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Direct Limits of Zuckerman Derived Functor Modules

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Abstract. We construct representations of certain direct limit Lie groups $G = \lim G^n$ via direct limits of Zuckerman derived functor modules of the groups G^n. We show such direct limits exist when the degree of cohomology can be held constant, and discuss some examples for the groups $Sp(p, \infty)$ and $SO(2p, \infty)$, relating to the discrete series and ladder representations. We show that our examples belong to the "admissible" class of Ol′shanskiĭ, and also discuss the globalizations of the Harish-Chandra modules obtained by the derived functor construction. The representations constructed here are the first ones in cohomology of non-zero degree for direct limits of non-compact Lie groups.

1. Introduction

The theme of this article is the construction of irreducible unitary representations of certain direct limit Lie groups. Such groups are not locally compact and their representation theory may be ill behaved. The obvious way to construct their representations is to build them as direct limits of irreducible unitary representations of the approximating groups. However, one may not be able to obtain every irreducible unitary representation of the limit group by this procedure.

Ol′shanskiĭ [18, 19] has established a representation theory for the classical direct limit Lie groups under a restricting condition, that we call "O-admissibility," and which is equivalent to continuity of the group action in the strong operator topology. One striking fact is that Ol′shanskiĭ completely describes the O-admissible unitary dual of the limit groups even though the unitary dual remains unknown for the finite dimensional groups. In particular he has shown that the O-admissible representations arise as direct limits of irreducible unitary representations of the constituent groups. It is of interest to see which parts of the unitary duals (in terms of the standard classifications) of the finite dimensional groups contribute to the O-admissible unitary dual of the limit group. Ol′shanskiĭ himself has addressed this question via an analog of Howe duality. (See also Neretin and Ol′shanskiĭ [17].)

In another direction, Natarajan [11, 12] has classified the unitarizable highest weight representations of $U(p, \infty)$, $SO(2, \infty)$ and $Sp(\infty, \mathbb{R})$. These turn out

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to be parametrized by highest weights $\lambda^\infty = (\lambda_1, \ldots, \lambda_i, \ldots)$ where the λ_i are eventually constant and within certain bounds. The O-admissible ones, however, are only those for which the eventual value of λ_i is 0. More generally, Neeb and Ørsted [16] have classified the unitary highest weight representations of Banach-Lie groups of automorphisms of symmetric domains in Hilbert spaces and the corresponding Banach-Lie algebras. Recently, Natarajan, Rodríguez-Carrington and Wolf [15] have constructed a version of the Bott-Borel-Weil Theorem for direct limits of compact groups.

The appropriate generalization of these results to the non-Hermitian symmetric case is to consider direct limits of Zuckerman's derived functor modules. A particular advantage of such a construction is the link it establishes with the Langlands classification of the irreducible $(\mathfrak{g}, K)$-modules for finite dimensional real reductive Lie groups.

In this article, we arrange derived functor modules into direct systems to generate unitary representations for the limit groups $SO(2p, \infty)$ and $Sp(p, \infty)$. As in [15], we need to hold the degree of cohomology constant over the direct system.

The derived functor approach proceeds in two steps. The first involves the construction of highest weight modules and we discuss this in Section 2. Section 3 contains the main theorem about the existence of direct limits of Zuckerman derived functor modules. In Section 4 we consider the natural requirement that the cohomology of the limit should equal the limit of the cohomologies. Section 5 lifts the obtained direct systems of Harish-Chandra modules to their globalizations to obtain representations of the limit groups. Finally, Sections 6 and 7 provide examples where the hypotheses of our theorems are satisfied.

The example of Section 6 is obtained by considering the coherent continuations of the Borel-de Siebenthal discrete series. An interesting aspect of this case is that our direct systems eventually consist of the continuations alone, so that the discrete series itself does not contribute to the limit representations. In Section 7 we line up ladder representations of $Sp(p, n)$ to get ladder representations of $Sp(p, \infty)$. Both situations yield examples of O-admissible representations.

We finish this section by collecting some standard definitions and notations. Let G be a real Lie group and M a compact subgroup of G. Write $\mathfrak{g}_0$ and $\mathfrak{m}_0$ for the Lie algebras of G and M. Here, and subsequently, we denote complexification by dropping the $_0$ subscript.

By a $(\mathfrak{g}, M)$-module we mean a complex vector space V which is simultaneously a representation space for $\mathfrak{g}$ and M (both representations being denoted by π) such that:

1. The differential of the M action equals the action of $\mathfrak{m}_0$ as a subalgebra of $\mathfrak{g}_0$.

2. $\pi(\mathrm{Ad}(m)X) = \pi(m)\pi(X)\pi(m)^{-1}$, for every $m \in M$, $X \in \mathfrak{g}$.

3. Every vector $v \in V$ is M-finite: the span of the M-orbit of v is finite dimensional. Further, the representation of M on this finite dimensional space is continuous with respect to its natural topology.

The category of $(\mathfrak{g}, M)$-modules is denoted $\mathcal{C}(\mathfrak{g}, M)$. Every member V of

$\mathcal{C}(\mathfrak{g}, M)$ has a direct sum decomposition

$$V = \bigoplus_{\gamma \in \widehat{M}} n_\gamma V_\gamma$$

where $\widehat{M}$ is the unitary dual of M, V_γ is a representation space corresponding to γ, and n_γ is its multiplicity. We say V is *admissible* if each n_γ is finite. The category of admissible $(\mathfrak{g}, M)$-modules is denoted $\mathcal{A}(\mathfrak{g}, M)$.

Now we restrict to the case where G is a connected real semisimple Lie group and K is a maximal compact subgroup of G. Let $(\pi, \mathcal{H})$ be a continuous representation of G in a separable Hilbert space $\mathcal{H}$ (i.e., the map $G \times \mathcal{H} \to \mathcal{H}$, $(x, v) \mapsto \pi(x)v$, is continuous). Then the space of K-finite analytic vectors in $\mathcal{H}$, denoted $\mathcal{H}_K$, is a $(\mathfrak{g}, K)$-module and is dense in $\mathcal{H}$. We call $\mathcal{H}_K$ the underlying *Harish-Chandra module* of $\mathcal{H}$, while $\mathcal{H}$ is said to be a globalization of $\mathcal{H}_K$. By a theorem of Harish-Chandra every admissible $(\mathfrak{g}, K)$-module arises as the $\mathcal{H}_K$ of some $\mathcal{H}$.

A $(\mathfrak{g}, K)$-module V is called *unitarizable* if it has an inner product $\langle , \rangle$ such that:

1. $\langle kv, kw \rangle = \langle v, w \rangle$, for every $k \in K$, $v, w \in V$.
2. $\langle Xv, w \rangle = -\langle v, Xw \rangle$, for every $X \in \mathfrak{g}_0$, $v, w \in V$.

If $\mathcal{H}$ is a unitary representation space of G then $\mathcal{H}_K$ is clearly a unitarizable $(\mathfrak{g}, K)$-module. Conversely, given a unitarizable admissible $(\mathfrak{g}, K)$-module V, its Harish-Chandra globalization mentioned above is a unitary representation of G on the completion of V.

Thus we have a framework for algebraicizing our problem. It is often simpler to work with the category $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ of $(\mathfrak{g}, \mathfrak{m})$-modules. These are representation spaces of $\mathfrak{g}$ in which every vector is $\mathfrak{m}$-finite, i.e., the space $\mathcal{U}(\mathfrak{m})v$ is finite dimensional for every vector v. A $(\mathfrak{g}, \mathfrak{m})$-module is a $(\mathfrak{g}, M)$-module if all its $\mathfrak{m}$ types are actually M types. For $(\mathfrak{g}, \mathfrak{m})$-modules we have again the notions of admissible and unitarizable, adapted in the obvious way from the above.

If V is some $\mathfrak{g}$ module, and $\mathfrak{m}$ a Lie subalgebra of $\mathfrak{g}$ such that V is $\mathfrak{m}$-semisimple, then $V_\mathfrak{m}$ denotes the submodule of $\mathfrak{m}$-finite vectors.

Let L be a Levi subgroup of G with Lie algebra $\mathfrak{l}_0$. The *Zuckerman functor* Γ takes $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ to the submodule $\Gamma V \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ of its $\mathfrak{k}$-finite vectors. $\mathcal{C}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ has enough injectives [1], so the right derived functors Γ^i of Γ are defined. For instance, we start with the Koszul resolution $X_i = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} \Lambda^i(\mathfrak{g}/\mathfrak{k})$ of the trivial representation and then set $\mathrm{I}_i = \mathrm{Hom}_\mathbb{C}(X_i, V)_{\mathfrak{l}\cap\mathfrak{k}}$ to get an injective resolution of V. Applying Γ to each element of the injective resolution $0 \to V \to \mathrm{I}_*$ we get the complex $\Gamma\mathrm{I}_0 \to \Gamma\mathrm{I}_1 \to \ldots$ and $\Gamma^i V$ is the i-th cohomology of this complex.

For $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$, ($\mathfrak{m} = \mathfrak{k}$ or $\mathfrak{l} \cap \mathfrak{k}$) define the contragredient representation space V^c to be the subspace of $\mathfrak{m}$-finite maps in the algebraic dual V^*. Similarly the conjugate dual $\hat{V}$ is defined to be the subspace of $\mathfrak{m}$-finite maps in the algebraic conjugate-dual $\bar{V}^*$. Then $V^c, \hat{V} \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$.

It is useful to consider the full subcategory $\mathcal{A}(\mathfrak{g}, \mathfrak{m})$ of $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ which consists of the $\mathfrak{m}$-admissible modules. $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$ contains the irreducible $(\mathfrak{g}, \mathfrak{k})$-modules. Γ and Γ^i map $\mathcal{A}(\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k})$ into $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Further, $\mathcal{A}(\mathfrak{g}, \mathfrak{m})$ is closed under $V \mapsto \hat{V}$ and $V \mapsto V^c$. (For details on Zuckerman functors see Knapp [8].)

Let θ be a Cartan involution of G, with corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Suppose $\mathfrak{k}_0$ has a Cartan subalgebra $\mathfrak{h}_0$ whose complexification $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Choose a set of positive roots $\triangle^+(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{q}$ be a corresponding parabolic subalgebra of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. Write $\mathfrak{u}_c = \mathfrak{u} \cap \mathfrak{k}$ and $s = \dim \mathfrak{u}_c$. Finally, denote by $\mathfrak{b}_L$ the Borel subalgebra of $\mathfrak{l}$ defined by $\triangle^+(\mathfrak{l}, \mathfrak{h}) = \triangle^+(\mathfrak{g}, \mathfrak{h}) \cap \triangle(\mathfrak{l}, \mathfrak{h})$.

We will work with a sequence G^n of semisimple Lie groups, with choices of maximal compact subgroups K^n and θ-stable Levi subgroups L^n (θ-stability makes the Levi subgroups reductive). We assume the presence of homomorphic embeddings $i_{m,n} : G^n \to G^m$, $n < m$, which also map K^n into K^m and L^n into L^m, and satisfy $i_{l,m} \circ i_{m,n} = i_{l,n}$. We denote their derivatives also by $i_{m,n}$.

By a *direct system* of $(\mathfrak{g}^n, \mathfrak{m}^n)$-modules we mean a collection $\{V^n, \phi_{m,n}\}$ where $V^n \in \mathcal{C}(\mathfrak{g}^n, \mathfrak{m}^n)$ and $\phi_{m,n} : V^n \to V^m$ $(n < m)$ are linear maps such that $\phi_{l,m} \circ \phi_{m,n} = \phi_{l,n}$ and the diagram

$$\begin{array}{ccc} \mathfrak{g}^n \times V^n & \longrightarrow & V^n \\ \big\downarrow & & \big\downarrow \\ \mathfrak{g}^m \times V^m & \longrightarrow & V^m \end{array}$$

commutes. Here $n < m$, the horizontal arrows are given by the action of $\mathfrak{g}^n$ on V^n, and the vertical ones by $i_{m,n} : \mathfrak{g}^n \hookrightarrow g^m$ and $\phi_{m,n} : V^n \hookrightarrow V^m$. Given such a direct system we get a limit representation for the direct limit algebra $\mathfrak{g}^\infty$ in an obvious way. We similarly define direct limits of representations at the group level.

We will also need the dual notion of an *inverse system* of $(\mathfrak{g}^n, \mathfrak{m}^n)$-modules. This is a collection $\{W^n, \psi_{m,n}\}$ where $W^n \in \mathcal{C}(\mathfrak{g}^n, \mathfrak{m}^n)$ and $\psi_{m,n} : W^m \to W^n$ $(n < m)$ are linear maps such that

$$\psi_{m,n}(i_{m,n}(X) \cdot v) = X \cdot \psi_{m,n}(v), \quad \forall X \in \mathfrak{g}^n, \ v \in W^m.$$

Such an inverse system leads to an *inverse limit* representation of $\mathfrak{g}^\infty$.

For details on direct and inverse limit constructions in this context, the reader should consult the articles [13, 14, 15] of Natarajan, Rodríguez-Carrington, and Wolf.

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2. Direct Limits of Highest Weight Modules

Let (G^n, K^n, L^n) be a sequence as described above, with the Lie algebras $(\mathfrak{g}_0^n, \mathfrak{k}_0^n, \mathfrak{l}_0^n)$. The basic construction in this article is to take a direct system of $(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$-modules and apply Γ^s to get a direct system of $(\mathfrak{g}^n, \mathfrak{k}^n)$-modules. The $(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$-modules that we will consider are quotients of generalized Verma modules, which we describe in this section. In this section, and hereafter, we should keep in mind that G^n is $Sp(p,n)$ or $SO(2p,2n)$, while the embeddings $i_{m,n} : \mathfrak{g}^n \hookrightarrow \mathfrak{g}^m$ are the ones from Section 6.

Extend the $i_{m,n}$ to the universal enveloping algebras $\mathcal{U}(\mathfrak{g}^n)$. Let $\mathfrak{h}^n \subset \mathfrak{g}^n$ be choices of Cartan subalgebras such that $i_{m,n}$ maps $\mathfrak{h}^n$ into h^m. Let an element λ^n of $\mathfrak{h}^{n*}$ be represented by a tuple $(\lambda_1, \dots, \lambda_{m_n})$. Consider an ∞-tuple $\lambda^\infty = (\lambda_1, \dots, \lambda_m, \dots)$ such that each truncation $\lambda^n = (\lambda_1, \dots, \lambda_{m_n})$ is a dominant integral weight of $\mathfrak{l}^n$. Define the $\mathcal{U}(\mathfrak{l}^n)$-module $M(\lambda^n) = \mathcal{U}(\mathfrak{l}^n) \otimes_{\mathcal{U}(\mathfrak{b}^n_L)} \mathbb{C}_{\lambda^n}$ and let $F(\lambda^n)$ be the (finite-dimensional) irreducible quotient. $F(\lambda^n)$ becomes a representation of $\mathfrak{q}^n$ by letting $\mathfrak{u}^n$ act trivially, and then the *generalized Verma module* $N(\lambda^n) = \mathcal{U}(\mathfrak{g}^n) \otimes_{\mathcal{U}(\mathfrak{q}^n)} F(\lambda^n)$ is a $\mathcal{U}(\mathfrak{g}^n)$-module with highest weight λ^n. Its irreducible quotient is denoted $L(\lambda^n)$.

All the modules $M(\lambda^n)$, $F(\lambda^n)$, $N(\lambda^n)$, and $L(\lambda^n)$ have a canonical invariant Hermitian form known as the *Jantzen Hermitian form* [7]. Let us denote this form by ξ_n in all cases. The kernel of this form is always the maximal submodule of the module under consideration. Dividing by this kernel therefore gives the unique irreducible quotient.

Proposition 2.1. *The modules $N(\lambda^n)$ and $L(\lambda^n)$ form direct systems.*

Proof. It is easy to see that the $M(\lambda^n)$ form a direct system of modules under the embeddings $e_{m,n} : M(\lambda^n) \to M(\lambda^m)$ whose action is $X \otimes 1 \mapsto i_{m,n}(X) \otimes 1$. It was shown by Natarajan in [12, §3] that the $e_{m,n}$ preserve the form ξ_n and hence fall through to the quotients $F(\lambda^n)$. Now define $j_{m,n} : N(\lambda^n) \to N(\lambda^m)$ by $X \otimes v \mapsto i_{m,n}(X) \otimes e_{m,n}(v)$. By the same argument as before, these fall through to the irreducible quotients $L(\lambda^n)$ to give us a direct system of irreducible $\mathcal{C}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$-modules. ■

3. Direct Limits of Derived Functor Modules

Consider a sequence of triples $(\mathfrak{g}^n, \mathfrak{k}^n, \mathfrak{l}^n)$, as in Section 2, with embeddings $i_{m,n} : \mathfrak{g}^n \hookrightarrow \mathfrak{g}^m$ $(n \leq m)$ which map $\mathfrak{k}^n \hookrightarrow \mathfrak{k}^m$ and $\mathfrak{l}^n \hookrightarrow \mathfrak{l}^m$. Suppose also that $\mathfrak{u}^n_c$ is mapped isomorphically onto $\mathfrak{u}^m_c$. (Thus the "middle dimension" $s = \dim(\mathfrak{u}^n_c)$ is independent of n. This is the **main assumption** of this paper and is made throughout.) Let V^n be a sequence of $\mathcal{C}(\mathfrak{g}^n, \mathfrak{l}^n)$-modules with embeddings $j_{m,n}$ which are compatible with the $i_{m,n}$. Let $\mathcal{F}$ denote the forgetful functor from $\mathcal{C}(\mathfrak{g}^m, \mathfrak{l}^m \cap \mathfrak{k}^m)$ to $\mathcal{C}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$, or from $\mathcal{C}(\mathfrak{g}^m, \mathfrak{k}^m)$ to $\mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$.

Proposition 3.1. *Let $V \in \mathcal{C}(\mathfrak{g}^m, \mathfrak{l}^m \cap \mathfrak{k}^m)$. Then $\Gamma^i \circ \mathcal{F}(V) \cong \mathcal{F} \circ \Gamma^i(V)$.*

Proof. We first show that $\Gamma \circ \mathcal{F}(V) = \mathcal{F} \circ \Gamma(V)$. On the left side, we pick out the $\mathfrak{k}^n$-finite vectors of V, while on the right side we pick out the $\mathfrak{k}^m$-finite vectors of V. Since the spaces are already $\mathfrak{l}^m \cap \mathfrak{k}^m$-finite (hence $\mathfrak{l}^n \cap \mathfrak{k}^n$-finite), and $\mathfrak{k}^m = (\mathfrak{l}^m \cap \mathfrak{k}^m) + \mathfrak{u}^m_c$, we are picking out the $\mathfrak{u}^n_c$ and $\mathfrak{u}^m_c$-finite vectors. The isomorphism shows the two choices are the same. And then, since the $\mathcal{F}$ functors are exact, we have $\Gamma^i \circ \mathcal{F} \cong (\Gamma \circ \mathcal{F})^i \cong (\mathcal{F} \circ \Gamma)^i \cong \mathcal{F} \circ \Gamma^i$. ■

Proposition 3.2. *The injections $j_{m,n}$ induce a direct limit of the modules $\Gamma^i V^n$.*

Proof. The map $j_{m,n} : V^n \to V^m$ induces a homomorphism $j_{m,n} : V^n \to \mathcal{F}(V^m)$ in the category $\mathcal{C}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$. Applying the Zuckerman derived functors, we get a map $\Gamma^i j_{m,n} : \Gamma^i V^n \to \Gamma^i \circ \mathcal{F}(V^m)$ in the category $\mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$, giving a homomorphism $\Gamma^i V^n \to \mathcal{F} \circ \Gamma^i V^m$ (by Proposition 3.1) in $\mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$, as needed. ∎

Note that the induced maps $\Gamma^i j_{m,n}$ have not been shown to be injective. Now suppose the representations V^n carry invariant Hermitian forms ϕ_n and that the maps $j_{m,n}$ preserve these forms. We shall show that $\Gamma^s j_{m,n}$ preserves the induced forms. The following duality result is useful:

Theorem 3.3. [2, Theorem 7.3] *Let $V^n \in \mathcal{A}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$. Then we have a natural isomorphism $\Gamma^i \hat{V}^n \cong (\Gamma^{2s-i} V^n)\hat{}$.*

The form ϕ_n induces a map $\hat{\phi}_n : V^n \to \hat{V}^n$ by $\hat{\phi}_n(u)(v) = \phi_n(u,v)$. Applying the functor Γ^s, we get a map $\Gamma^s \hat{\phi}_n : \Gamma^s V^n \to \Gamma^s \hat{V}^n \cong (\Gamma^s V^n)\hat{}$. This induces an invariant Hermitian form $\Gamma^s \phi_n$ on $\Gamma^s V^n$.

Proposition 3.4. *Let $V^n \in \mathcal{A}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$ form a direct system given by maps $j_{m,n} : V^n \to V^m$. Let ϕ_n be invariant Hermitian forms on V^n which are preserved by the $j_{m,n}$. Then the $\Gamma^s j_{m,n}$ preserve the induced forms $\Gamma^s \phi_n$. Hence they are injections when the induced forms are positive-definite (i.e., when the $\Gamma^s V^n$ are unitarizable).*

Proof. Set $j = j_{m,n}$. Let $j^* : \hat{V}^m \to \hat{V}^n$ be the adjoint map. For any $u, v \in V^n$ we have $(j^*(\hat{\phi}_m(ju)))v = (\hat{\phi}_m(ju))(jv) = \phi_m(ju, jv)$. The fact that j is an isometry gives us the commuting diagram on the left, below. This induces a commuting diagram in the category $\mathcal{A}(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$, drawn on the right.

$$\begin{CD} V^n @>{j}>> V^m \\ @V{\hat{\phi}_n}VV @VV{\hat{\phi}_m}V \\ \hat{V}^n @<<{j^*}< \hat{V}^m \end{CD} \qquad\qquad \begin{CD} V^n @>{j}>> \mathcal{F}V^m \\ @V{\hat{\phi}_n}VV @VV{\mathcal{F}\hat{\phi}_m}V \\ \hat{V}^n @<<{j^*}< \mathcal{F}\hat{V}^m \end{CD}$$

Γ^s is a functor, so we get the following commuting diagram in the category $\mathcal{A}(\mathfrak{g}^n, \mathfrak{k}^n)$:

$$\begin{CD} \Gamma^s V^n @>{\Gamma^s j}>> \Gamma^s \circ \mathcal{F}V^m \\ @V{\Gamma^s \hat{\phi}_n}VV @VV{\Gamma^s \circ \mathcal{F}\hat{\phi}_m}V \\ \Gamma^s \hat{V}^n @<<{\Gamma^s j^*}< \Gamma^s \circ \mathcal{F}\hat{V}^m \end{CD}$$

Theorem 3.3 implies $(\Gamma^s V^n)\hat{} \cong \Gamma^s \hat{V}^n$.
Proposition 3.1 implies $\Gamma^s \circ \mathcal{F} V^m \cong \mathcal{F} \circ \Gamma^s V^m$.
Similarly, $\Gamma^s \circ \mathcal{F} \hat{V}^m \cong \mathcal{F} \circ \Gamma^s \hat{V}^m \cong \mathcal{F} \circ (\Gamma^s V^m)\hat{}$.
Putting these together, we get the commuting diagram

$$\begin{array}{ccc}
\Gamma^s V^n & \xrightarrow{\quad \Gamma^s j \quad} & \mathcal{F} \circ \Gamma^s V^m \\
{\scriptstyle \Gamma^s \hat{\phi}_n} \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{F} \circ \Gamma^s \hat{\phi}_m} \\
(\Gamma^s V^n)\hat{} & \xleftarrow[\quad \Gamma^s j^* \quad]{} & \mathcal{F} \circ (\Gamma^s V^m)\hat{}
\end{array}$$

which is the desired result. ∎

Thus we have a way of constructing direct limits of $\mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$-modules. Of course, what we really need are $\mathcal{C}(\mathfrak{g}^n, K^n)$-modules. In our examples, at least, it is easy to see that we have also achieved the latter goal.

4. Zuckerman Functors at the Limit

A natural question that arises in such a situation is whether the limit of the cohomologies is the cohomology of the limit. To address this issue we first discuss inverse limits.

Suppose we have an inverse system W^n of $(\mathfrak{g}^n, \mathfrak{l}^n \cap \mathfrak{k}^n)$-modules. Take any W^n and let $W^n \to I_n^*$ be an injective resolution. Following the proof of [6, Chapter 1, Lemma 4.6] we can find an injective resolution $W^{n+1} \to I_{n+1}^*$ such that we have a commuting diagram

$$\begin{array}{ccc}
I_n^* & \xleftarrow{\quad \beta^* \quad} & I_{n+1}^* \\
\big\uparrow & & \big\uparrow \\
W^n & \longleftarrow & W^{n+1}
\end{array}$$

with $I_{n+1}^p = I_n^p \oplus K^p$ and β^p being the projection onto the first factor. By induction we get a double complex in which the rows are surjective. Applying the Zuckerman functor to each member, we get the following double complex

$$\begin{array}{ccccccc}
& & \big\uparrow & & \big\uparrow & & \\
& \longleftarrow & \Gamma I_n^1 & \longleftarrow & \Gamma I_{n+1}^1 & \longleftarrow & \\
& & \big\uparrow & & \big\uparrow & & \\
& \longleftarrow & \Gamma I_n^0 & \longleftarrow & \Gamma I_{n+1}^0 & \longleftarrow &
\end{array}$$

Note that Γ simply extracts the $\mathfrak{u}_c^n$-finite vectors and these algebras are identified with each other by our stability assumption. So the rows of this double complex are again surjective, while the i-th cohomologies of the columns give the derived functor modules $\Gamma^i(W^n)$. We can therefore apply [6, Chapter 1, Proposition 4.4] to obtain for each i the following exact sequence:

$$0 \to \varprojlim{}^{(1)} \Gamma^{i-1}(W^n) \to \Gamma^i(\varprojlim W^n) \to \varprojlim \Gamma^i(W^n) \to 0$$

where $\varprojlim^{(1)}$ is the first derived functor of the inverse limit functor.

Proposition 4.1. *Suppose we have a direct system V^n of $(\mathfrak{g}^n, \mathfrak{l}^n)$-modules, such that $\Gamma^i(V^n) = 0$ for $i \neq s = \dim(\mathfrak{u}_c^n)$. Then*

$$\varinjlim(\Gamma^s V^n) \cong \Gamma^s(\varinjlim V^n).$$

Proof. The adjoint maps give us an inverse system of the contragredients $W^n = (V^n)^c$. The vanishing hypothesis combined with the above exact sequence gives us

$$\varprojlim \Gamma^s(W^n) \cong \Gamma^s(\varprojlim W^n).$$

Taking contragredients and applying the isomorphism $\Gamma^s(W^n) \cong (\Gamma^s V^n)^c$ (an analog of Theorem 3.3 with the same proof) we get the desired result. ■

Remark A. The vanishing hypothesis may not be necessary, if one can show that $\varprojlim^{(1)} = 0$. In the compact case of [15] this is trivial because the finite dimensionality of the representations ensures that the "Mittag-Lefler condition" [6, Chapter 1, Corollary 4.3] is satisfied.

5. Globalizations

Once a direct system W^n of $\mathcal{C}(\mathfrak{g}^n, K^n)$-modules is obtained we would like to globalize its members so as to get a direct limit representation of the direct limit group. In this section we set up an appropriate notion of underlying Harish-Chandra module for the direct limit representations of direct limit groups. The results of this section do not require the stability assumption imposed in the rest of this article.

The classical limit groups G^∞ have a natural analytic differential structure, relative to the direct limit topology [13, 15]. If we define the exponential map for the limit group to be the direct limit of the exponential maps for the component groups G^n, then it is a local analytic diffeomorphism at 0 relative to the limit analytic structures. It follows that a vector v is analytic with respect to the G^∞ action if it is analytic with respect to each G^n action, the action on v of $X \in \mathfrak{g}^\infty$ being its action as a member of some $\mathfrak{g}^n$.

Definition 5.1. Let V be a representation space, without topology, of K^∞. A vector $v \in V$ is said to be K^∞-*finite* if it is K^n-finite for each n.

Definition 5.2. A $(\mathfrak{g}^\infty, K^\infty)$*-module* is a vector space V on which there are compatible $\mathfrak{g}^\infty$ and K^∞ actions (in the usual sense) and every vector is K^∞-finite.

Similarly, we have $(\mathfrak{g}^\infty, \mathfrak{k}^\infty)$-modules. The corresponding categories are denoted $\mathcal{C}(\mathfrak{g}^\infty, K^\infty)$ and $\mathcal{C}(\mathfrak{g}^\infty, \mathfrak{k}^\infty)$.

Definition 5.3. Let G^∞ have a continuous representation in a complete, locally convex, Hausdorff topological vector space $\mathcal{V}$. The subspace $\mathcal{HV}$ of analytic K^∞-finite vectors in $\mathcal{V}$ is called the *underlying Harish-Chandra module* of $\mathcal{V}$.

The corresponding results for finite dimensional groups imply that $\mathcal{HV}$ is dense in $\mathcal{V}$ and is a member of $\mathcal{C}(\mathfrak{g}^\infty, K^\infty)$. We call $\mathcal{V}$ a *globalization* of $\mathcal{HV}$. A given member of $\mathcal{C}(\mathfrak{g}^\infty, K^\infty)$ may have many globalizations.

Suppose the modules W^n in $\mathcal{C}(\mathfrak{g}^n, K^n)$ are unitarizable and the injections $j : W^n \hookrightarrow W^{n+1}$ preserve the invariant forms. Then they extend to the Hilbert space completions, and we get a direct limit of unitary Hilbert space globalizations. In the non-unitarizable case we can use Schmid's minimal globalization functor [20], as follows.

Let $W \mapsto W_{\min}$ be the minimal globalization functor from $\mathcal{C}(\mathfrak{g}^n, K^n)$ into the category $\mathcal{C}(G^n)$ of continuous representations of G^n in complete, locally convex, Hausdorff topological vector spaces. Let $\mathcal{H}$ be the functor which picks out the underlying Harish-Chandra module of a representation of G^n. $\mathcal{F}$ denotes the appropriate forgetful functors.

Proposition 5.4. *Let $W^n \in \mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$ form a direct system. Then the modules $W^n_{\min}$ also form a direct system.*

Proof. Since $K^n \subset K^{n+1}$, $\mathcal{F}W^{n+1} \subset \mathcal{HF}(W^{n+1}_{\min})$. Then, the inclusion $W^n \hookrightarrow \mathcal{F}W^{n+1}$ induces $W^n_{\min} \hookrightarrow (\mathcal{F}W^{n+1})_{\min} \hookrightarrow (\mathcal{HF}(W^{n+1}_{\min}))_{\min} \hookrightarrow \mathcal{F}(W^{n+1}_{\min})$; the last step following from the minimality property. ■

The direct limit of a countable sequence of complete, locally convex, Hausdorff topological vector spaces is also complete, locally convex and Hausdorff in the direct limit topology. (See Köthe[10].) It is clear, therefore, that W^∞ is the underlying Harish-Chandra module of the globalization $W^\infty_{\min} = \lim W^n_{\min}$.

Proposition 5.5. *Given a direct system of minimal globalizations, their direct limit is minimal among the complete, locally convex, Hausdorff globalizations.*

Proof. Consider a direct system $W^n \in \mathcal{C}(\mathfrak{g}^n, K^n)$ with direct limit W^∞. Let $\widetilde{W}^\infty$ be a globalization of W^∞ and $\widetilde{W}^m$ = closure of $\mathrm{im}(W^m)$ in $\widetilde{W}^\infty$. Then $\widetilde{W}^m$ is a globalization of W^m. The canonical injections $W^m_{\min} \hookrightarrow \widetilde{W}^m$ lead to the inclusions $W^\infty_{\min} \hookrightarrow \widetilde{W}^\infty$, with $\widetilde{W}^\infty$ given the direct limit topology. Since the original topology on $\widetilde{W}^\infty$ is necessarily weaker then the direct limit topology, this inclusion is continuous with respect to the original topology. ■

Remark B. The *maximal globalization functor* (Schmid [20]) can be analogously used for inverse limit representations. This fits neatly with the fact that the Dolbeault cohomology constructions give the maximal globalizations of the Zuckerman functor constructions (Wong [21]). One can then proceed as in [15] to set up representations in inverse limit sheaf cohomologies.

6. Continuation of Discrete Series

The classical groups $G^n = Sp(p,n)$ and $SO(2p,2n)$, $p < n$, satisfy the following two conditions:

1. $\operatorname{rank} G^n = \operatorname{rank} K^n = p$.

2. There is only one non-compact simple root and its coefficient in the expansion of the maximal root as a sum of simple roots is two. [1] (With $p > 1$ for $SO(2p,2n)$.)

Let $\mathfrak{q}^n = \mathfrak{l}^n + \mathfrak{u}^n$ be the parabolic whose reductive part $\mathfrak{l}^n$ is generated by the compact simple roots. This situation was studied in Section 13 of the article [2] by Enright, Parthasarathy, Wallach and Wolf, with λ^n restricted as follows:

Let $\zeta \in \mathfrak{h}^{n*}$ be orthogonal to $\triangle(\mathfrak{l}^n)$ and normalized by $2\langle\alpha,\zeta\rangle/\langle\alpha,\alpha\rangle = 1$, where α is the non-compact simple root. For $z \in \mathbb{R}$, define $\lambda^n = z\zeta$. Let z_0^n be the value of z such that $\lambda_0 + \rho$, where $\lambda_0 = z_0^n\zeta$, lies on a wall of the Weyl chamber $\mathcal{C}$ corresponding to the positive system $\triangle^+(\mathfrak{l}^n) \cup -\triangle(\mathfrak{u}^n)$. Then it was shown in [2] that $z < z_0^n$ implies $N(\lambda^n)$ is an irreducible $(\mathfrak{g},\mathfrak{l})$-module, and $\Gamma^s N(\lambda)$ is a discrete series representation of G^n (assuming λ^n is $\mathfrak{k}^n$-integral). A continuity argument shows that $N(\lambda^n)$ is irreducible for $z < a^n$, where a^n is called the *first reduction point* and is greater than z_0^n; and also that $\Gamma^s N(\lambda^n)$ is irreducible unitary for $z < a^n$. Moreover $\Gamma^i N(\lambda^n) = 0$ for $i \neq s$ in this range.

The values of a^n were calculated in [3] by Enright and Wolf. In [4], Frajria studied behaviour at and beyond a^n. Two constants c_0^n and c_1^n were calculated such that for $z < c_0^n$ (and $\mathfrak{k}^n$-integral), $\Gamma^s L(\lambda^n)$ is unitary and for z between c_0^n and c_1^n it is zero. In the cases of interest to us, it happens that $a^n < c_0^n = c_1^n$.

We will see below that the λ^n, as n increases, are compatible in the way described in Section 2, and give a limit weight λ^∞. Therefore Proposition 2.1 implies that the modules $L(\lambda^n)$ form a direct system. We say λ^∞ is $\mathfrak{k}^\infty$ integral if each λ^n is $\mathfrak{k}^n$ integral.

Proposition 6.1. *There exists a constant C (depending on G^∞) such that for every choice of $\mathfrak{k}^\infty$ integral λ^∞ satisfying $z < C$, the modules $\Gamma^s L(\lambda^n)$ form a direct system of unitary modules and so yield a unitary representation $\Gamma^s L(\lambda^\infty)$ of the limit group G^∞. Further, the sequence $\Gamma^s L(\lambda^n)$ eventually consists of representations which are not in the discrete series.*

Proof. We verify below, for each case, the following facts:

1. s is independent of n.
2. c_0^n is independent of n.
3. $z_0^n \to -\infty$ as $n \to \infty$.

Fact 2 ensures that the sequence consists eventually of unitary modules, with $C = c_0^n$. Fact 1 allows us (via Propositions 3.2 and 3.4) to line them up with isometric embeddings to get a unitary representation of the limit group. Fact 3 implies that the sequence eventually moves out of the discrete range.

We now give the values of s, c_0^n, and z_0^n for our two cases.

---

[1] Direct limits for the Hermitian symmetric situation, when the coefficient is one, were studied by Natarajan [12].

A. $G = Sp(p,n) = \{g \in Sp(p+n,\mathbb{C}) : g^tQ\bar{g} = Q\}$ where Q is the form

$$\begin{pmatrix} -I_p & & & \\ & I_n & & \\ & & -I_p & \\ & & & I_n \end{pmatrix}.$$

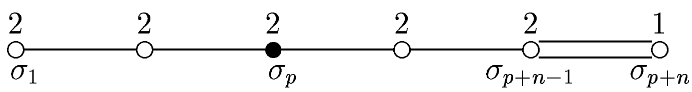
The Lie algebra of G is denoted $\mathfrak{g}_0 = \mathfrak{sp}(p,n)$. The Cartan involution is $\theta : X \mapsto -X^*$, with a corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. $\mathfrak{k}$ has a Cartan subalgebra $\mathfrak{h}$ which is also a Cartan subalgebra for $\mathfrak{g}$. It consists of the diagonal matrices with entries $(a_1,\ldots,a_{p+n},-a_1\ldots,-a_{p+n})$. We can form a direct system of Lie algebras

$$\cdots \longrightarrow \mathfrak{sp}(p,n) \longrightarrow \mathfrak{sp}(p,n+1) \longrightarrow \cdots \longrightarrow \mathfrak{sp}(p,\infty)$$

Let e_i be the functional on $\mathfrak{h}$ which extracts the i-th diagonal entry. Then the roots of $(\mathfrak{g},\mathfrak{h})$ are $e_i - e_j$ $(i \neq j)$, and $\pm(e_i + e_j)$, with $1 \leq i,j \leq p+n$. Let $E_{r,s}$ be the $2p \times 2n$ matrix with 1 as the (r,s) entry. Then the root spaces are:

$$\begin{aligned} e_i - e_j &\ :\ E_{i,j} - E_{p+n+j,p+n+i} \\ e_i + e_j &\ :\ E_{i,p+n+j} + E_{j,p+n+i} \\ -e_i - e_j &\ :\ E_{p+n+i,j} + E_{p+n+j,i} \end{aligned}$$

Taking the simple roots to be $\sigma_1 = e_1 - e_2, \ldots,\ \sigma_{p+n-1} = e_{p+n-1} - e_{p+n},\ \sigma_{p+n} = 2e_{p+n}$, the only non-compact simple root is σ_p. The corresponding Dynkin diagram is drawn below, with the numbers above the simple roots giving their coefficients in the maximal root.



Let $\triangle(\mathfrak{l})$ be the root system generated by the compact simple roots, $\triangle(\mathfrak{u}) = \triangle^+ \setminus \triangle(\mathfrak{l})$ and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ the corresponding parabolic subalgebra. We find that $\zeta = (1,\ldots,1,0,\ldots,0)$ and so $\lambda^n = (z,\ldots,z,0,\ldots,0)$, with 0's after the first p entries. So we have $\lambda^\infty = (z,\ldots,z,0,\ldots,0,\ldots)$. Here the i-th component of the vector is the coefficient of e_i. We also have $\rho = (p+n, p+n-1, \ldots, 1)$. The condition for $\mathfrak{k}$ integrality of λ^n is $z \in \mathbb{Z}$.

$\triangle(\mathfrak{u}_c) = \{e_i + e_j\ :\ i \leq j \leq p\}$ and so $s = \dim(\mathfrak{u}_c) = p(p+1)/2$ is independent of n.

The set of simple roots for the positive system $\triangle^+(\mathfrak{l}) \cup -\triangle(\mathfrak{u})$ is $\{\sigma'_1, \ldots, \sigma'_{p+n}\}$, where $\sigma'_i = \sigma_i$ for $i \neq p$, and $\sigma'_p = -e_1 - e_{p+1}$. It follows that $\langle \lambda^n + \rho, \sigma'_i \rangle = 1$ if $i \neq p$, and $-z - p - 2n$ if $i = p$. Therefore $z_0^n = -p - 2n$.

For n sufficiently large, [4] gives $c_0^n = -p + 1$.

B. $G = SO(2p,2n) = \{g \in GL(2p+2n,\mathbb{R}) : g^tJg = J,\ \det g = 1\}$ where J is the form

$$\begin{pmatrix} I_{2p} & 0 \\ 0 & -I_{2n} \end{pmatrix}.$$

The Lie algebra of G is denoted $g_0 = \mathfrak{so}(2p,2n)$. A Cartan involution is $\theta : X \mapsto -X^t$ and this gives the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The Lie algebra $\mathfrak{k}$ has a

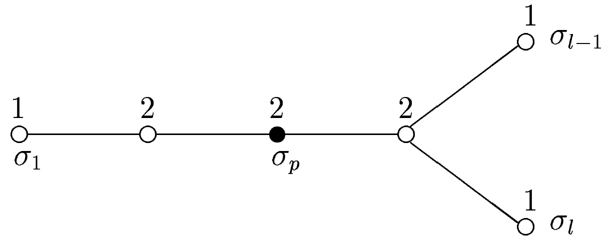
Cartan subalgebra $\mathfrak{h}$ which is also a Cartan subalgebra for $\mathfrak{g}$: $\mathfrak{h} = \mathrm{diag}(A_1, \ldots, A_l)$ where $p+n=l$, and each A_i is a 2×2 block:

$$A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} .$$

We can form a direct limit of Lie algebras

$$\cdots \longrightarrow \mathfrak{so}(2p, 2n) \longrightarrow \mathfrak{so}(2p, 2n+2) \longrightarrow \cdots \longrightarrow \mathfrak{so}(2p, \infty)$$

The roots of $(\mathfrak{g}, \mathfrak{h})$ are $\pm e_i \pm e_j$, $1 \leq i < j \leq p+n$, where e_i picks out the a_i entry of a member of $\mathfrak{h}$. Choosing our simple roots to be $\sigma_1 = e_1 - e_2, \ldots, \sigma_{l-1} = e_{l-1} - e_l, \sigma_l = e_{l-1} + e_l$, the only non-compact simple root is σ_p. The case $p=1$ was dealt with in [12], so we assume $p > 1$. The Dynkin diagram is:



Again, we have $\zeta = (1, \ldots, 1, 0, \ldots, 0)$ and $\lambda^n = (z, \ldots, z, 0, \ldots, 0)$, with 0's after the first p entries, so that $\lambda^\infty = (z, \ldots, z, 0, \ldots, 0, \ldots)$. λ^n is $\mathfrak{k}$ integral if $2z \in \mathbb{Z}$.

$\triangle(\mathfrak{u}_c) = \{e_i + e_j \ : \ i, j \leq p\}$ and so $s = \dim(\mathfrak{u}_c) = p(p-1)/2$. As before, we calculate $z_0^n = 2 - p - 2n$ and $c_0^n = -p - 3/2$ (for large n). ■

Proposition 6.2. *Each limit representation in the previous proposition is O-admissible.*[2]

Proof. We write out the proof in detail for $G^\infty = SO(2p, \infty)$. It is similar for $Sp(p, \infty)$. First, we list some definitions and results due to Ol′shanskiĭ [18, 19]. Let $K_n \subset K^\infty = SO(2p) \times SO(\infty)$ be the operators which fix the first $2p+2n$ basis vectors in the standard representation $\mathbb{C}^{2p} + \mathbb{C}^\infty$ of $Sp(p, \infty)$. If $\mathcal{H}$ is a representation of K^∞, then define $\mathcal{H}_n$ to be the subspace of vectors fixed by K_n. $\mathcal{H}$ is said to be *tame* if $\cup_n \mathcal{H}_n$ is dense in it. A unitary representation of G^∞ is *O-admissible* if it is tame as a representation of K^∞.

In our examples, the groups K_n are connected and so the requirement for O-admissibility can be phrased in terms of the Lie algebras and the underlying Harish-Chandra modules: Let V_n be the subspace of vectors in $V^\infty = \lim V^n$, $V^n \in \mathcal{C}(\mathfrak{g}^n, \mathfrak{k}^n)$, which are killed by the Lie algebra $\mathfrak{k}_n$ of K_n. Then the $\mathcal{H}$ corresponding to V^∞ is O-admissible if $\cup_n V_n$ equals V^∞. We need to look at $V^n = \Gamma^s L(\lambda^n)$.

Let $L_n = L^\infty \cap K_n$. From an argument of Natarajan [11, Theorem 6] we know that $\mathfrak{l}_n$ acts trivially on $N(\lambda^n)$ and on $L(\lambda^n)$. But from the block structure

[2] Natarajan and Rodríguez-Carrington have found O-admissibility criteria for highest weight representations in some other situations (unpublished).

of our groups, it is obvious that $\mathfrak{l}_n = \mathfrak{k}_n$. Now $\mathfrak{k}_n$ also acts trivially on the Koszul resolution of X_i^n of the trivial representation (see Section 1), and hence on $\mathrm{Hom}_{\mathbb{C}}(X_i^n, V^n)_{\mathfrak{l}^n}$ and the subspace $\Gamma\mathrm{Hom}_{\mathbb{C}}(X_i^n, V^n)_{\mathfrak{l}^n}$. Consequently, it acts trivially on the quotient $\Gamma^i V^n$. Since $\Gamma^s V$ is the direct limit of the $\{\Gamma^s V^n\}$, it follows that it is O-admissible. ∎

7. Ladder Representations of $Sp(p, \infty)$

A representation of G is called a ladder representation when the $\mathfrak{k}$-types have highest weights that lie along a line in $\mathfrak{h}^*$. We use Section 9 of [2] to set up direct limits of ladder representations of $Sp(p, n)$.

Let $\mathfrak{g}_0 = \mathfrak{sp}(p, n)$ and $\mathfrak{h}$ as defined in the proof of Proposition 4.1. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the parabolic subalgebra with $\triangle^+(\mathfrak{l})$ having simple roots $\sigma_2, \ldots, \sigma_{p+n}$. Decompose $\mathfrak{u}$ into $\mathfrak{u}_1 \oplus \mathfrak{u}^n$, $\triangle(\mathfrak{u}_2) = \{2e_1\}$ and $\triangle(\mathfrak{u}_1) = \{e_1 \pm e_j \,:\, 2 \le j \le p+n\}$. Split $\mathfrak{u}_1$ into its compact and non-compact parts $\mathfrak{u}_1 = \mathfrak{u}_{1,c}^n \oplus \mathfrak{u}_{1,n}$. Let γ be the lowest weight in $\triangle(\mathfrak{u}_{1,n})$, and $\lambda^n = (z, 0, \ldots, 0)$, $z \in \mathbb{R}$. Let $\mathcal{C}_c$ be the Weyl chamber corresponding to the positive system $\triangle^+(\mathfrak{l}_c) \cup -\triangle(\mathfrak{u}_c)$ for $\triangle(\mathfrak{k}, \mathfrak{h})$. Let $r_0 \in \mathcal{W}(\mathfrak{k}, \mathfrak{h})$ be the Weyl group element such that $r_0\mathcal{C}_c$ is the positive chamber for $\triangle^+(\mathfrak{k}, \mathfrak{h})$, and let ρ_c be half the sum of the elements of $\triangle^+(\mathfrak{k}, \mathfrak{h})$. Then we have:

Theorem 7.1. [2, Proposition 9.4] *For each integer $z < 0$, $\Gamma^i N(\lambda^n) = 0$ for $i \neq s$ while $\Gamma^s N(\lambda^n)$ is a unitarizable representation of $Sp(p, n)$. The $\mathfrak{k}$-types are multiplicity free and their highest weights are those elements $r_0(\lambda^n - r\gamma + \rho_c) - \rho_c$, $r \in \mathbb{N}$, which are $\triangle^+(\mathfrak{k})$-dominant.*

Proposition 7.2. *For each integer $z < 0$ the representations $\Gamma^s N(\lambda^n)$, as above, form a direct system of $Sp(p, n)$ representations and their limit is O-admissible. Further, the limit representation $\Gamma^s N(\lambda^\infty)$ is itself a ladder representation of $Sp(p, \infty)$, with the $\mathfrak{k}^\infty$-types having highest weights $(-z+r-p-1)e_1 + re_{p+1}$, $r \in \mathbb{N}$, $r \ge z+p+1$.*

Proof. Observe that $\triangle(\mathfrak{u}_c) = \{e_1 \pm e_j \,:\, 1 < j \le p\} \cup \{2e_1\}$ and so $s = 2p-1$ is independent of n. Also the highest weights described above are easily calculated to be $(-z+r-p-1)e_1 + re_{p+1}$ and the condition for one of these to be $\triangle^+(\mathfrak{k})$-dominant is $r \ge z+p+1$. Note the absence of n from these numbers. So Propositions 3.2 and 3.4, and the proof of 6.2, apply to give the result. ∎

Remark C. A question we have not addressed is the classification, if any, of the representations which can be constructed by our method. The difficulty is that a particular representation can be constructed by cohomological induction in various ways, by starting from different parabolics, and the choice affects whether our method applies. For instance, the discrete series representations can also be obtained by starting from a Borel subalgebra, but then the middle dimension does not stabilize. A way out may be offered by Knapp's method for finding the Langlands parameters of a cohomologically induced representation (see Knapp [9] and Friedman [5]).

References

[1] Enright, T. J., and N. R. Wallach, *Notes on homological algebra and representations of Lie algebras*, Duke Math. J. **47:1** (1980), 1–15.

[2] Enright, T. J., R. Parthasarathy, N. R. Wallach, and J. A. Wolf, *Unitary derived functor modules with small spectrum*, Acta Mathematica **154** (1985), 105–136.

[3] Enright, T. J., and J. A. Wolf, *Continuation of unitary derived functor modules out of the canonical chamber*, Memoires Math. Soc. France **15** (1984), 139–156.

[4] Frajria, P. M., *A note on the continuation of the Borel-De Siebenthal discrete series out of the canonical chamber*, J. Algebra **154** (1993), 95–107.

[5] Friedman, P. D., *The Langlands parameters of subquotients of certain derived functor modules*, J. Funct. Anal. **157:1** (1998), 210–241.

[6] Hartshorne, R., *On the de Rham cohomology of algebraic varieties*, Publ. Math. IHES **45** (1976), 5-99.

[7] Jantzen, J. C., *Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren*, Math. Ann. **226** (1977), 53–65.

[8] Knapp, A. W., "Lie Groups, Lie Algebras, and Cohomology," Mathematical Notes **34**, Princeton University Press, 1988.

[9] —, *Exceptional unitary representations*, Representation Theory **1** (1997), 1–24.

[10] Köthe, G., "Topological Vector Spaces I," Die Grundlehren der mathematischen Wissenschaften **159**, Springer-Verlag, 1969.

[11] Natarajan, L., *Unitary highest weight representations of certain infinite-dimensional Lie groups and Lie algebras*, PhD thesis, University of California, Berkeley, 1991.

[12] —, *Unitary highest weight-modules of inductive limit Lie algebras and groups*, J. Algebra **167:1** (1994), 9–28.

[13] Natarajan, L., E. Rodríguez-Carrington, and J. A. Wolf, *Differentiable structure for direct limit groups*, Lett. Math. Phys. **23:2** (1991), 99–109.

[14] —, *Locally convex Lie groups*, Nova J. Algebra Geom. **2:1** (1993), 59–87.

[15] —, *The Bott-Borel-Weil theorem for direct limit groups*, Trans. Amer. Math. Soc., to appear.

[16] Neeb, K.-H., and B. Ørsted, *Unitary highest weight representations in Hilbert spaces of holomorphic functions on infinite dimensional domains*, Journal of Funct. Anal. **156** (1998), 263–300.

[17] Neretin, Y. A., and G. I. Ol′shanskiĭ, *Boundary values of holomorphic functions, singular unitary representations of $O(p,q)$, and their limits as $q \to \infty$*, Journal of Mathematical Sciences **87:6** (1997), 3983–4035.

[18] Ol′shanskiĭ, G. I., *Infinite-dimensional classical groups of finite R-rank: description of representations and asymptotic theory*, Functional Anal. Appl. **18:1** (1984), 22–34.

[19] —, *Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians*, In "Topics in Representation Theory," Advances in Soviet Mathematics **2**, Amer. Math. Soc., 1991.

[20] Schmid, W., *Boundary value problems for group invariant differential equations*, Astérisque (Numero Hors Serie) (1985), 311–321.

[21] Wong, H. W., *Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations*, J. Funct. Anal. **129** (1995), 428–454.

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