $\mathfrak{g}(0)$, and we have $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_D$. Let $N_0 := \exp \mathfrak{n}_0$ and $N := \exp \mathfrak{n}$. It is also clear that $G = N \rtimes A$ and $G(0) = N_0 \rtimes A$. We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let us define functions $\Delta_{\mathbf{s}}$ ($\mathbf{s} \in \mathbb{C}^r$) on Ω by

$$\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h) \qquad (h \in G(0)). \tag{13}$$

Evidently it holds that

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \qquad (h \in G(0),\ x \in \Omega). \tag{14}$$

In particular, putting $h = \exp tH \in A$ with $t = \log \lambda$ ($\lambda > 0$), we see that $\Delta_{\mathbf{s}}(\lambda x) = \lambda^{|\mathbf{s}|}\Delta_{\mathbf{s}}(x)$, where $|\mathbf{s}| := s_1 + \cdots + s_r$ for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. Furthermore, we know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [10, Corollary 2.5]).

For $h \in G(0)$, let $\operatorname{Ad}_{\mathfrak{g}(k)}(h) := (\operatorname{Ad} h)|_{\mathfrak{g}(k)}$ for $k = 1/2, 1$. Similarly we define $\operatorname{ad}_{\mathfrak{g}(k)}(T) = (\operatorname{ad} T)|_{\mathfrak{g}(k)}$ for $T \in \mathfrak{g}(0)$.

Lemma 1.1. *Let* $\mathbf{d} := (d_1, \ldots, d_r)$ *with* d_j *as in* (7). *One has*

$$\det \operatorname{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h) \qquad (h \in G(0)).$$

Proof. Since N_0 acts on V unipotently and since $G(0) = N_0 \rtimes A$, it is enough to show the lemma for $h = \exp H_j$ for $j = 1, \ldots, r$ by the definition of $\chi_{\mathbf{s}}$. Now

$$\det \operatorname{Ad}_{\mathfrak{g}(1)}(\exp H_j) = \det\big(\exp \operatorname{ad}_{\mathfrak{g}(1)}(H_j)\big) = \exp\big(\operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1)}(H_j)\big).$$

Since $\operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1)}(H_j) = d_j$ by virtue of (3) and (7), the proof is complete. ∎

Let $\operatorname{Ad}_U(h)$ ($h \in G(0)$) stand for the *complex* linear operator on U defined by the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$, and $\det \operatorname{Ad}_U(h)$ its determinant as a complex linear operator. Then, putting $\mathbf{b} := (b_1, \ldots, b_r)$ with b_j as in (7), we have

Lemma 1.2. $|\det \mathrm{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h)$ *for* $h \in G(0)$.

Proof. The lemma is clear from the fact that $|\det \mathrm{Ad}_U(h)|^2 = \det \mathrm{Ad}_{\mathfrak{g}(1/2)}(h)$ and $\mathrm{tr}\,\mathrm{ad}_{\mathfrak{g}(1/2)}(H_i) = 2b_i \times (1/2) = b_i$. The details are similar to the proof of Lemma 1.1. ■

2. Pseudoinverse map

2.1. Introduction of the pseudoinverse map. Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra and D the corresponding Siegel domain (see (9)). By [9, §5] or [18, §II.6], it is known that D has a Bergman kernel κ. The function κ has the following property: if $\mathrm{Hol}\,(D)$ denotes the Lie group of the holomorphic automorphisms of D, then

$$\kappa(z_1, z_2) = \kappa(g \cdot z_1,\, g \cdot z_2) \det g'(z_1)\, \overline{\det g'(z_2)} \quad (g \in \mathrm{Hol}\,(D),\ z_1, z_2 \in D), \tag{15}$$

where $g'(z)$ is the complex Jacobian map of g at $z \in D$. The discussion in 1.2 of the simple transitive action of G on D together with the property (15) and Lemmas 1.1, 1.2 shows that

$$\kappa(z_1, z_2) = C \cdot \Delta_{-2\mathbf{d}-\mathbf{b}}\big(w_1 + w_2^* - Q(u_1, u_2)\big) \qquad (z_j = (u_j, w_j) \in D) \tag{16}$$

with $C = \kappa(\mathsf{e}, \mathsf{e})\Delta_{2\mathbf{d}+\mathbf{b}}(2E) > 0$.

We put $\eta := \Delta_{-2\mathbf{d}-\mathbf{b}}$ in what follows for simplicity. Let D_v be the directional derivative in the direction $v \in V$:

$$D_v f(x) = \frac{d}{dt} f(x + tv)\Big|_{t=0}.$$

For every $x \in \Omega$ we define $\mathcal{I}(x) \in V^*$ to be $-\nabla \log \eta(x)$, that is,

$$\langle v, \mathcal{I}(x)\rangle = -D_v \log \eta(x) \qquad (v \in V). \tag{17}$$

$\mathcal{I}$ is called the *pseudoinverse map.* By [4, §2], $\mathcal{I}$ gives a diffeomorphism of Ω onto the dual cone Ω^* in V^*, where

$$\Omega^* := \big\{\xi \in V^*\ ;\ \langle x, \xi\rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}\big\}.$$

Now the group $G(0)$ acts on V^* by the coadjoint action: $h \cdot \xi = \xi \circ h^{-1}$, where $h \in G(0)$ and $\xi \in V^*$. It is easy to show by using (14) that $\mathcal{I}$ is $G(0)$-equivariant:

$$\mathcal{I}(hx) = h \cdot \mathcal{I}(x) \qquad (h \in G(0),\ x \in \Omega). \tag{18}$$

In particular, $\mathcal{I}(\lambda x) = \lambda^{-1}\mathcal{I}(x)$ for all $\lambda > 0$, and $G(0)$ acts on Ω^* simply transitively.

In order to write down the image $\mathcal{I}(E)$ of E, we define $E_1^*, \ldots, E_r^* \in V^*$ by

$$\Big\langle \sum_{j=1}^{r} x_j E_j + \sum_{m>k} X_{mk},\ E_i^* \Big\rangle = x_i \qquad (x_j \in \mathbb{R},\ X_{mk} \in \mathfrak{n}_{(\alpha_m+\alpha_k)/2}).$$

Elements of V^* are canonically considered as elements of W^*, the space of complex linear forms on W. On the other hand, we extend every α_j to a linear form on $\mathfrak{g}(0) = \mathfrak{a} + \mathfrak{n}_0$ by setting $\langle T, \alpha_j\rangle = 0$ for $T \in \mathfrak{n}_0$, and then to a complex linear form on $\mathfrak{g}(0)_{\mathbb{C}}$ naturally. Now for every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ we set

$$\alpha_{\mathbf{s}} := s_1\alpha_1 + \cdots + s_r\alpha_r \in \mathfrak{g}(0)_{\mathbb{C}}^*, \qquad E_{\mathbf{s}}^* := s_1E_1^* + \cdots + s_rE_r^* \in W^*.$$

Clearly $\chi_{\mathbf{s}}(\exp T) = \exp \alpha_{\mathbf{s}}(T)$ $(T \in \mathfrak{g}(0))$ and $\langle Jv, \alpha_{\mathbf{s}}\rangle = \langle v, E_{\mathbf{s}}^*\rangle$ $(v \in V)$.

Lemma 2.1. (1) *Let* $\mathbf{s} \in \mathbb{C}^r$. *Then,* $D_v \Delta_{\mathbf{s}}(E) = \langle v, E^*_{\mathbf{s}} \rangle$ *for every* $v \in V$.
(2) *One has* $\mathcal{I}(E) = E^*_{\mathbf{2d+b}}$.

Proof. (1) Let $v \in V = \mathfrak{g}(1)$ and consider $T = Jv \in \mathfrak{g}(0)$. Then, by (6) we have

$$(\mathrm{Ad}\,\exp tT)E = E + t[T, E] + O(t^2) = E + tv + O(t^2) \qquad (t \in \mathbb{R}). \tag{19}$$

Hence we get by (13)

$$\begin{aligned} D_v \Delta_{\mathbf{s}}(E) &= \frac{d}{dt}\, \Delta_{\mathbf{s}}\big((\mathrm{Ad}\,(\exp tT))E\big)\Big|_{t=0} = \frac{d}{dt}\, \chi_{\mathbf{s}}(\exp tT)\Big|_{t=0} \\ &= \langle T, \alpha_{\mathbf{s}} \rangle = \langle v, E^*_{\mathbf{s}} \rangle. \end{aligned}$$

(2) By (1) we have for $v \in V$

$$\langle v, \mathcal{I}(E) \rangle = -\frac{1}{\Delta_{-\mathbf{2d-b}}(E)}\, D_v \Delta_{-\mathbf{2d-b}}(E) = \langle v, E^*_{\mathbf{2d+b}} \rangle,$$

which completes the proof. ∎

We note here that

$$\langle\, v_1 \,|\, v_2 \,\rangle_\eta := D_{v_1} D_{v_2} \log \eta(E) \qquad (v_1, v_2 \in V) \tag{20}$$

defines an inner product on V (see [4, §2]). Let us write this inner product in a more concrete way.

Lemma 2.2. $\langle\, v_1 \,|\, v_2 \,\rangle_\eta = \langle [Jv_1, v_2], E^*_{\mathbf{2d+b}} \rangle$ *for all* $v_1, v_2 \in V$.

Proof. It is enough to show the equality for $v_1 = v_2 = v$ by polarization. By definition we have

$$\langle\, v \,|\, v \,\rangle_\eta = D_v^2 \log \eta(E) = -\frac{d}{dt}\, \langle v,\, \mathcal{I}(E + tv) \rangle\Big|_{t=0}.$$

By (18) and (19) it holds that

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(E + tv)\Big|_{t=0} &= \frac{d}{dt}\mathcal{I}\big((\mathrm{Ad}\,\exp tJv)E\big)\Big|_{t=0} = (\mathrm{ad}^* Jv)\big(\mathcal{I}(E)\big) \\ &= -\langle [Jv, \cdot],\, \mathcal{I}(E) \rangle. \end{aligned}$$

Since $\mathcal{I}(E) = E^*_{\mathbf{2d+b}}$ by Lemma 2.1, the proof is complete. ∎

For every $f \in V^*$, we denote by $\widetilde{f}$ the element in V determined by

$$\langle v, f \rangle = \langle\, v \,|\, \widetilde{f}\, \rangle_\eta \qquad (\text{for all } v \in V). \tag{21}$$

For $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{C}^r$ and $\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{C}^r$ such that none of c_j is zero, we put $\mathbf{a}/\mathbf{c} = (a_1/c_1, \ldots, a_r/c_r)$. Then

Lemma 2.3. *One has* $(E^*_{\mathbf{s}})^{\sim} = E_{\mathbf{s}/(\mathbf{2d+b})}$ *for any* $s \in \mathbb{R}^r$.

Proof. Note that by (3), (4) and Lemma 2.2, the root spaces $\mathfrak{n}_{(\alpha_m+\alpha_k)/2}$ for $m \geqq k$ are still orthogonal relative to the inner product $\langle \cdot \,|\, \cdot \rangle_\eta$. Thus $(E^*_{\mathbf{s}})^{\sim}$ is a linear combination of $E_1, \ldots, E_r$, and it suffices to prove the proposition for the linear forms E^*_j $(j = 1, 2, \ldots, r)$. But this is clear from $\langle E_i \,|\, E_j \rangle_\eta = \delta_{ij}(2d_j + b_j)$, which is seen by Lemma 2.2. ■

2.2. Birationality of $\mathcal{I}$. The fact that $\mathcal{I}$ is a birational map is stated in the paper Dorfmeister [4, Lemma 2.5]. However, for our later use, we would like to make the matters more explicit here by using the normal j-algebra structure. To see first that $\mathcal{I}$ is a rational map, we introduce a (non-associative) product $\star$ in V by

$$v_1 \star v_2 := [Jv_1, v_2] = \big(\mathrm{ad}\,(Jv_1)\big)v_2 \qquad (v_1, v_2 \in V). \tag{22}$$

Note that by (6), the map $v \mapsto Jv$ is just an inverse map to the linear isomorphism $\mathfrak{g}(0) \ni T \to [T, E] \in \mathfrak{g}(1)$, which is the differential of the orbit map $G(0) \ni h \mapsto hE \in \Omega$. We shall write $R_J(v_2)v_1 = v_1 \star v_2$. Then, $R_J(E)v = [Jv, E] = v$ and $R_J(E) = I$. Therefore $v \mapsto \det R_J(v)$ is a non-zero polynomial function on V. Hence the subset $\mathcal{O} := \{v \in V \,;\, \det R_J(v) \neq 0\}$ is a non-empty Zariski-open set. Although the following lemma is a simple translation of [5, Satz I.3.3] into the present context, we write down a proof for reader's convenience.

Lemma 2.4. *If $v \in \mathcal{O}$, then $\mathcal{I}(v) = E^*_{\mathbf{2d+b}} \circ R_J(v)^{-1}$, so that $\mathcal{I}$ is a rational map.*

Proof. By the definition (22) of $\star$, we have for $v_1, v_2 \in V$ and $t \in \mathbb{R}$

$$\big(\mathrm{Ad}\,\exp tJv_1\big)v_2 = v_2 + tv_1 \star v_2 + O(t^2).$$

This together with (14) and (19) gives

$$\begin{aligned} D_{v_1 \star v_2}(\log\eta)(v_2) &= \frac{d}{dt}\,(\log\eta)\big((\mathrm{Ad}\,\exp tJv_1)v_2\big)\Big|_{t=0} \\ &= \frac{d}{dt}\,(\log\eta)\big((\mathrm{Ad}\,\exp tJv_1)E\big)\Big|_{t=0} = D_{v_1}(\log\eta)(E). \end{aligned}$$

This implies $\langle v_1 \star v_2, \mathcal{I}(v_2)\rangle = \langle v_1, \mathcal{I}(E)\rangle$ by the definition (17) of $\mathcal{I}$. Now we get the lemma immediately from (2) of Lemma 2.1. ■

In order to find an inverse map of $\mathcal{I}$, we need to dualize the matters concerning $\mathcal{I}$. First for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, we put $\mathbf{s}^* := (s_r, \ldots, s_1)$ and set

$$\chi^*_{\mathbf{s}} := \chi_{-\mathbf{s}^*}, \qquad \Delta^*_{\mathbf{s}}(h \cdot E^*_{\mathbf{2d+b}}) := \chi^*_{\mathbf{s}}(h) \quad (h \in G(0)).$$

$\Delta^*_{\mathbf{s}}$ is a function on Ω^* such that $\Delta^*_{\mathbf{s}}(h \cdot \xi) = \chi^*_{\mathbf{s}}(h)\Delta^*_{\mathbf{s}}(\xi)$ for $h \in G(0)$ and $\xi \in V^*$. We define $\eta^* := \Delta^*_{-\mathbf{2d}^*-\mathbf{b}^*}$ and

$$\langle \mathcal{I}^*(\xi), f\rangle := -D_f \log\eta^*(\xi) \qquad (\xi \in \Omega^*,\ f \in V^*).$$

Thus $\mathcal{I}^*(\xi) \in V$ and $\mathcal{I}^*$ gives a diffeomorphism of Ω^* onto Ω. Moreover, $\mathcal{I}^*$ is $G(0)$-equivariant, that is, $\mathcal{I}^*(h \cdot \xi) = h(\mathcal{I}^*(\xi))$ for any $h \in G(0)$. Considering the inverse map of the linear isomorphism $\mathfrak{g}(0) \ni T \mapsto (\mathrm{ad}^* T)f \in V^*$, we see that $\mathcal{I}^*$ is a rational map on V^*. The following is the counterpart of Lemma 2.1.

Lemma 2.5. (1) *Let* $\mathbf{s} \in \mathbb{C}^r$. *Then for every* $f \in V^*$ *one has*

$$D_f \Delta^*_{\mathbf{s}}(E^*_{2\mathbf{d}+\mathbf{b}}) = \langle E_{\mathbf{s}^*/(2\mathbf{d}+\mathbf{b})}, f \rangle.$$

(2) $\mathcal{I}^*(E^*_{2\mathbf{d}+\mathbf{b}}) = E$.

Proof. Given $f \in V^*$, we consider the element $T := J\widetilde{f} \in \mathfrak{g}(0)$. We have

$$\langle v, \mathrm{Ad}^*(\exp -tT)E^*_{2\mathbf{d}+\mathbf{b}} \rangle = \langle v, E^*_{2\mathbf{d}+\mathbf{b}} \rangle + t \langle [J\widetilde{f}, v], E^*_{2\mathbf{d}+\mathbf{b}} \rangle + O(t^2) \quad (v \in V).$$

Since the definition (21) of $\widetilde{f}$ and Lemma 2.2 give $\langle [J\widetilde{f}, v], E^*_{2\mathbf{d}+\mathbf{b}} \rangle = \langle v, f \rangle$, we get

$$D_f \Delta^*_{\mathbf{s}}(E^*_{2\mathbf{d}+\mathbf{b}}) = \frac{d}{dt} \Delta^*_{\mathbf{s}}\big(\mathrm{Ad}^*(\exp -tT)E^*_{2\mathbf{d}+\mathbf{b}}\big)\Big|_{t=0} = \langle T, \alpha_{\mathbf{s}^*} \rangle = \langle \widetilde{f}, E^*_{\mathbf{s}^*} \rangle.$$

Since $\langle \widetilde{f}, E^*_{\mathbf{s}^*} \rangle = \langle E_{\mathbf{s}^*/(2\mathbf{d}+\mathbf{b})}, f \rangle$ by (21) and Lemma 2.3, the assertion (1) follows. The proof of (2) is left to the reader. ∎

Proposition 2.6. $\mathcal{I}^*(\mathcal{I}(x)) = x$ *for any* $x \in \Omega$, *and* $\mathcal{I}(\mathcal{I}^*(\xi)) = \xi$ *for any* $\xi \in \Omega^*$.

Proof. This is a direct consequence of the formulas $\mathcal{I}(E) = E^*_{2\mathbf{d}+\mathbf{b}}$ in (2) of Lemma 2.1 and $\mathcal{I}^*(E^*_{2\mathbf{d}+\mathbf{b}}) = E$ in (2) of Lemma 2.5 together with the fact that both $\mathcal{I}$ and $\mathcal{I}^*$ are $G(0)$-equivariant. ∎

By analytic continuation we have thus shown that $\mathcal{I}$ is a birational map $W \to W^*$ with $\mathcal{I}^{-1} = \mathcal{I}^*$.

2.3. Holomorphy of $\mathcal{I}$ on $\Omega + iV$. We now consider the complexification $G_{\mathbb{C}}$ of G as in Penney [14]. Since G is split solvable and of trivial center, we can think of G as a triangular linear group sitting inside $GL(\mathfrak{g})$ by passing to $\mathrm{Ad}(G)$. Thus the complexification $G_{\mathbb{C}}$ of G is inside $GL(\mathfrak{g}_{\mathbb{C}})$. Furthermore we consider the complexification $G(0)_{\mathbb{C}}$, $(N_D)_{\mathbb{C}}$, so that $G_{\mathbb{C}} = (N_D)_{\mathbb{C}} \rtimes G(0)_{\mathbb{C}}$. Here $N(D)_{\mathbb{C}}$ is the complex nilpotent Lie group with group law described like (10), where $a, a' \in \mathfrak{g}(1)_{\mathbb{C}}$ and $b, b' \in \mathfrak{g}(1/2)_{\mathbb{C}}$, and Q_I is extended by complex bilinearity. The complex group $G(0)_{\mathbb{C}}$ acts on $W = V_{\mathbb{C}}$ by adjoint action. On the other hand, the product $\star$ in V introduced by (22) extends to W by complex bilinearity, where J is continued to a complex linear operator on $\mathfrak{g}_{\mathbb{C}}$. Let $R_J(w)$ be the right multiplication by $w \in W$. We put $\mathcal{O}_{\mathbb{C}} := \{w \in W \; ; \; \det R_J(w) \neq 0\}$. By Lemma 2.4 we have $\mathcal{I}(w) = E^*_{2\mathbf{d}+\mathbf{b}} \circ R_J(w)^{-1}$ for $w \in \mathcal{O}_{\mathbb{C}}$. so that $\mathcal{I}$ is holomorphic on $\mathcal{O}_{\mathbb{C}}$.

Lemma 2.7. *If* $h \in G(0)_{\mathbb{C}}$, *then* $hE \in \mathcal{O}_{\mathbb{C}}$. *In particular,* $\mathcal{I}$ *is holomorphic at* hE.

Proof. Let $w \in W$. By definition we have

$$R_J(hE)w = w \star hE = [Jw, hE] = h[h^{-1}Jw, E] = -hJh^{-1}Jw,$$

where the last equality follows from (6). Hence we get $R_J(hE) = -\mathrm{Ad}_{\mathfrak{g}(1)_{\mathbb{C}}}(h) \circ J \circ \mathrm{Ad}_{\mathfrak{g}(0)_{\mathbb{C}}}(h^{-1}) \circ J$, so that $\det R_J(hE) \neq 0$. ∎

Let $\mathcal{O}^*_{\mathbb{C}}$ be the open subset in W^* defined similarly to $\mathcal{O}_{\mathbb{C}}$. Then $\mathcal{I}^*$ is holomorphic on $\mathcal{O}^*_{\mathbb{C}}$. Moreover, if $h \in G(0)_{\mathbb{C}}$, then we have $h \cdot E^*_{2\mathbf{d}+\mathbf{b}} \in \mathcal{O}^*_{\mathbb{C}}$ just in the same way as above. Hence $\mathcal{I}^*$ is holomorphic at $h \cdot E^*_{2\mathbf{d}+\mathbf{b}}$.

Proposition 2.8. *For every $y \in V$ there exists $h \in G(0)_{\mathbb{C}}$ such that $hE = E + iy$.*

To prove the proposition, we need some more structural description of our normal j-algebra. Put

$$\mathfrak{b} := \mathbb{R}H_r \oplus \mathbb{R}E_r \oplus \mathfrak{n}_{\alpha_r/2} \oplus \sum_{k=1}^{r-1} \mathfrak{n}_{(\alpha_r-\alpha_k)/2} \oplus \sum_{k=1}^{r-1} \mathfrak{n}_{(\alpha_r+\alpha_k)/2},$$

$$\mathfrak{g}' := \sum_{k=1}^{r-1} \mathbb{R}H_k \oplus \sum_{k=1}^{r-1} \mathbb{R}E_k \oplus \sum_{k=1}^{r-1} \mathfrak{n}_{\alpha_k/2} \oplus \sum_{k<m<r} \mathfrak{n}_{(\alpha_m-\alpha_k)/2} \oplus \sum_{k<m<r} \mathfrak{n}_{(\alpha_m+\alpha_k)/2}.$$

Then $\mathfrak{b}$ (resp. $\mathfrak{g}'$) itself is a rank 1 (resp. rank $r-1$) normal j-algebra, and we have a J-invariant decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{g}'$. Moreover, it is easy to see that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. We also note that

$$[\mathfrak{g}', H_r] = \{0\}, \qquad [\mathfrak{g}', E_r] = \{0\}, \tag{23}$$

where the latter follows from (3). Let $\mathfrak{g}'(j)$ ($j = 0, 1/2, 1$) be the j-eigenspace of $\operatorname{ad}(H_1 + \cdots + H_{r-1})$ on $\mathfrak{g}'$, and put

$$\mathfrak{m} := \sum_{k=1}^{r-1} \mathfrak{n}_{(\alpha_r+\alpha_k)/2}.$$

Clearly we have $\mathfrak{g}(0) = \mathfrak{g}'(0) \oplus \mathbb{R}H_r \oplus J\mathfrak{m}$ and $\mathfrak{g}(1) = \mathfrak{g}'(1) \oplus \mathbb{R}E_r \oplus \mathfrak{m}$. Moreover, it is evident that both $\mathfrak{m}$ and $J\mathfrak{m}$ are invariant under $G'(0) := \exp \mathfrak{g}'(0)$.

Now we are ready to begin the proof of Proposition 2.8.

Proof. The proof is done by induction on r. When $r = 1$, then $\mathfrak{g}(0) = \mathbb{R}H$ and $\mathfrak{g}(1) = \mathbb{R}E$ with $[H, E] = E$. Thus, given $(1+iy)E$ ($y \in \mathbb{R}$), we have only to take $h = \exp(\log(1+iy))H \in G(0)_{\mathbb{C}}$.

Now suppose that the lemma is true for $r-1$. Given $y \in V$, we write $y = y' + y_r E_r + z$ with $y' \in \mathfrak{g}'(1)$, $y_r \in \mathbb{R}$ and $z \in \mathfrak{m}$. Setting $h = h'(\exp T)(\exp \alpha H_r)$ with $h' \in G'(0)_{\mathbb{C}}$, $T \in J\mathfrak{m}_{\mathbb{C}}$ and $\alpha \in \mathbb{C}$, we shall look for h', T and α such that $hE = E + iy$.

Let $E' = E_1 + \cdots + E_{r-1}$. We have $hE = h'(\exp T)(E' + e^{\alpha}E_r)$. Since $(\exp T)E_r = E_r$ and

$$(\exp T)E' = E' + [T, E'] + \frac{1}{2}[T, [T, E']] = E' - JT - \frac{1}{2}[T, JT],$$

we get

$$hE = h'E' - h'(JT) + \Big(e^{\alpha} + \frac{1}{2}\langle [JT, T], E_r^* \rangle\Big) E_r,$$

where note $[JT, T] \in \mathbb{C}E_r$ and (23). Therefore, solving $hE = E + iy$ is equivalent to finding h', T, α such that

$$h'E' = E' + iy', \tag{24}$$

$$-h'(JT) = iz, \tag{25}$$

$$e^{\alpha} + \frac{1}{2}\langle [JT, T], E_r^* \rangle = 1 + iy_r. \tag{26}$$

Now by induction hypothesis, we find $h' \in G'(0)_{\mathbb{C}}$ that satisfies (24). Then, (25) gives $T = iJ(h')^{-1}z \in J\mathfrak{m}_{\mathbb{C}}$. We note here that with ω_r as in (7)

$$\langle [Jz_1, z_2], E_r^* \rangle = \omega_r^{-1} \langle z_1 \,|\, z_2 \rangle_\omega \qquad (z_1, z_2 \in \mathfrak{m}).$$

This means that $\langle z_1 \,|\, z_2 \rangle_{\mathfrak{m}} := \langle [Jz_1, z_2], E_r^* \rangle$ defines an inner product on $\mathfrak{m}$, which we extend to a complex bilinear form on $\mathfrak{m}_{\mathbb{C}}$, denoted by the same symbol $\langle \cdot \,|\, \cdot \rangle_{\mathfrak{m}}$. Let us put $A(h') = (\mathrm{Ad}\, h')\big|_{\mathfrak{m}_{\mathbb{C}}}$. Then solving α in (26) with the already fixed h', T is equivalent to determining $\alpha \in \mathbb{C}$ by

$$e^\alpha = 1 + iy_r + \frac{1}{2} \langle {}^tA(h')^{-1} A(h')^{-1} z \,|\, z \rangle_{\mathfrak{m}}, \tag{27}$$

where ${}^tS$ stands for the adjoint of an operator S on $\mathfrak{m}_{\mathbb{C}}$ relative to the non-degenerate bilinear form $\langle \cdot \,|\, \cdot \rangle_{\mathfrak{m}}$. Since $z \in \mathfrak{m}$, it is enough to show that the real part of the complex symmetric operator $A(h')\, {}^tA(h')$ is positive definite, because then so is its inverse ${}^tA(h')^{-1} A(h')^{-1}$ and the real part of the right hand side of (27) is non-zero.

Before proceeding further, we put $\mathfrak{u} := \mathfrak{m} + J\mathfrak{m}$ and $\mathfrak{h} := \mathfrak{u} + \mathbb{R}E_r$. Then $\mathfrak{h}$ is a Heisenberg algebra. In fact

$$[\mathfrak{m}, \mathfrak{m}] = [J\mathfrak{m}, J\mathfrak{m}] = \{0\}, \qquad [\mathfrak{m}, J\mathfrak{m}] = \mathbb{R}E_r.$$

Moreover the alternating bilinear form σ on $\mathfrak{u} \times \mathfrak{u}$ defined by $[u_1, u_2] = \sigma(u_1, u_2)E_r$ is non-degenerate and we have

$$\sigma(Jx_1 + y_1, Jx_2 + y_2) = \langle x_1 \,|\, y_2 \rangle_{\mathfrak{m}} - \langle x_2 \,|\, y_1 \rangle_{\mathfrak{m}} \qquad (x_j, y_j \in \mathfrak{m}). \tag{28}$$

We extend σ to $\mathfrak{u}_{\mathbb{C}} \times \mathfrak{u}_{\mathbb{C}}$ by complex bilinearity, so that (28) remains valid for $x_j, y_j \in \mathfrak{m}_{\mathbb{C}}$. For every $Z \in \mathfrak{g}'(1)_{\mathbb{C}}$, we set

$$\psi(Z) := \mathrm{ad}\,(JZ)\big|_{\mathfrak{m}_{\mathbb{C}}} + {}^t\big(\mathrm{ad}\,(JZ)\big|_{\mathfrak{m}_{\mathbb{C}}}\big).$$

Thus $\psi(Z)$ is a complex symmetric operator on $\mathfrak{m}_{\mathbb{C}}$ such that $\psi(E')$ is the identity operator.

Lemma 2.9. (1) $\psi(Z)x = -(\mathrm{ad}\, Z)(Jx)$ *for all* $x \in \mathfrak{m}_{\mathbb{C}}$.
(2) *If* $h_0 \in G'(0)_{\mathbb{C}}$, *then* $\psi(h_0 Z) = A(h_0)\, \psi(Z)\, {}^tA(h_0)$.

Proof. Let $x_1, x_2 \in \mathfrak{m}_{\mathbb{C}}$.
(1) By definition we have

$$\begin{aligned} \langle {}^t(\mathrm{ad}\, JZ)x_1 \,|\, x_2 \rangle_{\mathfrak{m}} &= \langle x_1 \,|\, [JZ, x_2] \rangle_{\mathfrak{m}} = \langle [Jx_1, [JZ, x_2]], E_r^* \rangle \\ &= \langle [[Jx_1, JZ], x_2], E_r^* \rangle + \langle [JZ, [Jx_1, x_2]], E_r^* \rangle, \end{aligned}$$

where we have used the Jacobi identity in the last equality. Since $[Jx_1, x_2] \in \mathbb{C}E_r$, we get $[JZ, [Jx_1, x_2]] = 0$ by (23). On the other hand, (1) together with $[x_1, Z] = 0$ gives $[Jx_1, JZ] = J[Jx_1, Z] + J[x_1, JZ]$. Hence it holds that

$$\begin{aligned} \langle {}^t(\mathrm{ad}\, JZ)x_1 \,|\, x_2 \rangle_{\mathfrak{m}} &= \langle [J[Jx_1, Z], x_2], E_r^* \rangle + \langle [J[x_1, JZ], x_2], E_r^* \rangle \\ &= \langle [Jx_1, Z] \,|\, x_2 \rangle_{\mathfrak{m}} + \langle [x_1, JZ] \,|\, x_2 \rangle_{\mathfrak{m}}. \end{aligned}$$

This implies ${}^t(\operatorname{ad} JZ) + \operatorname{ad} JZ = -(\operatorname{ad} Z) \circ J$ on $\mathfrak{m}_{\mathbb{C}}$.
(2) Since $\operatorname{ad}(h_0 Z) = h_0 (\operatorname{ad} Z) h_0^{-1}$, it is sufficient to show that $J\, {}^t\!A(h_0) = \operatorname{Ad} h_0^{-1} \circ J$ on $\mathfrak{m}_{\mathbb{C}}$ by (1). We have

$$\begin{aligned}\langle [J\, {}^t\!A(h_0) x_1, x_2], E_r^* \rangle &= \langle {}^t\!A(h_0) x_1 \,|\, x_2 \rangle_{\mathfrak{m}} = \langle [Jx_1, \operatorname{Ad}(h_0) x_2], E_r^* \rangle \\ &= \langle [\operatorname{Ad}(h_0)^{-1} J x_1, x_2], E_r^* \rangle,\end{aligned}$$

where we have used $\operatorname{Ad}(h_0) E_r = E_r$ by (23). Therefore, by the definition of σ

$$\sigma(J\, {}^t\!A(h_0) x_1, x_2) = \sigma(\operatorname{Ad}(h_0)^{-1} J x_1, x_2).$$

Since σ is non-degenerate on $J\mathfrak{m}_{\mathbb{C}} \times \mathfrak{m}_{\mathbb{C}}$, the proof is complete. ■

Now we return to the proof Proposition 2.8. Since $h'E' = E' + iy'$ by (24), we have by Lemma 2.9

$$A(h')\, {}^t\!A(h') = \psi(h'E') = I - i\,(\operatorname{ad} y') \circ J\big|_{\mathfrak{m}_{\mathbb{C}}}.$$

Since $y' \in \mathfrak{g}'(1)$, it follows that $\operatorname{Re} A(h')\, {}^t\!A(h') = I$, which is positive definite. This is what we had to show. ■

Theorem 2.10. *$\mathcal{I}$ is holomorphic on $\Omega + iV$, and $\mathcal{I}^*$ is holomorphic on $\Omega^* + iV^*$.*

Proof. Since $\Omega = G(0)E$, Proposition 2.8 says that $\Omega + iV \subset G(0)(E + iV) \subset G(0)_{\mathbb{C}} E$. Now Theorem 2.10 follows from Lemma 2.7. The proof for $\mathcal{I}^*$ is similar and omitted. ■

Theorem 2.11. *One has $\mathcal{I}(\Omega + iV) \subset \mathcal{O}_{\mathbb{C}}^*$ and $\mathcal{I}^*(\Omega^* + iV^*) \subset \mathcal{O}_{\mathbb{C}}$.*

Proof. Owing to (18) and Lemma 2.1 we have $\mathcal{I}(hE) = h \cdot E^*_{2\mathbf{d}+\mathbf{b}}$ for any $h \in G(0)$. By analytic continuation this equality holds for all $h \in G(0)_{\mathbb{C}}$. Therefore we obtain $\mathcal{I}(\Omega + iV) \subset G(0)_{\mathbb{C}} \cdot E^*_{2\mathbf{d}+\mathbf{b}} \subset \mathcal{O}_{\mathbb{C}}^*$. ■

Remark 2.12. In general we cannot have $\mathcal{I}(\Omega + iV) \subset \Omega^* + iV^*$ if Ω is no longer selfdual. We present an example in section 5.

3. Cayley transform

Let D be our Siegel domain (9). We put

$$C(w) := E^*_{2\mathbf{d}+\mathbf{b}} - 2\,\mathcal{I}(w + E) \in W^* \qquad (w \in W).$$

It is evident that C is a rational mapping $W \to W^*$ which is holomorphic on $\Omega + iV$ by Theorem 2.10. Let $U^\dagger$ denote the space of all antilinear forms on U. We set

$$\mathcal{C}(z) := \big(2\,\mathcal{I}(w + E) \circ Q(u, \cdot),\, C(w)\big) \in U^\dagger \times W^* \quad (z = (u, w) \in U \times W). \tag{29}$$

Clearly $\mathcal{C}$ is a rational map $U \times W \to U^\dagger \times W^*$. It should be noted that if $z = (u, w) \in D$, then we have $w \in \Omega + iV$, so that $\mathcal{C}(z)$ is holomorphic on D. We shall call $\mathcal{C}$ a *Cayley transform.* This is a slight modification of the Cayley transform defined by Penney [14]. First we state the following.

Lemma 3.1. *The image $\mathcal{C}(D)$ of D under $\mathcal{C}$ is bounded.*

Proof. As the proof proceeds with a verbal translation of [14], we only give here its sketch. Let us denote by $\mathfrak{g}^{\pm}$ the $\pm i$-eigenspaces of the operator J on $\mathfrak{g}_{\mathbb{C}}$. By the integrability condition (1) of J, it is clear that $\mathfrak{g}^{\pm}$ are subalgebras of $\mathfrak{g}_{\mathbb{C}}$. Let $G^{\pm} := \exp \mathfrak{g}^{\pm}$ be the analytic subgroups of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}^{\pm}$. Since $\langle [\mathfrak{g}^{\pm}, \mathfrak{g}^{\pm}], E^*_{2\mathbf{d}+\mathbf{b}} \rangle = 0$ as is easily seen, the formula

$$\chi^{\pm}(\exp z) := \exp i \, \langle z, E^*_{2\mathbf{d}+\mathbf{b}} \rangle \qquad (z \in \mathfrak{g}^{\pm})$$

defines one-dimensional representations $\chi^{\pm}$ of $G^{\pm}$. We have $G \subset G^- G^+$ as Penney proved in [14, Theorem 3], and we set $g = g_- g_+$ with $g_{\pm} \in G^{\pm}$ for every $g \in G$. We now define a function Φ on G by

$$\Phi(g) := \chi^-(g_-)\chi^+(g_+) \qquad (g \in G).$$

We remark that Penney's Φ in [14, (18)] is defined out of $E^*_{\mathbf{d}}$ in our notation, where one should note $|\mathbf{d}| = \dim V$ by (7). Let $n(a,b) \in N_D$, $h \in G(0)$, and put, with π_W the projection $U \times W \to W$,

$$w_0 := hE + ia + \tfrac{1}{2} Q(b,b) = \pi_W\big(n(a,b)h \cdot \mathsf{e}\big) \qquad \text{(cf. (11))}.$$

Then the key formulas are

$$\begin{aligned} \Phi(n(a,b)h)\chi_{2\mathbf{d}+\mathbf{b}}(h) &= 4^{-(2|\mathbf{d}|+|\mathbf{b}|)}\eta(E+w_0)^{-2}, \\ X \log \Phi((n(a,b)h) &= -2i \, \langle X, \mathcal{I}(E+w_0) \rangle \quad (X \in \mathfrak{g}(1)), \\ Y \log \Phi(n(a,b)h) &= 2 \, \langle Q(b,Y), \mathcal{I}(E+w_0) \rangle \quad (Y \in \mathfrak{g}(1/2)), \end{aligned}$$

where $Zf(g) = (d/dt) f((\exp -tZ)g)\big|_{t=0}$ for $Z \in \mathfrak{g}$. In this way the lemma is a consequence of the fact that $Z \log \Phi$ is a bounded function on G for every fixed $Z \in \mathfrak{g}$, which is shown in [14, p. 310]. ∎

In order to give a formula for $\mathcal{C}^{-1}$, we first recall the inner product $\langle \cdot \,|\, \cdot \rangle_\eta$ on V defined by (20). Extending $\langle \cdot \,|\, \cdot \rangle_\eta$ to $W \times W$ by complex bilinearity, we define

$$(u_1 \,|\, u_2)_\eta := \langle\, Q(u_1,u_2) \,|\, E \,\rangle_\eta \qquad (u_1, u_2 \in U).$$

Lemma 3.2. *The sesquilinear form $(\cdot \,|\, \cdot)_\eta$ defines a Hermitian inner product on U.*

Proof. Let us write every $u \in U$ as $u = \sum_k u_k$ with $u_k \in \mathfrak{n}_{\alpha_{k/2}}$. Then we have $[Ju, u] = \sum [Ju_k, u_k] + X$ with $X \in \sum_{k<m} \mathfrak{n}_{(\alpha_m+\alpha_k)/2}$. Therefore (8) and Lemma 2.3 imply

$$\begin{aligned} 2\,(u \,|\, u)_\eta = 2\,\langle\, Q(u,u) \,|\, E \,\rangle_\eta &= \langle [Ju,u], E^*_{2\mathbf{d}+\mathbf{b}} \rangle \\ &= \sum (2d_k + b_k) \langle [Ju_k, u_k], E^*_k \rangle = \sum \omega_k^{-1}(2d_k + b_k) \|u_k\|^2_\omega, \end{aligned}$$

where ω_k's are as in (7). The lemma is now clear. ∎

We define linear maps $F \mapsto \widetilde{F}$ from $U^\dagger$ to U and $u \mapsto \widehat{u}$ from U to $U^\dagger$ by

$$(\widetilde{F} \mid u')_\eta = \langle u', F \rangle, \qquad \langle u', \widehat{u} \rangle = (u \mid u')_\eta \qquad (u' \in U). \tag{30}$$

It is obvious that they are inverse to one another. Similarly we define $\widetilde{f} \in W$ and $\widehat{w} \in W^*$ for $f \in W^*$ and $w \in W$ respectively by (cf. (21))

$$\langle w' \mid \widetilde{f} \rangle_\eta = \langle w', f \rangle, \qquad \langle w', \widehat{w} \rangle = \langle w' \mid w \rangle_\eta \qquad (w' \in W). \tag{31}$$

Moreover, for every $w \in W$, let $\varphi(w)$ be the complex linear operator on U defined through

$$(\varphi(w) u_1 \mid u_2)_\eta = \langle Q(u_1, u_2) \mid w \rangle_\eta \qquad (u_1, u_2 \in U). \tag{32}$$

Clearly $\varphi(E)$ is the identity operator, and it is easy to see that $\varphi(w^*) = \varphi(w)^*$. Let us set

$$B(f) := 2\mathcal{I}^*(E^*_{\mathbf{2d+b}} - f) - E \in W \qquad (f \in W^*), \tag{33}$$
$$\mathcal{B}(F, f) := \big(\varphi(E - \widetilde{f})^{-1}\widetilde{F},\, B(f)\big) \in U \times W \quad ((F, f) \in U^\dagger \times W^*). \tag{34}$$

It is evident that both B and $\mathcal{B}$ are rational mappings.

Proposition 3.3. *One has $\mathcal{B}\mathcal{C}(z) = z$ for all $z \in U \times W$, and $\mathcal{C}\mathcal{B}(\phi) = \phi$ for all $\phi \in U^\dagger \times W^*$.*

Proof. First of all we have for $w \in W$ and $f \in W^*$

$$BC(w) = 2\mathcal{I}^*(E^*_{\mathbf{2d+b}} - C(w)) - E = \mathcal{I}^*\big(\mathcal{I}(w + E)\big) - E = w,$$
$$CB(f) = E^*_{\mathbf{2d+b}} - 2\mathcal{I}(B(f) + E) = E^*_{\mathbf{2d+b}} - \mathcal{I}\big(\mathcal{I}^*(E^*_{\mathbf{2d+b}} - f)\big) = f.$$

Before proceeding further we note that if $u \in U$, then

$$f \circ Q(u, \cdot) = \langle Q(u, \cdot), f \rangle = \langle Q(u, \cdot) \mid \widetilde{f} \rangle_\eta = (\varphi(\widetilde{f}) u \mid \cdot)_\eta.$$

Hence $\big(f \circ Q(u, \cdot)\big)^\sim = \varphi(\widetilde{f})u$, so that $\mathcal{C}$ is rewritten as

$$\mathcal{C}(u, w) = \big(2\,[\varphi(\mathcal{I}(w + E)^\sim) u]^\wedge,\, C(w)\big). \tag{35}$$

From this we see immediately that $\mathcal{B}\mathcal{C}(u, w) = (u, w)$. The proof for $\mathcal{C}\mathcal{B}$ is similar and omitted here. ∎

For each linear operator T on V, we denote by ${}^tT$ its transpose relative to the inner product $\langle \cdot \mid \cdot \rangle_\eta$.

Lemma 3.4. *$\varphi\big({}^t(\mathrm{Ad}_{\mathfrak{g}(1)} h) x\big) = (\mathrm{Ad}_U h)^* \varphi(x) (\mathrm{Ad}_U h)$ for all $h \in G(0)$ and $x \in V$.*

Proof. By definition we have

$$\big(\varphi({}^t(\mathrm{Ad}_{\mathfrak{g}(1)}h)x)u \,\big|\, u'\big)_\eta = \langle\, (\mathrm{Ad}\,h)Q(u,u') \,|\, x \,\rangle_\eta \qquad (u, u' \in U).$$

Since $(\mathrm{Ad}\,h)Q(u,u') = Q((\mathrm{Ad}\,h)u, (\mathrm{Ad}\,h)u')$, the lemma follows easily. ∎

We need to extend Lemma 3.4 to $x \in W$ and $h \in G(0)_{\mathbb{C}}$. First we still denote by ${}^tT$ the transpose of a complex linear operator T on W relative to the nondegenerate symmetric bilinear form $\langle \cdot \,|\, \cdot \rangle_\eta$. Next we put $\langle\, u_1 \,|\, u_2 \,\rangle := \mathrm{Re}\,(u_1 \,|\, u_2)_\eta$ $(u_1, u_2 \in \mathfrak{g}(1/2))$. This is a real inner product on $\mathfrak{g}(1/2)$, which we extend to a complex bilinear form on $\mathfrak{g}(1/2)_{\mathbb{C}} \times \mathfrak{g}(1/2)_{\mathbb{C}}$ written by the same symbol. Let L be a complex linear operator on $\mathfrak{g}(1/2)_{\mathbb{C}}$. Its transpose with respect to $\langle \cdot \,|\, \cdot \rangle$ is expressed as ${}^tL$. Note that every complex linear operator S on U can be naturally extended to a complex linear operator S_C on $\mathfrak{g}(1/2)_{\mathbb{C}}$ commuting with J. It is clear that ${}^tS_C u = S^* u$ for $u \in \mathfrak{g}(1/2)$. We set $\varphi_C(w) = \varphi(w)_C$ $(w \in W)$. Consider Lemma 3.4 first as an identity of operators on the real vector space $\mathfrak{g}(1/2)$ and thus on its complexification $\mathfrak{g}(1/2)_{\mathbb{C}}$ canonically. Then analytic continuation gives the following corollary with the obvious notation $\mathrm{Ad}_{\mathfrak{g}(k)_{\mathbb{C}}}h$ for $h \in G(0)_{\mathbb{C}}$ and $k = 1/2, 1$.

Corollary 3.5. *If $h \in G(0)_{\mathbb{C}}$ and $w \in W$, then*

$$\varphi_C\big({}^t(\mathrm{Ad}_{\mathfrak{g}(1)_{\mathbb{C}}}h)w\big) = {}^t(\mathrm{Ad}_{\mathfrak{g}(1/2)_{\mathbb{C}}}h)\varphi_C(w)(\mathrm{Ad}_{\mathfrak{g}(1/2)_{\mathbb{C}}}h).$$

We are now able to prove the main theorem.

Theorem 3.6. *The Cayley transform $\mathcal{C}$ is a birational map which sends D biholomorphically onto $\mathcal{C}(D)$.*

Proof. First of all we show that if $(F, f) \in \mathcal{C}(D)$, then the operator $\varphi(E - \widetilde{f})$ is invertible. Since $f = C(w)$ for some $w \in \Omega + iV$, it is enough to prove the invertibility of the operator $\varphi\big(\mathcal{I}(w+E)^{\sim}\big)$. By the proof of Theorem 2.11, we know $\mathcal{I}(w+E) \in G(0)_{\mathbb{C}} \cdot E^*_{\mathbf{2d+b}}$. Hence the invertibility of $\varphi\big((h \cdot E^*_{\mathbf{2d+b}})^{\sim}\big)$ for $h \in G(0)_{\mathbb{C}}$ suffices. Now if $w \in W$, we have

$$\begin{aligned}\langle\, w \,|\, (h \cdot E^*_{\mathbf{2d+b}})^{\sim} \,\rangle_\eta &= \langle w,\, h \cdot E^*_{\mathbf{2d+b}} \rangle = \langle h^{-1}w, E^*_{\mathbf{2d+b}} \rangle \\ &= \langle\, h^{-1}w \,|\, E \,\rangle_\eta = \langle\, w \,|\, {}^t(\mathrm{Ad}_{\mathfrak{g}(1)_{\mathbb{C}}}h^{-1})E \,\rangle_\eta.\end{aligned}$$

Therefore we get $\varphi\big((h \cdot E^*_{\mathbf{2d+b}})^{\sim}\big) = \varphi\big({}^t(\mathrm{Ad}_{\mathfrak{g}(1)_{\mathbb{C}}}h^{-1})E\big)$, and the right hand side is an invertible operator in view of Corollary 3.5.

The rest is a direct consequence of Theorems 2.10, 2.11 and Proposition 3.3 together with Lemma 2.7 and the remark thereafter. ∎

Remark 3.7. Unless D is quasisymmetric, it is not true that $\varphi\big(\mathcal{I}(w)^{\sim}\big) = \varphi(w)^{-1}$ in general. This can be verified for the unique 4-dimensional non-quasisymmetric Siegel domain due to Pjatetskii-Shapiro [15, p. 26] (see also [7]). Details are left to the reader.

4. Comparison with Dorfmeister's Cayley transform

Let D be a quasisymmetric Siegel domain. This means that the cone Ω in the defining data of D in (9) is selfdual with respect to the inner product (20) (see [3, Theorem 2.1]). Let us suppose further that D is irreducible, so that the cone Ω is also irreducible (see [11, Theorem 6.3]). Then Proposition 3 in [1] says that the constants n_{mk} (resp. b_i) in (7) are independent of m, k (resp. i). Thus d_j is also independent of j. We set $d := d_j$ and $b := b_i$ for simplicity. Moreover, by the proof of that proposition in [1], the product $\circ$ defined by

$$\langle\, v_1 \circ v_2 \,|\, v_3 \,\rangle_\eta := -\frac{1}{2} D_{v_1} D_{v_2} D_{v_3} \log \eta(E) \qquad (v_1, v_2, v_3 \in V)$$

is a Jordan algebra product. It is clear by definition that the inner product $\langle \cdot \,|\, \cdot \rangle_\eta$ is associative, that is, every Jordan multiplication $L(x) : v \mapsto x \circ v$ for $x \in V$ is a symmetric operator on V with respect to this inner product. Thus V is a Euclidean Jordan algebra in the sense of [6].

In order to describe the Jordan algebra structure in terms of the normal j-algebra structure that we started with, we introduce positive definite symmetric operators $H(x)$ $(x \in \Omega)$ by (cf. [4, §2])

$$\langle\, H(x) v_1 \,|\, v_2 \,\rangle_\eta = D_{v_1} D_{v_2} \log \eta(x) \qquad (v_1, v_2 \in V).$$

It is clear that $H(E) = I$. Moreover it is easy to see that

$$H((\mathrm{Ad}\, g)x) = {}^t\big(\mathrm{Ad}_{\mathfrak{g}(1)}(g^{-1})\big) H(x) \mathrm{Ad}_{\mathfrak{g}(1)}(g^{-1}) \qquad (g \in G(0)). \tag{36}$$

In particular we have $H(\lambda x) = \lambda^{-2} H(x)$ for all $\lambda > 0$.

Lemma 4.1. *$2\, x \circ y = [Jx, y] + {}^t\big(\mathrm{ad}_{\mathfrak{g}(1)} Jx\big) y$ for all $x, y \in V$.*

Proof. By definition we have $-2 \langle\, x \circ y \,|\, z \,\rangle_\eta = (d/dt) D_y D_z \log \eta(E + tx)\big|_{t=0}$ for any $z \in V$. By (19) we know $(\mathrm{Ad}\, \exp tJx)E = E + tx + O(t^2)$. Hence

$$\begin{aligned} -2 \langle\, x \circ y \,|\, z \,\rangle_\eta &= \frac{d}{dt} D_y D_z \log \eta\big((\mathrm{Ad}\, \exp tJx)E\big)\Big|_{t=0} \\ &= \frac{d}{dt} \big\langle H\big((\mathrm{Ad}\, \exp tJx)E\big) y \,|\, z \,\big\rangle_\eta \Big|_{t=0}. \end{aligned}$$

Using (36), we obtain $2 \langle\, x \circ y \,|\, z \,\rangle_\eta = \langle\, [Jx, y] \,|\, z \,\rangle_\eta + \langle\, y \,|\, [Jx, z] \,\rangle_\eta$, from which the lemma follows immediately. ∎

Corollary 4.2. (1) *One has $L(E_j) = \mathrm{ad}_{\mathfrak{g}(1)} H_j$ for $j = 1, \ldots, r$, so that E is the unit element.*
(2) *$\{E_j\}_{j=1}^r$ is a complete system of orthogonal primitive idempotents in V, and the corresponding Peirce spaces are $\mathfrak{n}_{(\alpha_m + \alpha_k)/2}$ $(1 \leqq k < m \leqq r)$.*

In particular, the Jordan algebra rank of V is r. Let $\langle \cdot \,|\, \cdot \rangle_0$ denote the trace inner product of V: $\langle\, v_1 \,|\, v_2 \,\rangle_0 = \mathrm{tr}\,(v_1 \circ v_2)$, where $\mathrm{tr}\, v$ stands for the Jordan algebra trace of the element $v \in V$. Since V is simple owing to the irreducibility of Ω, $\langle \cdot \,|\, \cdot \rangle_\eta$ is a constant multiple of the trace inner product [6, III.4.1].

Lemma 4.3. $(2d+b)\langle v_1 \,|\, v_2 \rangle_0 = \langle v_1 \,|\, v_2 \rangle_\eta$ *for all* $v_1, v_2 \in V$.

Proof. It suffices to show $\|E_j\|_\eta^2 = 2d + b$. But this is immediate from Lemma 2.2. ■

Before proceeding, we note that every element $w \in \Omega + iV$ is invertible in the complexified Jordan algebra W.

Proposition 4.4. *For each* $w \in \Omega + iV$, *one has* $\mathcal{I}(w)^{\sim} = w^{-1}$, *where the right hand side is the Jordan algebra inverse of* w.

Proof. By analytic continuation it is enough to prove the proposition for $w = x \in \Omega$. Let ϕ_0 be the characteristic function of Ω:

$$\phi_0(x) := \int_\Omega \exp -\langle x \,|\, y \rangle_0 \, dy \qquad (x \in \Omega). \tag{37}$$

Let $g \in G(0)$. A change of variable yields $\phi_0(gE) = \gamma \cdot \chi_{\mathbf{d}}(g)^{-1}$ by Lemma 1.1, where $\gamma > 0$ is a constant. Since $\mathbf{d} = (d, \cdots, d)$ and $\mathbf{b} = (b, \cdots, b)$, it holds that $\Delta_{-\mathbf{d}}(x) = \eta(x)^{d/(2d+b)}$ for all $x \in \Omega$, so that $\phi_0(x) = \gamma \cdot \eta(x)^{d/(2d+b)}$. Hence we get

$$D_v \log \phi_0(x) = \frac{d}{2d+b} D_v \log \eta(x) = -\frac{d}{2d+b} \langle v, \mathcal{I}(x) \rangle \quad (v \in V).$$

On the other hand, Proposition III.4.3 in [6] together with Lemma 4.3 shows

$$-D_v \log \phi_0(x) = \frac{\dim V}{r} \langle x^{-1} \,|\, v \rangle_0 = \frac{d}{2d+b} \langle x^{-1} \,|\, v \rangle_\eta,$$

where note that $\dim V = |\mathbf{d}| = rd$ in the present case. Thus $\langle v, \mathcal{I}(x) \rangle = \langle x^{-1} \,|\, v \rangle_\eta$ for each $x \in \Omega$, which implies $\mathcal{I}(x)^{\sim} = x^{-1}$. ■

Thus it follows that for any $w \in \Omega + iV$

$$C(w)^{\sim} = E - 2(w+E)^{-1} = (w - E) \circ (w+E)^{-1}. \tag{38}$$

Let us recall the complex linear operators $\varphi(w)$ $(w \in W)$ on U defined by (32).

Proposition 4.5. *The linear map* $w \mapsto \varphi(w)$ *is a* $*$*-representation of the Jordan algebra* W. *In other words, one has* $\varphi(w^*) = \varphi(w)^*$ *and*

$$\varphi(w_1 \circ w_2) = \frac{1}{2} \big(\varphi(w_1)\varphi(w_2) + \varphi(w_2)\varphi(w_1) \big) \qquad (w_1, w_2 \in W).$$

This proposition is due to Dorfmeister. However, instead of making a mere reference to [3, Theorem 2.1 (6)] which forces the reader into pursuing part of Dorfmeister's work, we would like to indicate here the way to get to Proposition 4.5 in our language of normal j-algebra for the readability of this paper.

Lemma 4.6. *One has* $\varphi(x) = \mathrm{ad}_U Jx + (\mathrm{ad}_U Jx)^*$ *for all* $x \in V$, *where* $\mathrm{ad}_U Jx$ *denotes the complex linear operator on* U *defined by the adjoint action of* $Jx \in \mathfrak{g}(0)$ *on* $\mathfrak{g}(1/2)$.

Proof. The definition (8) of Q together with the fact that $\mathrm{ad}_{\mathfrak{g}(1/2)} Jx$ commutes with J implies

$$(\mathrm{ad}\, Jx)Q(u,u') = Q((\mathrm{ad}\, Jx)u, u') + Q(u, (\mathrm{ad}\, Jx)u') \qquad (u, u' \in U),$$

so that we have

$$\langle\, Q(u,u') \,|\, {}^t(\mathrm{ad}_{\mathfrak{g}(1)} Jx)E \,\rangle_\eta = ((\mathrm{ad}_U Jx)u \,|\, u')_\eta + ((\mathrm{ad}_U Jx)^* u \,|\, u')_\eta. \tag{39}$$

Here Lemma 4.1 and (6) give

$${}^t(\mathrm{ad}_{\mathfrak{g}(1)} Jx)E = 2\, x \circ E - [Jx, E] = 2x - x = x.$$

Hence the left hand side of (39) is equal to $(\varphi(x)u \,|\, u')_\eta$. ∎

Lemma 4.7. *It holds that* $\mathrm{tr}\, \varphi(x) = b\, \mathrm{tr}\, x$ *for all* $x \in V$. *In particular,* $\mathrm{tr}\, \varphi(x \circ y)$ *defines an associative inner product on the Jordan algebra* V.

Proof. We note that by Lemma 4.6

$$\mathrm{tr}\, \varphi(x) = 2\, \mathrm{Re}\, \mathrm{tr}\, (\mathrm{ad}_U Jx) = \mathrm{tr}\, \mathrm{ad}_{\mathfrak{g}(1/2)} Jx.$$

Thus $\mathrm{tr}\, \varphi(E_j) = 2b \times (1/2) = b = b\, \mathrm{tr}\, E_j$ for all j, because E_j's are primitive idempotents. If $x \in \mathfrak{n}_{(\alpha_m+\alpha_k)/2}$ $(m > k)$, then $Jx \in \mathfrak{n}_{(\alpha_m-\alpha_k)/2}$ and the operator $\mathrm{ad}_{\mathfrak{g}(1/2)} Jx$ is nilpotent. Hence we get $\mathrm{tr}\, \varphi(x) = 0 = \langle\, x \,|\, E \,\rangle_0 = \mathrm{tr}\, x$. ∎

Lemma 4.8. *One has* $\det \varphi(x) = (\det x)^b$ *for any* $x \in V$, *where* $\det x$ *on the right hand side is the Jordan algebra determinant of* $x \in V$.

Proof. Lemmas 1.2 and 3.4 give $\det \varphi({}^t(\mathrm{Ad}_{\mathfrak{g}(1)} h)E) = \chi_{\mathbf{b}}(h)$ for any $h \in G(0)$. On the other hand, [6, III.4.3] and Lemma 1.1 imply $\det({}^t(\mathrm{Ad}_{\mathfrak{g}(1)} h)E) = \chi_{\mathbf{d}}(h)^{r/\dim V} = \chi_{\mathbf{1}}(h)$, where $\mathbf{1} = (1, \ldots, 1)$. Since Ω is selfdual with respect to the inner product $\langle \cdot \,|\, \cdot \rangle_\eta$, there exists, for a given $x \in \Omega$, an element $h \in G(0)$ such that ${}^t(\mathrm{Ad}_{\mathfrak{g}(1)} h)E = x$. Hence $\det \varphi(x) = (\det x)^b$ for any $x \in \Omega$, and thus for any $x \in V$, because both sides are polynomial functions on V. ∎

Let $G(\Omega)$ denote the linear automorphism group of the cone Ω. It is a reductive Lie group, as Ω is selfdual. Let $G(\Omega)^\circ$ be the connected component of the identity of $G(\Omega)$. Since $\det(gx) = (\det g)^{r/\dim V} \det x$ for $g \in G(\Omega)^\circ$ and $x \in V$ by [6, III.4.3], we get the following corollary, though we do not yet have the formula of the form $\varphi(gx) = T\varphi(x)T^*$ with an operator T on U at this stage when g is a general element of $G(\Omega)^\circ$.

Corollary 4.9. $\det \varphi(gx) = (\det g)^{b/d} \det \varphi(x)$ *for all* $g \in G(\Omega)^\circ$ *and* $x \in V$.

With Corollary 4.9 and Lemma 4.7 in hand, one can trace the proofs of Lemma 5.2 and Theorem 5.4 in [2] easily to give a proof to the displayed identity of Proposition 4.5. The other identity $\varphi(w^*) = \varphi(w)^*$ has been already mentioned just after (32).

Now we return to the Cayley transform in the quasisymmetric case. The formula turns out to be the same as the one given by Dorfmeister [3, (2.8)] except for a minor modification. For symmetric domains, the formula appeared in [12, 10.3] and [18, Exercise III.7.3].

Theorem 4.10. *For any* $(u, w) \in U \times W$ *one has*

$$\mathcal{C}(u,w)^{\sim} = \big(2\varphi(w+E)^{-1}u,\ (w-E)\circ(w+E)^{-1}\big),$$
$$\mathcal{B}(\widehat{u},\widehat{w}) = \big(\varphi(E-w)^{-1}u,\ (E+w)\circ(E-w)^{-1}\big).$$

Proof. By (35), (38) and Proposition 4.4 we have

$$\mathcal{C}(u,w)^{\sim} = \big(2\varphi((w+E)^{-1})u,\ (w-E)\circ(w+E)^{-1}\big).$$

Since φ is a Jordan algebra representation, we have $\varphi((w+E)^{-1}) = \varphi(w+E)^{-1}$. On the other hand Proposition 2.6 implies $\mathcal{I}^*(\widehat{w}) = w^{-1}$. Thus we get $B(\widehat{w}) = (E+w)\circ(E-w)^{-1}$, and the expression for $\mathcal{B}(\widehat{u},\widehat{w})$ is obtained from (34). ∎

5. Pseudoinverses related to the Vinberg cone

In this section we exhibit an example in which $\mathcal{I}(\Omega + iV)$ is not contained in $\Omega^* + iV^*$. Let V be the 5-dimensional real vector space presented by the following form of symmetric matrices:

$$V = \left\{ v = \begin{pmatrix} v_1 & v_2 & v_4 \\ v_2 & v_3 & 0 \\ v_4 & 0 & v_5 \end{pmatrix} \ ;\ v_1, \ldots, v_5 \in \mathbb{R} \right\}.$$

The vector space V is equipped with the inner product $\langle v \mid v' \rangle = \operatorname{tr}(vv')$. Whenever we identify the dual vector space V^* with V, we use this inner product throughout this section. Now the *Vinberg cone* Ω is defined to be

$$\Omega := \{v \in V\ ;\ v_1 > 0,\ \Delta_{12}(v) := v_1v_3 - v_2^2 > 0,\ \Delta_{13}(v) := v_1v_5 - v_4^2 > 0\}.$$

Its dual cone Ω^* is described as

$$\Omega^* = \{v \in V\ ;\ v \text{ is positive definite}\}.$$

One knows that there is no linear isomorphism of V that maps Ω onto Ω^*. In particular, Ω is *not* selfdual (see [19, §8] or [6, Exercise I.10]). Set for every $v \in V$

$$v_{(1)} := \begin{pmatrix} v_1 & v_2 \\ v_2 & v_3 \end{pmatrix}, \qquad v_{(2)} := \begin{pmatrix} v_1 & v_4 \\ v_4 & v_5 \end{pmatrix}.$$

If $h_j := \begin{pmatrix} a & 0 \\ b_j & c_j \end{pmatrix} \in GL(2,\mathbb{R})$ $(j = 1,2)$, we define $h = (h_1, h_2) \in GL(V)$ by the formula $(hv)_{(j)} := h_j v_{(j)} {}^t h_j$, where ${}^t h_j$ denotes the transposed matrix of h_j. Let

$$H := \{h = (h_1,h_2) \in GL(V)\ ;\ a > 0,\ c_j > 0\ (j = 1,2)\}.$$

Then H acts on Ω simply transitively.

Now we consider the tube domain $D = \Omega + iV$. Let us describe the corresponding normal j-algebra $\mathfrak{g}$. Let $\mathfrak{h} := \operatorname{Lie}(H)$, the Lie algebra of H, and we form the semidirect product $\mathfrak{g} = V \rtimes \mathfrak{h}$. Define a linear map J on $\mathfrak{g}$ by the rule

$$J : \mathfrak{h} \ni \left(\begin{pmatrix} \alpha & 0 \\ \beta_1 & \gamma_1 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ \beta_2 & \gamma_2 \end{pmatrix} \right) \mapsto - \begin{pmatrix} 2\alpha & \beta_1 & \beta_2 \\ \beta_1 & 2\gamma_1 & 0 \\ \beta_2 & 0 & 2\gamma_2 \end{pmatrix} \in V,$$

$$J : V \ni v \mapsto \left(\begin{pmatrix} v_1/2 & 0 \\ v_2 & v_3/2 \end{pmatrix}, \begin{pmatrix} v_1/2 & 0 \\ v_4 & v_5/2 \end{pmatrix} \right) \in \mathfrak{h}.$$

It is clear that $J^2 = -I$. With $\omega = 0 \oplus \mathrm{tr} \in \mathfrak{h}^* \oplus V^* = \mathfrak{g}^*$, it is easy to see that the triple $(\mathfrak{g}, J, \omega)$ is a normal j-algebra. In order to present the root space decomposition of $\mathfrak{g}$, let e^k_{ij} $(k = 2, 3)$ be the $k \times k$ matrix unit with the only non-zero (i, j)-entry equal to 1. Set

$$\begin{aligned} &H_1 := \tfrac{1}{2}(e^2_{11},\, e^2_{11}), \quad H_2 := \tfrac{1}{2}(e^2_{22},\, 0), \quad H_3 := \tfrac{1}{2}(0,\, e^2_{22}), \\ &T_{21} := (e^2_{21},\, 0), \qquad T_{31} := (0,\, e^2_{21}), \\ &E_j := e^3_{jj}\ (j = 1, 2, 3), \quad X_{21} := e^3_{12} + e^3_{21}, \quad X_{31} = e^3_{13} + e^3_{31}. \end{aligned}$$

These 10 elements form a linear basis of $\mathfrak{g}$ such that $E_j = -JH_j$ $(j = 1, 2, 3)$ and $X_{k1} = -JT_{k1}$ $(k = 1, 2)$. Put $\mathfrak{a} := \mathbb{R}H_1 \oplus \mathbb{R}H_2 \oplus \mathbb{R}H_3$ and let $\alpha_1, \alpha_2, \alpha_3$ be the basis of $\mathfrak{a}^*$ dual to H_1, H_2, H_3: $\langle H_i, \alpha_j \rangle = \delta_{ij}$. An elementary calculation shows

$$\mathfrak{n}_{(\alpha_k - \alpha_1)/2} = \mathbb{R}T_{k1}, \qquad \mathfrak{n}_{(\alpha_k + \alpha_1)/2} = \mathbb{R}X_{k1} \qquad (k = 1, 2).$$

Since $\mathfrak{n}_{\alpha_k} = \mathbb{R}E_k$ $(k = 1, 2, 3)$, we have

$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{k=1}^{3} \mathfrak{n}_{\alpha_k} \oplus \sum_{k=1}^{2} \mathfrak{n}_{(\alpha_k - \alpha_1)/2} \oplus \sum_{k=1}^{2} \mathfrak{n}_{(\alpha_k + \alpha_1)/2}.$$

Note that the roots $(\alpha_3 \pm \alpha_2)/2$ are absent. The constants in (7) are

$$n_{21} = n_{31} = 1, \quad n_{32} = 0, \quad d_1 = 2, \quad d_2 = d_3 = 3/2, \quad b_i = 0.$$

We have $E^*_{2\mathbf{d}} = 4E^*_1 + 3E^*_2 + 3E^*_3$ and observe that $E^*_j = e^3_{jj}$ $(j = 1, 2, 3)$.

Put $E = E_1 + E_2 + E_3 \in \Omega$. For any $y \in \Omega$, let $h(y) \in H$ be the unique element such that $h(y)E = y$. Then we have $\mathcal{I}(y) = h(y) \cdot E^*_{2\mathbf{d}}$. A direct computation yields

$$h(y) = \left(\begin{pmatrix} \sqrt{y_1} & 0 \\ \dfrac{y_2}{\sqrt{y_1}} & \sqrt{\dfrac{\Delta_{12}(y)}{y_1}} \end{pmatrix}, \begin{pmatrix} \sqrt{y_1} & 0 \\ \dfrac{y_4}{\sqrt{y_1}} & \sqrt{\dfrac{\Delta_{13}(y)}{y_1}} \end{pmatrix} \right),$$

so that we have

$$\mathcal{I}(y) = \begin{pmatrix} \dfrac{1}{y_1}\left(4 + \dfrac{3y_2^2}{\Delta_{12}(y)} + \dfrac{3y_4^2}{\Delta_{13}(y)}\right) & -\dfrac{3y_2}{\Delta_{12}(y)} & -\dfrac{3y_4}{\Delta_{13}(y)} \\ -\dfrac{3y_2}{\Delta_{12}(y)} & \dfrac{3y_1}{\Delta_{12}(y)} & 0 \\ -\dfrac{3y_4}{\Delta_{13}(y)} & 0 & \dfrac{3y_1}{\Delta_{13}(y)} \end{pmatrix}.$$

Allowing every y_j being a complex number in the above formula gives an analytic continuation of $\mathcal{I}$ to $W = V_{\mathbb{C}}$. In particular, if a real vector $y \in V$ satisfies $y_1 = y_3 = y_5 = 0$, then

$$\mathcal{I}(E + iy) = \begin{pmatrix} 4 - \dfrac{3y_2^2}{1 + y_2^2} - \dfrac{3y_4^2}{1 + y_4^2} & -\dfrac{3iy_2}{1 + y_2^2} & -\dfrac{3iy_4}{1 + y_4^2} \\ -\dfrac{3iy_2}{1 + y_2^2} & \dfrac{3}{1 + y_2^2} & 0 \\ -\dfrac{3iy_4}{1 + y_4^2} & 0 & \dfrac{3}{1 + y_4^2} \end{pmatrix}.$$

Hence if $|y_2|$ and $|y_4|$ are sufficiently large, the diagonal matrix $\mathrm{Re}\,\mathcal{I}(E+iY)$ is no longer positive definite, that is, no longer belongs to Ω^*. This verifies Remark 2.12.

We conclude this paper by writing down the formula for $\mathcal{I}^*$:

$$\mathcal{I}^*(y) = \begin{pmatrix} \dfrac{4y_3y_5}{\det y} & -\dfrac{4y_2y_5}{\det y} & -\dfrac{4y_3y_4}{\det y} \\ -\dfrac{4y_2y_5}{\det y} & \dfrac{4y_2^2y_5}{y_3\det y} + \dfrac{3}{y_3} & 0 \\ -\dfrac{4y_3y_4}{\det y} & 0 & \dfrac{4y_4^2y_3}{y_5\det y} + \dfrac{3}{y_5} \end{pmatrix}.$$

Details are left to the reader.

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