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# Explicit formulae for cocycles of holomorphic vector fields with values in $\lambda$ densities

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**Abstract.** In this article, we give explicit formulae for the generators of $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ in terms of affine and projective connections, $\Sigma_r$ being a compact Riemann surface punctured in $r$ points This is done using the cocycles which have been evidenced by V. Ovsienko and C. Roger in [13] and globalizing them by their transformation property.

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## Introduction

The continuous cohomology of Lie algebras of $\mathcal{C}^\infty$-vector fields has been studied by I. M. Gelfand, D. B. Fuks, R. Bott, A. Haefliger and G. Segal in some outstanding papers [4], [9], [1].

B. L. Feigin [2] and N. Kawazumi [11], whose work is continued in [18], studied Gelfand-Fuks cohomology of Lie algebras of holomorphic vector fields $Hol(\Sigma)$ on an open Riemann surface $\Sigma$. Kawazumi calculated the cohomology spaces $H^*(Hol(\Sigma), \mathcal{F}_\lambda(\Sigma))$ of $Hol(\Sigma)$ with values in the space of (holomorphic) $\lambda$-densities on $\Sigma$, using a well known theorem of Goncharova, cf [3]. He expressed the generators of the cohomology spaces in terms of the nowhere-vanishing holomorphic vector field $\partial$ which exists on open Riemann surfaces, trivializing the holomorphic tangent bundle.

In this article, we give explicit formulae for the generators of $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ where $\Sigma_r = \Sigma \setminus \{p_1, \ldots, p_r\}$, $\Sigma$ being a compact Riemann surface, in terms of affine and projective connections. This is done using the cocycles which have been evidenced by V. Ovsienko and C. Roger in [13] and globalizing them by their transformation property.

The main reason to look for explicit formulae is the search for a generalization of the Krichever-Novikov algebras [12] to extensions of $Hol(\Sigma_r)$ by $\mathcal{F}_\lambda(\Sigma_r)$, cf [13] for the case of $Vect(S^1)$.

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## 1. Preliminairies, statement of the result

In this section, we state the theorems of Kawazumi and of Ovsienko-Roger which are the starting point of our work.

Let $Vect(S^1)$ denote the Lie algebra of differentiable vector fields on the circle $S^1$, and $\mathcal{F}_\lambda$ the $Vect(S^1)$-module of $\lambda$-densities, using the action

$$L_f a = (fa' + \lambda f'a)(dx)^\lambda, \tag{1}$$

where $f \in Vect(S^1)$ and $a \in \mathcal{F}_\lambda$ are both represented by their coefficient function. Here $\mathcal{F}_\lambda$ is identified with the space of $\mathcal{C}^\infty$ functions on the circle $S^1$ and only the $Vect(S^1)$-action changes with $\lambda$. Thus, $\lambda$ can be any complex number.

**Theorem 1.1.** (Theorem 3, [13]) *The cohomology groups $H^2(Vect(S^1), \mathcal{F}_\lambda)$ are non-zero only for $\lambda = 0, 1, 2, 5, 7$. They are two-dimensional for $\lambda = 0, 1, 2$ and one-dimensional for $\lambda = 5, 7$.*

*The cohomology classes are represented by generators, denoted by $c_\lambda$ or $\bar{c}_\lambda$ corresponding to the different values of $\lambda$, which read explicitly*

$$\begin{aligned}
\bar{c}_0(f,g) &= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \\
c_0(f,g) &= c_{GF}(f,g) \\
c_1(f,g) &= \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} dx \\
\bar{c}_1(f,g) &= \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} dx \\
c_2(f,g) &= \begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} (dx)^2 \\
\bar{c}_2(f,g) &= \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} (dx)^2 \\
c_5(f,g) &= \begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} (dx)^5 \\
c_7(f,g) &= \left( 2\begin{vmatrix} f''' & g''' \\ f^{(VI)} & g^{(VI)} \end{vmatrix} - 9 \begin{vmatrix} f^{(IV)} & g^{(IV)} \\ f^{(V)} & g^{(V)} \end{vmatrix} \right) (dx)^7
\end{aligned}$$

In the theorem, $c_{GF}$ denotes the Gelfand-Fuks cocycle, being a cocycle with values in the trivial module $\mathbb{C} \subset \mathcal{F}_0$. Its explicit formula is

$$c_{GF}(f,g) = \int_{S^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} (x) \quad dx.$$

Now, let $\Sigma_r$ - as in the rest of this article - denote an open Riemann surface, obtained from the compact Riemann surface $\Sigma$ by extraction of $r$ points: $\Sigma_r := \Sigma \setminus \{p_1, \ldots, p_r\}$.

Let $Hol(\Sigma_r)$ denote the (infinite dimensional) Lie algebra of holomorphic vector fields on $\Sigma_r$. Let $\mathcal{F}_\lambda(\Sigma_r)$ be the space of sections of the bundle of holomorphic $\lambda$-densities. As all bundles on $\Sigma_r$ are trivial, elements of $\mathcal{F}_\lambda(\Sigma_r)$ can be represented by holomorphic functions. $Hol(\Sigma_r)$ still acts on $\mathcal{F}_\lambda(\Sigma_r)$ according to

$$L_f a = (fa' + \lambda f'a)(dz)^\lambda,$$

where $f \in Hol(\Sigma_r)$ and $a \in \mathcal{F}_\lambda(\Sigma_r)$ are both represented by their coefficient function, the $(dz)^\lambda$ being the global section trivialising the bundle of $\lambda$-densities.

Recall that

$$H^p(\Sigma_r) = \begin{cases} \mathbb{C} & \text{for } p = 0 \\ \mathbb{C}^{2g+r-1} & \text{for } p = 1 \\ 0 & \text{for } p \geq 2 \end{cases},$$

if $g$ denotes the genus of $\Sigma$.

In [11], Kawazumi calculates the spaces $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ using the Rešetnikov [14] spectral sequence. This sequence has as $E_2$-term the sheaf cohomology of a sheaf whose stalk at $x \in \Sigma_r$ is the cohomology of $Hol(\Sigma_r)$ with values in $\mathcal{F}_\lambda(\Sigma_r)_x$, the fibre of $\mathcal{F}_\lambda(\Sigma_r)$ at $x$.

Furthermore, Kawazumi uses the main result of his article to express the stated $E_2$-term as a tensor product of some "covariant derivative" cocycles with the formal version of his cohomology, namely $H^*(W_1, T_\lambda)$. Here, $W_1$ is the Lie algebra of formal vector fields on the complex line and $T_\lambda$ is the corresponding module of formal $\lambda$-densities. $H^*(W_1, T_\lambda)$ is explicitly given, thanks to the theorem of Goncharova, cf [3]. Thus, he obtains a (collapsing) spectral sequence for the wanted cohomology.

Let us state his result just for the dimensions of $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ for the different $\lambda$.

**Theorem 1.2.** (consequence of (9.7), [11])

$$\begin{aligned}
\dim H^2(Hol(\Sigma_r), \mathcal{F}_0(\Sigma_r)) &= 2(2g + r - 1) \\
\dim H^2(Hol(\Sigma_r), \mathcal{F}_1(\Sigma_r)) &= 2g + r \\
\dim H^2(Hol(\Sigma_r), \mathcal{F}_2(\Sigma_r)) &= 2g + r \\
\dim H^2(Hol(\Sigma_r), \mathcal{F}_5(\Sigma_r)) &= 1 \\
\dim H^2(Hol(\Sigma_r), \mathcal{F}_7(\Sigma_r)) &= 1
\end{aligned}$$

*For all other values of $\lambda$, $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ is zero.*

To understand these dimensions, recall from [11] (9.7) p.701 that for $\lambda = 0$, $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ is generated by some classes, which we denote $c_0^\omega$ and $\bar{c}_0^\omega$, each depending on an element $\omega \in H^1(\Sigma_r) = \mathbb{C}^{2g+r-1}$. In the same manner, $H^2(Hol(\Sigma_r), \mathcal{F}_1(\Sigma_r))$ is generated by one family, denoted $\bar{c}_1^\omega$, and a cocycle $c_1$, $H^2(Hol(\Sigma_r), \mathcal{F}_2(\Sigma_r))$ is generated by a family, denoted $\bar{c}_2^\omega$, and a cocycle $c_2$ and $H^2(Hol(\Sigma_r), \mathcal{F}_5(\Sigma_r))$ and $H^2(Hol(\Sigma_r), \mathcal{F}_7(\Sigma_r))$ are each generated by a cocycle

$c_5$ and $c_7$. We have chosen the same notation as in the above theorem of Ovsienko and Roger, but their cocycles would not give globally defined objects on a Riemann surface $\Sigma_r$. The explicit construction in terms of connections of these cocycles, resp. families of cocycles, is not known and the subject of this article.

Partial results are known: namely, the holomorphic version of the Gelfand-Fuks cocycle leading to a meromorphic version of the Virasoro algebra appeared in work of Krichever and Novikov, further developped by Schlichenmaier and Sheinman, cf [16]. It reads (cf [17] where we applied Poincaré duality to write it as an integral over $\Sigma_r$):

$$c_0^\omega(f,g) = \frac{c}{24\pi\mathbf{i}} \int_{\Sigma_r} \left( \frac{1}{2} \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} - 2R \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \right) dz \wedge \bar{\omega},$$

where $\omega \in H^1(\Sigma_r)$ and $R$ is a projective connection. Recall that for a Stein manifold (in particular for an open Riemann surface) the subcomplex of holomorphic forms calculates all the de Rham cohomology, cf [6] p.449.

On this form, we already see how such a cocycle is constructed: the Gelfand-Fuks cocycle serves as symbol, and then one adds terms to have a globally defined 1-form. In other words, the 1-form without the term involving $R$ is globally defined only with respect to an atlas of charts from $PSl(2;\mathbb{C})$, to define it for a general holomorphic atlas, one has to use a projective connection.

Affine connections are more general than projective connections, namely, a manifold supporting an affine connection admits also a projective connection. These connections come from the corresponding structures, an affine (projective) structure being an (holomorphic) atlas such that the chart transitions are in the subgroup of affine (resp. projective) transformations. One sees that an affine structure is in particular a projective structure. We will state some well known facts about these objects in the next section.

Let us state the main result of this article:

**Theorem 1.3.** *Let $\Sigma_r$ be an open Riemann surface, $Hol(\Sigma_r)$ the Lie algebra of holomorphic vector fields on $\Sigma_r$ and $\mathcal{F}_\lambda(\Sigma_r)$ the space of holomorphic $\lambda$-densities.*

*The spaces $H^2(Hol(\Sigma_r), \mathcal{F}_\lambda(\Sigma_r))$ for $\lambda = 0,1,2,5,7$ are generated by the classes $\bar{c}_0, c_0^\omega, c_1, \bar{c}_1^\omega, c_2, \bar{c}_2^\omega, c_5, c_7$, where the subscript indicates the value of $\lambda$ and the superscript the dependence on (the class of) a holomorphic 1-form $\omega$ on $\Sigma_r$.*

*The explicit formulae are given in section 2 (in terms of affine and projective connections) and in section 4 (in terms of the covariant derivative).*

Note that the theorem asserts in particular that the formulae in section 2 and in section 4 for $\bar{c}_0, c_0^\omega, c_1, \bar{c}_1^\omega, c_2, \bar{c}_2^\omega, c_5, c_7$ coïncide.

## 2. Transformation behavior

In this section, we shall calculate the correction terms in order to make the cocycles of theorem 1 globally defined geometrical objects.

Let $X, Y$ denote holomorphic vector fields on $\Sigma_r$. Let $U_\alpha, U_\beta \subset \Sigma_r$ be open subsets such that $U_\alpha \cap U_\beta \neq \emptyset$. Let $X$ and $Y$ be given by local coefficient functions $f_\alpha, g_\alpha$ in $U_\alpha$ and $f_\beta, g_\beta$ in $U_\beta$. Denote by $z_\alpha$ and $z_\beta$ local coordinates in $U_\alpha$ and $U_\beta$, and by $h(z_\alpha) = z_\beta$ the holomorphic change of coordinates. We have

$$f_\beta = \frac{\partial h}{\partial z_\alpha} f_\alpha,$$

and similarly for $g_\beta$. Denote $\frac{\partial h}{\partial z_\alpha}$ just by $h'$.

Now, it is easy to transform derivatives on the coefficient functions:

$$f'_\beta = \frac{1}{h'}(h'' f_\alpha) + f'_\alpha,$$

and

$$f''_\beta = \frac{1}{h'}\left\{\left(\frac{h'''}{h'} - \frac{(h'')^2}{(h')^2}\right) f_\alpha + \frac{h''}{h'} f'_\alpha + f''_\alpha\right\}.$$

Observe that this kind of manipulations is particularly well suited for being treated by MAPLE.

Denote by $S := S(h)$ the Schwarzian derivative of $h$, i.e. the expression

$$S = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2.$$

It is easy to show and well known that we have:

$$f'''_\beta = \frac{1}{(h')^2}\left(f'''_\alpha + S' f_\alpha + 2S f'_\alpha\right).$$

Now recall some generalities on affine and projective structures, resp., connections, cf. [7] §9 p. 164 and [15] p. 137–138:

**Definition 2.1.** Let $\{U_\alpha, z_\alpha\}$ be a covering of $\Sigma_r$ by coordinate charts and $z_\beta = h(z_\alpha)$ the coordinate transitions for non-empty $U_\alpha \cap U_\beta$.

A (holomorphic) projective connection is a family of holomorphic functions $R_\alpha$ on $U_\alpha$ such that for non-empty $U_\alpha \cap U_\beta$, we have

$$R_\beta (h')^2 \;=\; R_\alpha \;+\; S.$$

In the same way, we have

**Definition 2.2.** A (holomorphic) affine connection is a family of holomorphic functions $T_\alpha$ on $U_\alpha$ such that for non-empty $U_\alpha \cap U_\beta$, we have

$$T_\beta h' \;=\; T_\alpha \;+\; \frac{h''}{h'}.$$

(Observe that $h' \neq 0$.) There is a 1-1 correspondance between connections and the corresponding structures, see [7] thm. 19, p. 170. See also [5], section 2, for a brief summary on these structures. Affine connections (thus affine structures, projective structures and projective connections) exist on any open Riemann surface, cf [8]. This is in contrast to compact Riemann surfaces where affine connections exist only for genus 1, see [7] p. 173.

We use these objects to compensate extra terms arising from the transition behaviour of the cocycles of theorem 1. One arrives at the following results (the first two are trivial; in the following, $R$ denotes a projective connection and $T$ an affine connection):

- $\bar{c}_0(f,g)$ is a well-defined global vector field, so $\bar{c}_0^\omega(f,g) := \bar{c}_0(f,g)\omega$ is a well-defined global function

- $c_0^\omega(f,g)$ is a well-defined global (constant) function

- $c_1(f,g) := \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} - T\begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} + (R - \frac{1}{2}T^2)\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ is a well-defined global 1-form

- $\bar{c}_1(f,g) := \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} - T\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ is a well-defined global function, so $\bar{c}_1^\omega(f,g) := \bar{c}_1(f,g)\omega$ is a well-defined global 1-form

- $c_2(f,g) := \begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} - T\begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} - (2TR - R')\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ is a well-defined global 2-form

- $\bar{c}_2(f,g) := \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} - 2R\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ is a well-defined global 1-form, so $\bar{c}_2^\omega(f,g) := \bar{c}_2(f,g) \otimes \omega$ is a well-defined global quadratic differential

- $c_5(f,g) := \begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} + R''\begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} + 3R'\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} + 2R\begin{vmatrix} f'' & g'' \\ f''' & g''' \end{vmatrix} + (2RR' - 3(R')^2)\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} - 2RR'\begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} - R'\begin{vmatrix} f & g \\ f^{(IV)} & g^{(IV)} \end{vmatrix} - 4R^2\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} - 2R\begin{vmatrix} f' & g' \\ f^{(IV)} & g^{(IV)} \end{vmatrix}$ is a well-defined global 5-form

Note that the assignment of holomorphic 1-forms $\omega$ to certain cocycles gives exactly the number of generators which is needed to generate the cohomology spaces. The formula for $c_7$ is given in the appendix.

These cocycles do not depend on the choice of the connections.

### 3. Cocycle property

Now, we have globalized the cocycles to individual cochains or families of cochains. But it is not clear whether the terms that we added will disturb the cocycle property. This is what we check in this section.

By writing explicitly the cocycle identity for the different expressions which we considered in the preceeding section to globalize the cocycles (the action depends on $\lambda$ (cf equation (1) in the preliminairies)), we get the following result:

(1) $\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$ is a cocycle for any value of $\lambda$

(2) $\begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}$ is a cocycle only for $\lambda = 1$

(3) $\begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix}$ is a cocycle only for $\lambda = 2$

(4) $\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}$ is a cocycle only when taking trivial action

(5) $\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix}$ is a cocycle for any value of $\lambda$

(6) $\begin{vmatrix} f & g \\ f^{(IV)} & g^{(IV)} \end{vmatrix}$ is never a cocycle

(7) $\begin{vmatrix} f' & g' \\ f^{(IV)} & g^{(IV)} \end{vmatrix}$ is never a cocycle

(8) $\begin{vmatrix} f'' & g'' \\ f''' & g''' \end{vmatrix}$ is a cocycle only for $\lambda = 3$

(9) $\begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix}$ is a cocycle only for $\lambda = 5$

The 6th and the 7th expressions arise as terms in $c_5(f,g)$. Note that in this formal calculation, $f, g$ can be interpreted as vector fields on the circle or on the open Riemann surface. In the latter case, the expressions are not globally defined geometric objects.

It is thus obvious that $\bar{c}_0^\omega$, $c_0^\omega$, $c_1$, $\bar{c}_1^\omega$ $c_2$, $\bar{c}_2^\omega$ are well-defined, global 2-cocycles for cohomology with values in $\mathcal{F}_\lambda$ with $\lambda = 0, 0, 1, 1, 2$ and 2 respectively.

For $c_5$ and $c_7$, we will take a different point of view.

## 4. Formulation in terms of the covariant derivative

The fundamental fact which assures the validity of our work is the existence of affine structures on open Riemann surfaces.

These connections are flat integrable connections in the sense of differential geometry, thus we can talk about associated covariant derivatives. The covariant derivative associated to the affine connection reads locally (on a $\lambda$-density $\phi$)

$$\nabla\phi \;=\; \phi' \;-\; \lambda\Gamma\phi.$$

In general, $\Gamma$ plays the role of the trace of the Christoffel symbols; in higher dimensions, we have

$$\nabla_i\phi \;=\; \partial_i\phi \;-\; \lambda\Gamma^j_{ij}\phi.$$

Actually, $\Gamma$ is nothing else than what we called before the affine connection $T$. $\nabla\phi$ is a globally defined object. On $\frac{1}{2}$-densities, one can exhibit a particular convenient choice of a projective connection associated to an affine connection:

$$\begin{aligned}
\nabla^2(\phi(dz)^{-\frac{1}{2}}) &= \nabla((\phi' + \frac{1}{2}\Gamma\phi)(dz)^{\frac{1}{2}}) \\
&= (\phi'' + \frac{1}{2}(\Gamma\phi)' - \frac{1}{2}\Gamma(\phi' + \frac{1}{2}\Gamma\phi))(dz)^{\frac{3}{2}} \\
&= (\phi'' - \frac{1}{4}\Gamma^2\phi + \frac{1}{2}\Gamma'\phi)(dz)^{\frac{3}{2}} \\
&= (\partial^2 + \frac{1}{2}R)\phi(dz)^{-\frac{1}{2}}
\end{aligned}$$

where we have put $R = -\frac{1}{2}\Gamma^2 + \Gamma'$.

Otherwise, this choice is justified by $\Gamma = \frac{h''}{h'}$ and $S = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2$, giving also $S = \Gamma' - \frac{1}{2}\Gamma^2$, cf [10] equation (10) p. 205.

Furthermore, we can set

$$L_f\phi(dx)^\lambda = f\nabla\phi + \lambda(\nabla f)\phi,$$

because this action coïncides with the action defined in equation (1). We have $[f,g] = f\nabla g - (\nabla f)g$ and a derivation property of $\nabla$ on tensor products. This corresponds to the product formula for the derivative. With this in mind, we have the same rules of manipulation as before for computations which concerned only ordinary derivatives of functions on the circle.

In conclusion, it is clear that we can formulate all cocycles in terms of the covariant derivative:

- $c_1(f,g) = \begin{vmatrix} \nabla f & \nabla g \\ \nabla^2 f & \nabla^2 g \end{vmatrix} dz$
- $\bar{c}_1(f,g) = \begin{vmatrix} f & g \\ \nabla^2 f & \nabla^2 g \end{vmatrix} dz^0$
- $c_2(f,g) = \begin{vmatrix} \nabla f & \nabla g \\ \nabla^3 f & \nabla^3 g \end{vmatrix} dz^2$
- $\bar{c}_2(f,g) = \begin{vmatrix} f & g \\ \nabla^3 f & \nabla^3 g \end{vmatrix} dz^1$
- $c_5(f,g) = \begin{vmatrix} \nabla^3 f & \nabla^3 g \\ \nabla^4 f & \nabla^4 g \end{vmatrix} dz^5$
- $c_7(f,g) := \left(2\begin{vmatrix} \nabla^3 f & \nabla^3 g \\ \nabla^6 f & \nabla^6 g \end{vmatrix} - 9\begin{vmatrix} \nabla^4 f & \nabla^4 g \\ \nabla^5 f & \nabla^5 g \end{vmatrix}\right) dz^7$

The cocycle $\bar{c}_2$ is the covariant derivative version of the Krichever-Novikov cocycle; we learnt this expression from D. Millionshshikov.

Obviously, this description is much simpler. To show at least in principle how the proof of this coïncidence looks like, take for example $\bar{c}_2$. We have to calculate

$$\begin{aligned}\nabla^3 f &= \nabla^2(f' + \Gamma f) \\ &= \nabla(f'' + \Gamma' f + \Gamma f') \\ &= f''' + \Gamma'' f + 2\Gamma' f' - \Gamma\Gamma' f - \Gamma^2 f'.\end{aligned}$$

Note that in the first line, $f$ is an element of $\mathcal{F}_{-1}(\Sigma_r)$, but $(f' + \Gamma f)$ is an element of $\mathcal{F}_0(\Sigma_r)$, in the second line, $(f'' + \Gamma' f + \Gamma f')$ is an element of $\mathcal{F}_1(\Sigma_r)$ and the result is in $\mathcal{F}_2(\Sigma_r)$.

This gives

$$f(\nabla^3 g) - g(\nabla^3 f) = fg''' - gf''' + (2\Gamma - \Gamma^2)(fg' - gf').$$

Identifying $(2\Gamma - \Gamma^2)$ with $(2\Gamma - \Gamma^2) = 2R$, we get the coincidence of the covariant derivative expression with $\bar{c}_2$.

As said before, it is clear that all covariant derivative expressions will be cocycles with values in the appropriate $\mathcal{F}_\lambda$ - the computations are straight forward.

## 5. Non-triviality of the cocycles

Let us scetch here an argument showing the non-triviality of the constructed cocycles (and their linear independence):

Choose an embedding $S^1 \to \Sigma_r$. The associated restriction of holomorphic vector fields (resp. holomorphic $\lambda$-densities) to $S^1$ gives a map $\phi : Hol(\Sigma_r) \to Vect(S^1)$ (resp. $\psi : \mathcal{F}_\lambda(\Sigma_r) \to \mathcal{F}_\lambda(S^1)$). These maps are injective Lie algebra homomorphisms with dense image (where the image is equipped with the induced topology from $Vect(S^1)$), cf [12].

There is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_\lambda(\Sigma_r) \arrow[r] \arrow[d, "\psi"] & Hol(\Sigma_r) \times \mathcal{F}_\lambda(\Sigma_r) \arrow[r] \arrow[d] & Hol(\Sigma_r) \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & \mathcal{F}_\lambda(S^1) \arrow[r] & Vect(S^1) \times \mathcal{F}_\lambda(S^1) \arrow[r] & Vect(S^1) \arrow[r] & 0
\end{tikzcd}$$

The non-triviality now follows from the non-triviality (see [13]) of the corresponding cocycles for $Vect(S^1)$.

## A. The formula for $c_7$

Let $T$ denote an affine connection on the open Riemann surface $\Sigma_r$. In this appendix, we give the formula for the global version of the cocycle $c_7$ in terms of $T$.

$$
\begin{aligned}
c_7(f,g) &= \left(2\begin{vmatrix} f''' & g''' \\ f^{(VI)} & g^{(VI)} \end{vmatrix} - 9\begin{vmatrix} f^{(IV)} & g^{(IV)} \\ f^{(V)} & g^{(V)} \end{vmatrix}\right) \\
&+ (2T^{(V)} - 22T'T''' - 2TT^{(IV)} - 20(T'')^2 + 14(T')^3 + 42TT'T'' \\
&\quad +7T^2T''' - 21T^2(T')^2 - 7T^3T'')\begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} \\
&+ (10T^{(IV)} - 100T'T'' - 10TT''' \\
&\quad +70T(T')^2 + 35T^2T'' - 35T^3T')\begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} \\
&+ (18T''' - 46(T')^2 - 18TT'' + 28T^2T' - 7T^4)\begin{vmatrix} f'' & g'' \\ f''' & g''' \end{vmatrix} \\
&+ (-37T''T^{(IV)} + 60T'(T'')^2 - 35TT''T''' - 167T(T')^2T'' - 6T^2(T'')^2 \\
&\quad +32T^3T'T'' + 16T^5T'' + 33TT'T^{(IV)} + 35T^2T'T''' + 32T^2(T')^3 \\
&\quad -32T^4(T')^2 + 4T^{(V)}T' - 152(T')^2T''' + 100(T')^4 - 2T^2T^{(V)} \\
&\quad +2T^3T^{(IV)} - 16T^4T''' + 135T'T''' + 36(T''')^2)\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \\
&+ (27T''T''' + 91(T')^2T'' - 27T(T'')^2 - 64T^2T'T'' + 16T^4T'' \\
&\quad -9TT'T''' - 37T(T')^3 + 64T^3(T')^2 - 16T^5T' - 18T'T^{(IV)} \\
&\quad +9T^2T^{(IV)} - 9T^3T''')\begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} \\
&+ (-9T^{(IV)} - 14T^2T'' + 14T^3T' + 55T'T'' - 28T(T')^2 \\
&\quad +9TT''')\begin{vmatrix} f & g \\ f^{(IV)} & g^{(IV)} \end{vmatrix} \\
&+ (-36T'T''' + 236(T')^3 - 162TT'T'' - 111(T')^2T^2 + 96T'T^4 \\
&\quad +54T^2T''' - 54T^3T'' - 16T^6 - 72T'T'' + 135(T'')^2)\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} \\
&+ (-56T^2T' + 92(T')^2 + 14T^4 - 36T''' + 36TT'')\begin{vmatrix} f' & g' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} \\
&+ (-5T^2 + 10T')\begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} \\
&+ (-2T'' + 2TT')\begin{vmatrix} f & g \\ f^{(VI)} & g^{(VI)} \end{vmatrix} \\
&+ (-9TT'' + 9T''' - 9(T')^2)\begin{vmatrix} f & g \\ f^{(V)} & g^{(V)} \end{vmatrix} \\
&+ (-4T' + 2T^2)\begin{vmatrix} f' & g' \\ f^{(VI)} & g^{(VI)} \end{vmatrix} \\
&+ (-27TT' + 27T'')\begin{vmatrix} f' & g' \\ f^{(V)} & g^{(V)} \end{vmatrix} \\
&+ (-45T'' + 45TT')\begin{vmatrix} f'' & g'' \\ f^{(IV)} & g^{(IV)} \end{vmatrix}
\end{aligned}
$$

$$+ \quad (18T' - 18T^2) \begin{vmatrix} f'' & g'' \\ f^{(V)} & g^{(V)} \end{vmatrix}$$

is a well-defined global 7-form.

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