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An Explicit Construction of the Metaplectic Representation over a Finite Field

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Abstract. This paper deals with the metaplectic representation of $\mathrm{Sp}\,(n,K)$, where K is a finite field. We give an explicit, elementary and largely self-contained construction of the metaplectic representation over finite fields of characteristic $\neq 2$. The important steps of the construction will be given for a system of generating elements of $\mathrm{Sp}\,(n,K)$.

1. Introduction

For a finite field the metaplectic representation is a representation with multiplier 1. This will be shown in an elementary manner. At first the dual of the Heisenberg group over a finite field is considered and it is shown that in analogy to the Stone–von Neumann theorem the representation is completely determined by the restriction to the center. In section 3. a system of generating elements for the symplectic group will be obtained. The idea for the proof explicitly uses the finiteness of the field. This system of generating elements will be important in the following, as the representation is given in terms of these elements. In [6] the representation was constructed in an abstract manner, from which the concrete matrix representation could not easily be deduced. In section 4. the representation is determined for the generating elements and it will be decomposed into irreducible representations with a new elementary proof. Also in this section the factors for all but one of the generating elements will be computed to obtain an ordinary representation, which is not found explicitly in the literature. As a by-product the absolute value of the character for the generating elements can be computed very easily in comparison with the calculations done in [8]. In section 5. the factor for the last generating element will be computed, which is the determinant of the inverse Fourier transform for finite fields. The computation of the determinant is reduced to the computation of the trace, which is a generalized Gauß sum. This result is an easy exercise for prime fields of odd characteristic, see for example [2]. For general finite fields it is new. In section 7. the dependence of the representation on a constant is considered, which yields all equivalence classes of metaplectic representations. This result appears also in [7].

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The results obtained here are in connection with classical quantum kinematics, whose underlying configuration spaces are vector spaces over finite fields. This subject is treated for example in [15] and [16]. In the case of the real numbers this was done for example in [5].

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2. Preliminaries

In this section we recall the representation theory of the Heisenberg group $H_n = H_n(K)$, K a finite field.

Let K be a finite field of characteristic $\neq 2$. Let $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ be a non-degenerate symmetric bilinear form on the finite dimensional vector space K^n.

The trace of the finite field K on its prime field P will be denoted by Tr. For $\alpha \in K$ this trace is the ordinary trace for finite dimensional linear transformations of the mapping $\beta \mapsto \alpha\beta$ of the vector space K over $P \cong \mathbb{Z}/|P|\mathbb{Z}$ (cf. for example [11, page 214]). Moreover, let ε denote a fixed primitive $|P|$-th root of unity in $\mathbb{C}$, for example $\varepsilon = e^{\frac{2\pi i}{|P|}}$. With $\chi : K \to \mathbb{C}$, $\chi(t) = \varepsilon^{\operatorname{Tr} t}$ we obtain a one dimensional representation of the additive group of K.

It is well-known, that $K^n \cong \widehat{K^n}$. An explicit isomorphism of K^n and $\widehat{K^n}$ yields the mapping $\chi : K^n \to \widehat{K^n}$, $\chi_x(y) = \varepsilon^{\operatorname{Tr}\langle x,y\rangle}$.

Let the Heisenberg group H_n be the set K^{2n+1} equipped with the group action

$$(v,t)(w,u) = \left(v + w, t + u + 2^{-1}[v,w]\right)$$

for $v, w \in K^{2n}$, $t, u \in K$, where

$$[v,w] = \langle p, s\rangle - \langle q, r\rangle$$

for $v = (p,q) \in K^{2n}$, $w = (r,s) \in K^{2n}$, $p,q,r,s \in K^n$ is a symplectic bilinear form on the space K^{2n}.

Let $f \in \mathbb{C}^{K^n}$, where $\mathbb{C}^{K^n}$ is the set of all functions $f : K^n \to \mathbb{C}$, and $\eta(p,q,t) f(x) = \chi(t + \langle q, x\rangle + 2^{-1}\langle p, q\rangle) f(x+p)$, where $\chi : K \to \mathbb{C}$ is a non-trivial character of K, i. e. $\chi(t) = e^{\frac{2\pi i}{|P|}\operatorname{Tr}\alpha t}$ for an $\alpha \in K \setminus \{0\}$ as was shown before.

An easy computation shows, that the mapping $\eta = \eta_\chi$ is a representation of H_n on $\mathbb{C}^{K^n}$. It also can be shown easily, that the center of the Heisenberg group is $\{(0,t) : t \in K\}$.

As clearly the center is a normal subgroup of H_n, we obtain further representations of H_n by

$$\rho_\psi(w,t) = \psi(w)$$

for $w \in K^{2n}$, $t \in K$ and ψ a character of K^{2n}.

Lemma 2.1. *The support of the character of η is the center.*

Proof. The identity

$$\eta(p,q,t) f(x) = \chi\left(t + \langle q,x\rangle + 2^{-1}\langle p,q\rangle\right) f(x+p)$$

yields $\operatorname{tr}\eta(p,q,t) = 0$, if $p \neq 0$, since $x+p \neq x$. If $p = 0$, the orthogonality relations for characters show

$$\operatorname{tr}\eta(0,q,t) = |K|^n \chi(t)\, \delta_{q,0},$$

as χ is a nontrivial character. So

$$\operatorname{tr}\eta(p,q,t) = 0$$

for $p \neq 0$ or $q \neq 0$ and $|\operatorname{tr}\eta(0,0,t)| = |K^n| \neq 0$. ■

It can now be deduced, that the representation η is irreducible, as

$$\begin{aligned} |H_n|^{-1} \sum_{(p,q,t)\in H_n} |\operatorname{tr}\eta(p,q,t)|^2 &= \left|K^{2n+1}\right|^{-1} \sum_{t\in K} |K^n|^2 \\ &= \left|K^{2n+1}\right|^{-1} |K|\, |K^n|^2 = 1. \end{aligned}$$

For an element of the center it was shown in the proof of Lemma 2.1 that $\operatorname{tr}\eta(0,0,t) = |K|^n \chi(t)$. So again the orthogonality relations for characters yield, that two representations η_χ with different χ are not equivalent. This shows that the representations are uniquely determined by the values of the center. This is the Stone–von Neumann theorem, see also [12, page 28] for arbitrary fields of characteristic ≥ 3. In the dual of H_n are the $|K|-1$ equivalence classes of the representations η_χ with nontrivial characters χ of K and the $|K^{2n}|$ equivalence classes of the representations ρ_ψ with characters ψ of K^{2n}. These are all irreducible representations of H_n, because $|H_n| = |K|^{2n+1} = |K|^{2n} + (|K|-1)(|K|^n)^2$.

3. The symplectic group

The group of isometries of the symplectic bilinear form $[\cdot,\cdot]$ will be referred to as the symplectic group and denoted by $\operatorname{Sp}(n,K)$. The main result of this section will be the determination of a set of matrices, which generate $\operatorname{Sp}(n,K)$. It holds, that $[v,w] = v^* J w$ with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where * denotes transposition. For the following see also [5, page 171].

Theorem 3.1. *For* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}(2n,K)$ *with* $a,b,c,d \in M(n,K)$ *the following are equivalent:*

1. $g \in \operatorname{Sp}(n,K)$
2. $a^*c = c^*a$, $b^*d = d^*b$ *and* $a^*d - c^*b = I$
3. $ab^* = ba^*$, $cd^* = dc^*$ *and* $ad^* - bc^* = I$

The proof is straightforward and will be omitted.

Let $u(b) = \begin{pmatrix} I & 0 \\ b & I \end{pmatrix}$ with $b^* = b$, $s(a) = \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$ with $a \in \mathrm{GL}(n, K)$. For $\mathrm{Sp}(1, K) = \mathrm{SL}(2, K)$ the following theorem is proven for arbitrary fields in [10]. In this case there is also a description in terms of generators and relations, see for example [4, page 300]. For the proof of the following theorem we need a lemma about the cardinality of the symplectic group for finite fields.

Lemma 3.2. *The cardinality of* $\mathrm{Sp}(n, K)$ *is* $|K|^{n^2} \cdot \prod_{k=1}^{n} \left(|K|^{2k} - 1\right)$.

A proof is given in [1, page 147].

The proof of the following theorem is apparently new.

Theorem 3.3. *The symplectic group* $\mathrm{Sp}(n, K)$ *for a finite field* K *is generated by the matrices* $u(b)$, $s(a)$ *and* J, *with* $b = b^*$ *and* $a \in \mathrm{GL}(n, K)$.

Proof. Let $G \subset \mathrm{Sp}(n, K)$ denote the group generated by these matrices. Clearly, the sets

$$N = \{u(b) : b \in M(n, K), b = b^*\}, D = \{s(a) : a \in \mathrm{GL}(n, K)\}$$

are subgroups of $G \subset \mathrm{Sp}(n, K)$. The set

$$T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, K) : \det a \neq 0 \right\}$$

is contained in G, more precisely $T = NDJNJ^{-1}$. We are going to show $|G| > \frac{1}{2} |\mathrm{Sp}(n, K)|$, which implies the theorem.

The index of G in $\mathrm{Sp}(n, K)$ is

$$[\mathrm{Sp}(n, K) : G] \leq \frac{|\mathrm{Sp}(n, K)|}{|T|},$$

as $T \subset G$. By lemma 3.2 the cardinality of $\mathrm{Sp}(n, K)$ is $|K|^{n^2} \cdot \prod_{k=1}^{n} \left(|K|^{2k} - 1\right)$ and that of T is $|N|\,|D|\,|N|$. Now $|N| = |K|^{\frac{n(n+1)}{2}}$ and $|D| = |\mathrm{GL}(n, K)| = \prod_{k=0}^{n-1} \left(|K|^n - |K|^k\right)$, as is shown in [1, page 169], so

$$|T| = |K|^{n(n+1)} \prod_{k=0}^{n-1} \left(|K|^n - |K|^k\right).$$

The index is

$$\begin{aligned} [\mathrm{Sp}(n, K) : G] &\leq \frac{\prod_{k=1}^{n} \left(|K|^k + 1\right) \left(|K|^k - 1\right)}{|K|^n \prod_{k=0}^{n-1} |K|^k \left(|K|^{n-k} - 1\right)} = \prod_{k=1}^{n} \left(1 + |K|^{-k}\right) \\ &\leq \exp\left(\sum_{k=1}^{n} |K|^{-k}\right) \leq \exp\left(\frac{3^{-1}}{1 - 3^{-1}}\right) < 2 \end{aligned}$$

and we are done. ∎

4. The metaplectic representation

The metaplectic representation is now constructed from the projective representation obtained from the representations of the Heisenberg group. The main result of this section will be the determination of the projective metaplectic representation and showing how it can be made an ordinary representation. The explicit construction also yields the absolute value of the trace of the metaplectic representation of the generating elements almost at once.

The symplectic group acts on the Heisenberg group by

$$(w,t)\mapsto(gw,t)$$

for $g\in\mathrm{Sp}\,(n,K)$. This action is clearly an automorphism of H_n, as g is an isometry of the symplectic bilinear form. With this action we obtain a representation η_g defined by $\eta_g\,(w,t)=\eta\,(gw,t)$. The center of the Heisenberg group is kept pointwise fixed, so the Stone–von Neumann theorem yields, that η_g is unitarily equivalent to η, see the remark after lemma 2.1. By Schur's lemma there is a matrix $\mu\,(g)$, uniquely determined up to a scalar multiple, with $\eta_g=\mu\,(g)\,\eta\mu\,(g)^{-1}$. This yields a projective representation μ of the symplectic group. By the uniqueness we get

$$\mu\,(gh)=c\,(g,h)\,\mu\,(g)\,\mu\,(h)$$

for all $g,h\in\mathrm{Sp}\,(n,K)$, where $c\,(g,h)$ is a scalar factor. The function c is a multiplier of the representation, which means that the equation $c\,(g,hk)\,c\,(h,k)=c\,(gh,k)\,c\,(g,h)$ for $g,h,k\in\mathrm{Sp}\,(n,K)$ holds. The projective representation μ is an obstruction of an ordinary representation ω, if and only if there is a scalar function κ such that $c\,(g,h)=\frac{\kappa(g)\kappa(h)}{\kappa(gh)}$, and then $\omega=\kappa\mu$.

Now μ is described on the generating set.

Theorem 4.1. *For μ we have the equations*

$$\begin{aligned}\mu\,(u\,(b))\,f\,(x)&=\chi\left(-2^{-1}\,\langle x,bx\rangle\right)f\,(x)\,,\\ \mu\,(s\,(a))\,f\,(x)&=f\left(a^{-1}x\right),\\ \mu\,(J)\,f\,(x)&=\frac{1}{\sqrt{|K^n|}}\sum_{y\in K^n}\chi\,(\langle x,y\rangle)\,f\,(y)\,.\end{aligned}$$

Here $\chi:K\to\mathbb{C}$ is the character associated with the representation $\eta=\eta_\chi$. The transformation $\mu\,(J)$ acts as the inverse Fourier transform on K^n.

Proof. Let $f\in\mathbb{C}^{K^n}$, $b=b^*$ and $\mu\,(u\,(b))\,f\,(x)=\chi\,(-2^{-1}\,\langle x,bx\rangle)\,f\,(x)$, then

$$\mu\,(u\,(b))^{-1}\,f\,(x)=\chi\left(2^{-1}\,\langle x,bx\rangle\right)f\,(x)$$

and

$$\begin{aligned}&\mu\,(u\,(b))\,\eta\,(p,q,t)\,\mu\,(u\,(b))^{-1}\,f\,(x)\\ =\;&\chi\left(-2^{-1}\,\langle x,bx\rangle+t+\langle q,x\rangle+2^{-1}\,\langle p,q\rangle+\right.\\ &\qquad\left.+\,2^{-1}\,\langle x,bx\rangle+\langle p,bx\rangle+2^{-1}\,\langle p,bp\rangle\right)f\,(x+p)\\ =\;&\eta\,(p,bp+q,t)\,f\,(x)\,.\end{aligned}$$

So $\mu(u(b))\,\eta(p,q,t)\,\mu(u(b))^{-1} = \eta_{u(b)}(p,q,t)$.

Now let $a \in \mathrm{GL}(n,K)$ and $\mu(s(a))\,f(x) = f(a^{-1}x)$, then

$$\mu(s(a))^{-1} f(x) = f(ax)$$

and

$$\begin{aligned}\mu(s(a))\,\eta(p,q,t)\,\mu(s(a))^{-1} f(x) &= \chi\left(t + \langle q, a^{-1}x\rangle + 2^{-1}\langle p,q\rangle\right) f(x+ap)\\ &= \eta\left(ap, a^{*-1}q, t\right) f(x).\end{aligned}$$

So $\mu(s(a))\,\eta(p,q,t)\,\mu(s(a))^{-1} = \eta_{s(a)}(p,q,t)$.

Now for $J \in \mathrm{Sp}(n,K)$. By $\mu(J)\,f(x) = \frac{1}{\sqrt{|K^n|}}\sum_{y\in K^n}\chi(\langle x,y\rangle)\,f(y)$ it can be deduced, that $\mu(J)^{-1} f(x) = \frac{1}{\sqrt{|K^n|}}\sum_{y\in K^n}\chi(-\langle x,y\rangle)\,f(y)$, as

$$\begin{aligned}&\frac{1}{\sqrt{|K^n|}}\sum_{y\in K^n}\chi(-\langle x,y\rangle)\,\mu(J)\,f(y)\\ =&\sum_{z\in K^n}\frac{1}{|K^n|}\left(\sum_{y\in K^n}\chi(\langle x,y\rangle)^{-1}\chi(\langle y,z\rangle)\right) f(z) = f(x)\end{aligned}$$

because of the orthogonality relations for characters. So the above equation holds for $\mu(J)^{-1}$, as $\mu(J)^{-1}$ is a finite dimensional linear transformation. Hence

$$\begin{aligned}&\mu(J)\,\eta(p,q,t)\,\mu(J)^{-1} f(x)\\ =&\sum_{z\in K^n}\chi\left(t + 2^{-1}\langle p,q\rangle - \langle p,z\rangle\right)\frac{1}{|K^n|}\left(\sum_{y\in K^n}\chi(\langle x+q,y\rangle)\,\chi(\langle y,z\rangle)^{-1}\right)\cdot\\ &\qquad\cdot f(z)\\ =&\chi\left(t + 2^{-1}\langle p,q\rangle - \langle p, x+q\rangle\right) f(x+q) = \eta(q,-p,t)\,f(x).\end{aligned}$$

So $\mu(J)\,\eta(p,q,t)\,\mu(J)^{-1} = \eta_J(p,q,t)$. ∎

So we now have a projective representation of $\mathrm{Sp}(n,K)$. To obtain an ordinary representation, the invariant subspaces of the projective representation are considered. The elementary proof of the irreducibility in the following theorem appears to be new. The theorem appears also in [7, page 17].

Theorem 4.2. *The subspaces of the even and odd functions of $\mathbb{C}^{K^n}$ are invariant, i. e.*

$$\begin{aligned}V^+ &= \left\{f \in \mathbb{C}^{K^n} : f(-x) = f(x)\,\forall x \in K^n\right\},\\ V^- &= \left\{f \in \mathbb{C}^{K^n} : f(-x) = -f(x)\,\forall x \in K^n\right\}\end{aligned}$$

are invariant subspaces. The vector spaces V^+ and V^- are irreducible and the representations of V^+ and V^- are not equivalent.

Proof. As $-I$ is in the center of $\mathrm{Sp}\,(n,K)$ and

$$\mu\,(-I)\,f\,(x) = \mu\,(s\,(-I))\,f\,(x) = f\,(-x)\,,$$

it holds, that

$$\mu\,(g)\,f\,(-x) = \mu\,(-I)\,\mu\,(g)\,f\,(x) = \mu\,(g)\,\mu\,(-I)\,f\,(x)\,.$$

So $\mu\,(g)\,f\,(-x) = \sigma\mu\,(g)\,f\,(x)$ for every $g \in \mathrm{Sp}\,(n,K)$, if $\mu\,(-I)\,f = \sigma f$, where $\sigma = \pm 1$. Let in the following $\mu^+\,(g) = \mu\,(g)|_{V^+}$, $\mu^-\,(g) = \mu\,(g)|_{V^-}$ for $g \in \mathrm{Sp}\,(n,K)$.

Now for irreducibility: By $\delta_x : K^n \to \mathbb{C}$, $\delta_x\,(y) = \delta_{x,y}$, $x \in K^n$ we have a basis of $\mathbb{C}^{K^n}$. Let $T : V \to V$ be a linear transformation defined by $T\delta_x\,(y) = t_{y,x}$ for $x, y \in V$, so $T\delta_x = \sum_{y\in V} t_{y,x}\delta_y$. We obtain

$$T\mu\,(u\,(b))\,\delta_x\,(y) = \chi\left(-2^{-1}\,\langle x, bx\rangle\right) t_{y,x}$$

and

$$\mu\,(u\,(b))\,T\delta_x\,(y) = \chi\left(-2^{-1}\,\langle y, by\rangle\right) t_{y,x}$$

for $x, y \in V$. If $T\mu\,(u\,(b)) = \mu\,(u\,(b))\,T$ for all b with $b = b^*$, then

$$\chi\left(-2^{-1}\,\langle x, bx\rangle\right) t_{y,x} = \chi\left(-2^{-1}\,\langle y, by\rangle\right) t_{y,x}.$$

If $\chi\,(-2^{-1}\,\langle x, bx\rangle) = \chi\,(-2^{-1}\,\langle y, by\rangle)$ for all b with $b = b^*$, it holds, that

$$-\operatorname{Tr} 2^{-1}\,\langle x, bx\rangle = -\operatorname{Tr} 2^{-1}\,\langle y, by\rangle$$

for all b with $b = b^*$ because of $\chi\,(-2^{-1}\beta) = \varepsilon^{-\operatorname{Tr} 2^{-1}\alpha\beta}$ for an $\alpha \in K \setminus \{0\}$. As the trace function is not degenerate, $\langle x, bx\rangle = \langle y, by\rangle$ for all b with $b = b^*$. By taking $b_{kk} = 1$ and 0 for the remaining coefficients of b, we have $x_k^2 = y_k^2$ for all k. If $x = 0$, then $y = 0$, and, if $x \neq 0$, there is a k, such that $x_k \neq 0$. So $y_k \neq 0$, too, and, by taking $b_{kj} = b_{jk} = 2^{-1}$ and 0 for the remaining coefficients, we obtain $x_k x_j = y_k y_j$ for $j \neq k$. So $y_j = y_k^{-1} x_k x_j$ for all j and $y = y_k^{-1} x_k x$. Because of $x_k^2 = y_k^2$, we have $y_k = \pm x_k$ and so $y = \pm x$. Hence $t_{y,x} = 0$ for $y \neq \pm x$. So we have $Tf\,(y) = \sum_{x\in\{\pm y\}} t_{y,x} f\,(x)$ for all $f \in V$ and this shows, that $V^\pm$ is invariant under T and T acts diagonal on $V^\pm$. Let in the following $T^+ = T|_{V^+}$ and $T^- = T|_{V^-}$.

Let S be a subset of $K^n \setminus \{0\}$, which contains from each pair $\{x, -x\}$, $x \in K^n \setminus \{0\}$ exactly one element. Hence we obtain a basis of V^+ by $\delta_0^+ = \delta_0$ and $\delta_x^+ = \delta_x + \delta_{-x}$, $x \in S$. Further

$$T^+\mu^+\,(J)\,\delta_0^+\,(y) = t_{y,y}^+ \frac{1}{\sqrt{|K^n|}}$$

and

$$\mu^+\,(J)\,T^+\delta_0^+\,(y) = \frac{1}{\sqrt{|K^n|}} t_{0,0}$$

for $y \in S$, where $t_{y,y}^+ = t_{y,y} + t_{y,-y}$ for $y \in S$. By $T^+\mu^+\,(J) = \mu^+\,(J)\,T^+$ we have $t_{y,y}^+ = t_{0,0}$ for all $y \in S$. So $T^+ = t_{0,0}I$. Now Schur's lemma yields, that the representation μ^+ is irreducible.

In a similar way it is shown, that μ^- is irreducible. As basis of V^- we choose $\delta_x^- = \delta_x - \delta_{-x}$ with $x \in S$. Again we have $T^- f(y) = t_{y,y}^- f(y)$ for $f \in V^-$ and so

$$T^- \mu^- (s(a)) \delta_x^- (y) = t_{y,y}^- \delta_x^- \left(a^{-1} y\right)$$

and

$$\mu^- (s(a)) T^- \delta_x^- (y) = t_{a^{-1}y,a^{-1}y}^- \delta_x^- \left(a^{-1} y\right)$$

for $x, y \in S$, where $t_{y,y}^- = t_{y,y} - t_{y,-y}$. By $T^- \mu^- (s(a)) = \mu^- (s(a)) T^-$ and $x = a^{-1} y$ we have $t_{y,y}^- = t_{a^{-1}y,a^{-1}y}^-$ for all $a \in \mathrm{GL}(n, K)$. So $t_{x,x}^- = t_{y,y}^-$ for all $x, y \in S$, because every vector $x \neq 0$ can be mapped to every other vector $y \neq 0$ by an $a \in \mathrm{GL}(n, K)$. So $T^- = tI$ for a $t \in \mathbb{C}$. By Schur's lemma V^- is irreducible, too.

So we obtain irreducible representations μ^+ of $\mathrm{Sp}(n, K)$ on V^+ and μ^- on V^-. Because of $\dim V^+ \neq \dim V^-$ these representations on V^+ and V^- are not equivalent. ■

With the help of the determinant of the finite dimensional linear transformations $\mu(g)$ we obtain the following.

Theorem 4.3. *There is a function $\kappa : \mathrm{Sp}(n, K) \to \mathbb{C}$, such that an ordinary representation $\omega : \mathrm{Sp}(n, K) \to \mathbb{C}^{K^n}$ is obtained from the projective representation μ by $\omega = \kappa\mu$. The function κ is defined by $\kappa(g) = \det \mu(g) \det \mu^+(g)^{-2}$, where μ^+ is the representation μ restricted to the invariant subspace $V^+ = \left\{f \in \mathbb{C}^{K^n} : f(-x) = f(x)\, \forall x \in K^n\right\}$.*

Proof. As μ is a projective representation,

$$\mu(gh) = c(g, h)\, \mu(g)\, \mu(h)$$

with $|c(g, h)| = 1$. Restriction to V^+ yields also

$$\mu^+(gh) = c(g, h)\, \mu^+(g)\, \mu^+(h).$$

Taking determinants we obtain

$$\det \mu(gh) = c(g, h)^{|K^n|} \det \mu(g) \det \mu(h)$$

and

$$\det \mu^+(gh) = c(g, h)^{\frac{1}{2}(|K^n|+1)} \det \mu^+(g) \det \mu^+(h),$$

as $\dim V^+ = \frac{1}{2}(|K^n| + 1)$. Let $\kappa(g) = \det \mu(g) \det \mu^+(g)^{-2}$, then $c(g, h) = \frac{\kappa(g)\kappa(h)}{\kappa(gh)}$ and $\omega = \kappa\mu$ is an ordinary representation. ■

Clearly $\det \mu(g) = \det \mu^+(g) \det \mu^-(g)$, where μ^- is the representation μ restricted to the space V^-. So we have

$$\kappa(g) = \det \mu^-(g) \det \mu^+(g)^{-1} = \det \mu^-(g)^2 \det \mu(g)^{-1}.$$

Now we calculate the values of the function κ for the generating elements, which we shall do here for $u(b)$ and $s(a)$. The case of J is more difficult and will be done in Section 5..

If $g = u(b)$, we have $\kappa(u(b)) = 1$. With the set $S \subset K^n \setminus \{0\}$ as in the proof of theorem 4.2, which contains exactly one of the two elements $x, -x \in K^n \setminus \{0\}$, we have

$$\begin{aligned}\kappa(u(b)) &= \det \mu^-(u(b)) \det \mu^+(u(b))^{-1} \\ &= \prod_{x \in S} \chi\left(-2^{-1}\langle x, bx\rangle\right) \prod_{x \in S \cup \{0\}} \chi\left(-2^{-1}\langle x, bx\rangle\right)^{-1} = 1.\end{aligned}$$

For $g = s(a)$ it holds, that $\kappa(s(a)) = \det \mu(s(a))$. The matrix $\mu^+(s(a))$ is like $\mu(s(a))$ a permutation matrix. So we have $\det \mu^+(s(a)) = \pm 1$ and hence we obtain $\kappa(s(a)) = \det \mu(s(a))$. By $\det \mu \circ s$ we have a homomorphism $\mathrm{GL}(n, K) \to \{\pm 1\}$ and the commutator subgroup of $\mathrm{GL}(n, K)$ is in the kernel of this homomorphism. This commutator equals $\mathrm{SL}(n, K)$, see for example [1, page 163]. As $H = \mathrm{SL}(n, K)$ is in the kernel of the above homomorphism, we obtain by $aH \mapsto \det(\mu(s(a)))$ a homomorphism $\mathrm{GL}(n, K)/H \to \{\pm 1\}$. With the determinant mapping on $\mathrm{GL}(n, K)$ the group of invertible elements of K and $\mathrm{GL}(n, K)/H$ is isomorphic. So we obtain a homomorphism $K \setminus \{0\} \to \{\pm 1\}$ by $\alpha \mapsto \det\left(\mu\left(s\begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}\right)\right)$. It holds, that $\mu\left(s\begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}\right) f\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} \alpha^{-1}x \\ y \end{pmatrix}$ with $x \in K$, $y \in K^{n-1}$. As the elements $y \in K^{n-1}$ are kept fixed, we can take $n = 1$ without loss of generality. Let α be a generating element of the group of invertible elements of $K \setminus \{0\}$. Choosing as basis of $\mathbb{C}^K$ the elements $\delta_0, \delta_1, \delta_\alpha, \delta_{\alpha^2}, \ldots, \delta_{\alpha^{|K|-2}}$ the determinant of the matrix $\mu(s(\alpha))$ equals the signature of the cyclic permutation $(0, 1, 2, \ldots, |K| - 2)$. This permutation is even, as $|K|$ is odd and so the permutation has signature -1. Hence $\kappa(s(a)) = \det(a)^{\frac{|K|-1}{2}}$.

It holds, that $\mu(J)^2 = \mu(-I)$. So $\mu^+(J)^2 = I$ and $\det(\mu^+(J))^{-2} = 1$. Now we have $\kappa(J) = \det \mu(J)$ and we only need to determine the value of $\det \mu(J)$. This will be done in the next section.

We will now compute an example.

Example 4.4. The metaplectic representation for $K = \mathbb{Z}/5\mathbb{Z}$ and $n = 1$. It holds, that $\mathrm{Sp}(1, K) = \mathrm{SL}(2, K)$. Let ε denote a primitive fifth root of unity in $\mathbb{C}$ and as basis δ_0, δ_1, δ_2, δ_3, δ_4 is chosen. With respect to this basis

$$\omega(u(b)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon^{2b} & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{3b} & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^{3b} & 0 \\ 0 & 0 & 0 & 0 & \varepsilon^{2b} \end{pmatrix}.$$

The set of invertible elements of K is cyclic and 2 is a generating element. So it is enough to give the representation of $s(2)$, which is

$$\omega(s(2)) = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

As

$$\mu(J) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 & \varepsilon^3 & \varepsilon^4 \\ 1 & \varepsilon^2 & \varepsilon^4 & \varepsilon & \varepsilon^3 \\ 1 & \varepsilon^3 & \varepsilon & \varepsilon^4 & \varepsilon^2 \\ 1 & \varepsilon^4 & \varepsilon^3 & \varepsilon^2 & \varepsilon \end{pmatrix},$$

we find $\kappa(J) = -1$ and the representation $\omega = \kappa\mu$ is obtained.

5. The determinant of the inverse Fourier transform

This section is devoted to the computation of $\kappa(J)$. At first it is shown how to obtain $\kappa(J)$ from the trace of $\mu(J)$ and then the trace is computed. The eigenvalues of $\mu(J)$ are fourth roots of unity $e^{\frac{\pi i}{2}k}$ occurring with multiplicities α_k. For the basis of the δ_x with $x \in V$ holds $\mu(J)\delta_x = \frac{1}{\sqrt{|K^n|}} \sum_{z\in K^n} \chi(\langle z, x\rangle)\delta_z$, as

$$\mu(J)\delta_x(y) = \frac{1}{\sqrt{|K^n|}} \sum_{z\in K^n} \chi(\langle z, x\rangle)\delta_z(y).$$

So $\frac{1}{\sqrt{|K^n|}} \sum_{x\in K^n} \varepsilon^{\mathrm{Tr}\langle x,x\rangle} = \sum_{k=0}^{3} \alpha_k e^{\frac{\pi i}{2}k}$, $\sum_{k=0}^{3} \alpha_k = |K^n|$ and

$$\det \mu(J) = \prod_{k=0}^{3} e^{\frac{\pi i}{2}k\alpha_k} = (-1)^{\alpha_2} i^{\alpha_1 - \alpha_3}.$$

By

$$\begin{aligned} |\mathrm{tr}\, \mu(J)|^2 &= \frac{1}{|K^n|} \sum_{x\in K^n} \sum_{y\in K^n} \chi(\langle x, x\rangle - \langle y, y\rangle) \\ &= \frac{1}{|K^n|} \sum_{z\in K^n} \sum_{y\in K^n} \chi(\langle z - 2y, z\rangle) = 1, \end{aligned}$$

we have

$$1 = |\alpha_0 - \alpha_2|^2 + |\alpha_1 - \alpha_3|^2.$$

We can decompose $V = V^+ \oplus V^-$ and $\det \mu(J) = \det \mu^+(J) \det \mu^-(J)$ by Theorem 4.2. Because of $\mu^+(J)^2 = I$ and $\mu^-(J)^2 = -I$, the matrix $\mu^+(J)$ has only the eigenvalues ± 1 and $\mu^-(J)$ only $\pm i$. So $\alpha_0 + \alpha_2 = \frac{|K^n|+1}{2}$ and $\alpha_1 + \alpha_3 = \frac{|K^n|-1}{2}$. If $|K^n| \equiv 1 \bmod 4$, it holds, that $\alpha_1 + \alpha_3 \equiv 0 \bmod 2$ and hence $\alpha_1 - \alpha_3 \equiv 0 \bmod 2$. As $|\alpha_1 - \alpha_3| \leq 1$, we have $\alpha_1 = \alpha_3$. So in this case $\det \mu(J) = (-1)^{\alpha_2} = \pm 1$. If $|K^n| \equiv 3 \bmod 4$, it holds, that $\alpha_1 + \alpha_3 \equiv 1 \bmod 2$ and hence $\alpha_1 - \alpha_3 \equiv 1 \bmod 2$. So $\alpha_1 - \alpha_3 = \pm 1$ and we have $\det \mu(J) = \pm i$. Now we obtain the following lemma, which will be used in the sequel to determine the value of the determinant. It reveals the connection of the trace of $\mu(J)$ to the determinant.

Lemma 5.1. *It holds, that* $\det \mu(J) = i^{\alpha}$, *where*

$$\alpha = \frac{|K^n| + 1}{2} - i^{(|K^n|-1)^2/4}\, \mathrm{tr}\, \mu(J) \in \mathbf{Z}.$$

Proof. If $|K^n| \equiv 1 \bmod 4$, we have $\det \mu(J) = (-1)^{\alpha_2} = i^{2\alpha_2}$ in the above notation, where $2\alpha_2 = \alpha_0 + \alpha_2 - (\alpha_0 - \alpha_2) = \frac{|K^n|+1}{2} - \operatorname{tr} \mu(J)$. It holds, that $|K^n| - 1 \equiv 0 \bmod 4$, so $\frac{(|K^n|-1)^2}{4} \equiv 0 \bmod 4$.

If $|K^n| \equiv 3 \bmod 4$, we have $\det \mu(J) = i^{2\alpha_2+\alpha_1-\alpha_3}$ and $2\alpha_2 = \frac{|K^n|+1}{2}$, $\alpha_1 - \alpha_3 = -i \operatorname{tr} \mu(J)$. As $|K^n| - 1 \equiv \pm 2 \bmod 8$, it holds, that $\frac{(|K^n|-1)^2}{4} \equiv 1 \bmod 4$. ∎

So if we knew the value of $\operatorname{tr} \mu(J)$, we could determine $\det \mu(J)$. By [3, page 333] holds the following.

Lemma 5.2. *For $[K:P] = 2m+1$*

$$\sum_{x \in K} \varepsilon^{\operatorname{Tr} x^2} = (-1)^{m(|P|-1)/2} \, i^{(|P|-1)^2/4} \sqrt{|K|}.$$

The first part of the theorem in [3, page 333] states, that $\sum_{x\in K} \varepsilon^{\operatorname{Tr} x^2} = (-1)^{m(|P|-1)/2} \sqrt{|K|}$ for $[K:P] = 2m$. But $-(-1)^{m(|P|-1)/2} \sqrt{|K|}$ is the correct value. This will be shown in the following theorem.

By [3, page 325] holds the following.

Lemma 5.3. *If $[K:P] = 2m$,*

$$\sum_{x \in K} e^{2\pi i \operatorname{Tr} x^2/|P|} = (-1)^{(|P|^m+1)/2} \sqrt{|K|}.$$

Now we obtain the following, which would also be the correct form of the theorem in [3, page 333]. This appears also in [13, page 486] without proof.

Theorem 5.4. *Let $\tau = \frac{1}{\sqrt{|K|}} \sum_{x\in K} \varepsilon^{\operatorname{Tr} x^2}$, then*

$$\tau = (-1)^{[K:P]+1} \, i^{[K:P](|P|-1)^2/4}.$$

Proof. We have

$$\sum_{x \in K} \varepsilon^{\operatorname{Tr} x^2} = \begin{cases} -(-1)^{m(|P|-1)/2} \sqrt{|K|}, & [K:P] = 2m \\ (-1)^{m(|P|-1)/2} \, i^{(|P|-1)^2/4} \sqrt{|K|}, & [K:P] = 2m+1 \end{cases}.$$

The equation for $[K:P] = 2m+1$ can be deduced from lemma 5.2. By lemma 5.3 holds $\sum_{x\in K} \varepsilon^{\operatorname{Tr} x^2} = -(-1)^{(|P|^m-1)/2} \sqrt{|K|}$ for $[K:P] = 2m$. Now $\frac{|P|^m-1}{2} = \frac{|P|-1}{2} \sum_{k=0}^{m-1} |P|^k \equiv \frac{|P|-1}{2} m \bmod 2$ and so $(-1)^{(|P|^m+1)/2} = -(-1)^{m(|P|-1)/2}$ for all m and $|P|$.

Let $[K:P] = 2m$, then $i^{[K:P]} = (-1)^m$. Because of $x^2 \equiv x \bmod 2$, $(-1)^{(|P|-1)^2/4} = (-1)^{(|P|-1)/2}$ can be deduced and hence the equation for $[K:P] = 2m$. For $[K:P] = 2m+1$ holds $i^{[K:P](|P|-1)^2/4} = (-1)^{m(|P|-1)/2} \, i^{(|P|-1)^2/4}$. ∎

Further we obtain the following corollary, which determines the trace of $\mu(J)$.

Corollary 5.5. *It holds, that*

$$\operatorname{tr}\mu(J) = \frac{1}{\sqrt{|K^n|}} \sum_{x \in K^n} \varepsilon^{\operatorname{Tr}\langle x,x\rangle} = (-1)^{n([K:P]+1)}\, i^{n[K:P](|P|-1)^2/4}.$$

Proof. ¿From $\frac{1}{\sqrt{|K^n|}} \sum_{x\in K^n} \varepsilon^{\operatorname{Tr}\langle x,x\rangle} = \left(\frac{1}{\sqrt{|K|}} \sum_{x\in K} \varepsilon^{\operatorname{Tr} x^2}\right)^n$ and the previous theorem this can be deduced at once. ■

This yields the value of $\kappa(J)$.

Theorem 5.6. *The value of $\kappa(J)$ is $(-1)^{n([K:P]+1)}(-i)^{n[K:P](|P|-1)/2}$.*

Proof. By lemma 5.1 we have $\det\mu(J) = (-i)^{\alpha}$, where $\alpha = -\frac{|K^n|+1}{2} + i^{(|K^n|-1)^2/4}\operatorname{tr}\mu(J)$. So $\alpha = -\frac{|K^n|+1}{2} + (-1)^{n([K:P]+1)}\, i^{(|K^n|-1)^2/4+n[K:P](|P|-1)^2/4}$ by corollary 5.5. As $(-1)^a \equiv 2a+1 \bmod 4$ we have

$$\begin{aligned}
&(-1)^{n([K:P]+1)}\, i^{(|K^n|-1)^2/4+n[K:P](|P|-1)^2/4}\\
\equiv\;& 2n([K:P]+1) + \frac{(|K^n|-1)^2}{4} + n[K:P]\frac{(|P|-1)^2}{4} + 1 \bmod 4.
\end{aligned}$$

So $\alpha \equiv 2n([K:P]+1) + \frac{|K^n|-1}{2}\frac{|K^n|-3}{2} + n[K:P]\frac{(|P|-1)^2}{4} \bmod 4$. Now

$$\begin{aligned}
\frac{|K^n|-1}{2}\frac{|K^n|-3}{2} &= \frac{|K^n|^2-1}{4} - (|K^n|-1)\\
&= \frac{|P|^2-1}{4}\sum_{k=0}^{n[K:P]-1}|P|^{2k} - \frac{|P|-1}{2}\sum_{k=0}^{n[K:P]-1} 2|P|^k\\
&\equiv \frac{|P|^2-1}{4}n[K:P] - (|P|-1)\,n[K:P] \bmod 4.
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{|K^n|-1}{2}\frac{|K^n|-3}{2} + n[K:P]\frac{(|P|-1)^2}{4}\\
\equiv\;& \frac{|P|-1}{2}(|P|-2)\,n[K:P] \bmod 4.
\end{aligned}$$

As

$$\frac{|P|-1}{2}(|P|-2) = \frac{|P|-1}{2} + \frac{|P|-1}{2}(|P|-3) \equiv \frac{|P|-1}{2} \bmod 4$$

and $\kappa(J) = \det\mu(J)$, the theorem is proven. ■

6. The character of the metaplectic representation

In [8, page 290] it was shown, that $|\operatorname{tr}\omega(g)| = |K|^{\frac{1}{2}\dim\ker(I-g)}$. With the explicit formula above for the projective representation μ we can derive this result easily for the set of generating elements. We even obtain the concrete value and not only the absolute value. For all elements of $\operatorname{Sp}(n,K)$ the character of the metaplectic representation has been determined in [14].

To state the following theorem, we need the following. For $b = b^*$ exists a $c \in \operatorname{GL}(n,K)$, such that c^*bc is a diagonal, see for example [9, page 358].

Theorem 6.1. *The values of the character of the metaplectic representation for the generating elements of* $\operatorname{Sp}(n,K)$ *are*

$$\begin{aligned} \operatorname{tr}\omega(s(a)) &= \det(a)^{\frac{|K|-1}{2}} |K|^{\dim\ker(I-a)}, \\ \operatorname{tr}\omega(u(b)) &= (-1)^{\sigma_-}\left((-1)^{[K:P]+1} i^{[K:P](|P|-1)^2/4}\right)^{\sigma_++\sigma_-} |K|^{(n+\sigma_0)/2}, \\ \operatorname{tr}\omega(J) &= (-1)^{n[K:P](|P|-3)(|P|-1)/8}, \end{aligned}$$

where σ_+ *denote the number of squares,* σ_- *the number of non-squares,* σ_0 *the number of* 0 *on the diagonal of* $-2^{-1}c^*bc$ *and* c *is as above.*

Proof. It holds, that $\operatorname{tr}\omega(g) = \kappa(g)\operatorname{tr}\mu(g)$ for $g \in \operatorname{Sp}(n,K)$. We have $\mu(s(a))f(x) = f(a^{-1}x)$. For the basis δ_x of $\mathbb{C}^{K^n}$ holds $\mu(s(a))\delta_x = \delta_{ax}$. Hence

$$\operatorname{tr}\mu(s(a)) = |\ker(I-a)|$$

and

$$\operatorname{tr}\omega(s(a)) = \det(a)^{\frac{|K|-1}{2}} |K|^{\dim\ker(I-a)},$$

so the statement holds for $s(a)$.

We obtain

$$\begin{aligned} \operatorname{tr}\mu(u(b)) &= \sum_{x\in K^n} \chi\left(-2^{-1}\langle x, bx\rangle\right) = \sum_{y\in K^n} \chi\left(-2^{-1}\langle cy, bcy\rangle\right) \\ &= \sum_{y\in K^n} \chi\left(-2^{-1}\langle y, c^*bcy\rangle\right) \\ &= \left(\sum_{z\in K}\chi(z^2)\right)^{\sigma_+} \left(\sum_{z\in K}\chi(\alpha z^2)\right)^{\sigma_-} |K|^{\sigma_0} \\ &= (-1)^{\sigma_-}\left((-1)^{[K:P]+1} i^{[K:P](|P|-1)^2/4}\right)^{\sigma_++\sigma_-} |K|^{(\sigma_++\sigma_-)/2+\sigma_0}, \end{aligned}$$

where α is a non-square. So we have

$$\operatorname{tr}\omega(u(b)) = (-1)^{\sigma_-}\left((-1)^{[K:P]+1} i^{[K:P](|P|-1)^2/4}\right)^{\sigma_++\sigma_-} |K|^{(n+\sigma_0)/2}.$$

For J

$$\operatorname{tr}\mu(J) = (-1)^{n([K:P]+1)} i^{n[K:P](|P|-1)^2/4}.$$

So

$$\begin{aligned} \operatorname{tr}\omega(J) &= \kappa(J)\operatorname{tr}\mu(J) \\ &= (-1)^{n([K:P]+1)}(-i)^{n[K:P](|P|-1)/2}(-1)^{n([K:P]+1)} i^{n[K:P](|P|-1)^2/4} \\ &= i^{n[K:P]\left((|P|-1)^2/4-(|P|-1)/2\right)} = (-1)^{n[K:P](|P|-3)(|P|-1)/8}. \end{aligned}$$

■

In [14] the character of the metaplectic representation has been determined for every $g \in \operatorname{Sp}(n,K)$. In can be shown, that the values of the character for the generating elements coincide with the ones obtained here.

7. Equivalence classes of metaplectic representations

The representation of the Heisenberg group is completely determined by the center, so the only possibility to obtain "different" metaplectic representations is to change the value of η on the center. In this section it will be shown this way yields just one more representation, which is not equivalent to the original one. So we have exactly two equivalence classes of metaplectic representations. This result was also obtained in [7], but the proof here is different.

For every $h \in K \setminus \{0\}$ we obtain a representation η_h, defined by

$$\eta_h(p, q, t) = \eta(hp, q, ht),$$

and so we have a representation ω_h of the symplectic group by the condition

$$\omega_h(g)\,\eta_h(w,t)\,\omega_h(g)^{-1} = \eta_h(gw, t).$$

The representation η_h for $h \neq 1$ is not equivalent to η, because the values on the center are different. In the following there are some lemmata, which are needed to prove the main theorem of this section, that the representation ω_h is up to equivalence only depending on the condition whether h is a square or not. This result was also obtained in [7, page 13].

Lemma 7.1. *Let $n \neq 1$ or $K \not\cong \mathbb{Z}/3\mathbb{Z}$, then $\mathrm{Sp}(n,K)$ equals its commutator subgroup $\mathrm{Sp}(n,K)'$, i. e.*

$$\mathrm{Sp}(n,K) = \mathrm{Sp}(n,K)'.$$

Proof. It holds, that

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}^{-1}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I & I \\ I & 2I \end{pmatrix}$$

and

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2I & I \\ I & I \end{pmatrix}$$

are in the commutator group of $\mathrm{Sp}(n,K)$. Because both elements are distinct and, as

$$\begin{pmatrix} I & I \\ I & 2I \end{pmatrix}\begin{pmatrix} 2I & I \\ I & I \end{pmatrix}\begin{pmatrix} I & I \\ I & 2I \end{pmatrix}^{-1}\begin{pmatrix} 2I & I \\ I & I \end{pmatrix}^{-1} = \begin{pmatrix} 5I & -6I \\ 6I & -7I \end{pmatrix} \neq I,$$

none of these elements is in the center and in particular the commutator group of $\mathrm{Sp}(n,K)$ is no subgroup of the center. So by [1, page 173] the commutator group equals $\mathrm{Sp}(n,K)$, except for the case $\mathrm{Sp}(1,\mathbb{Z}/3\mathbb{Z})$. ∎

Lemma 7.2. *Let $n \neq 1$ or $K \not\cong \mathbb{Z}/3\mathbb{Z}$ and ρ a representation of $\mathrm{Sp}(n,K)$ on $\mathbb{C}^{K^n}$, which fulfills the condition $\rho(g)\,\eta(w,t)\,\rho(g)^{-1} = \eta(gw,t)$ for all $g \in \mathrm{Sp}(n,K)$ and $(w,t) \in H_n$, then $\rho = \omega$.*

Proof. Let ρ be such a representation, then, because of the uniqueness of the intertwining operators for all $g \in \mathrm{Sp}(n, K)$, there is a scalar $c(g) \in \mathbb{C}$, $|c(g)| = 1$ with $\rho(g) = c(g)\,\omega(g)$. Now $c : \mathrm{Sp}(n, K) \to T$ is a homomorphism of $\mathrm{Sp}(n, K)$ into the torus group, and hence trivial, by Lemma 7.1. Thus Lemma 7.2 is proven. ■

Let $(K \setminus \{0\}, \cdot)$ be the multiplicative group of the field K, then $K \setminus \{0\}$ is cyclic. As the group $K \setminus \{0\}$ is abelian the set of all squares of $K \setminus \{0\}$ is a normal subgroup of $K \setminus \{0\}$. So the factor group $K \setminus \{0\}$ modulo the normal subgroup of squares is also cyclic. On the other hand all elements of this factor group have order 2. So we have that this factor group is isomorphic to $\{\pm 1\}$.

The following theorem can also be found in [7, page 13].

Theorem 7.3. *The representations ω_h and ω_k are equivalent, if and only if hk is a square.*

Proof. In case $n = 1$ and $K = \mathbb{Z}/3\mathbb{Z}$ the statement about the equivalence is clear, as 1 is the only square and 2 the only non-square in $\mathbb{Z}/3\mathbb{Z}$. Let in the following $n \neq 1$ or $K \not\cong \mathbb{Z}/3\mathbb{Z}$. As $k^{-1}h = \ell^2$ is a square,

$$\eta_h(p, q, t) = \eta\left(kk^{-1}hp, q, kk^{-1}ht\right) = (\eta_k \circ \delta_\ell \circ b_\ell)(p, q, t)$$

with $\delta_\ell(p, q, t) = (\ell p, \ell q, \ell^2 t)$ and $b_\ell(p, q, t) = (\ell p, \ell^{-1} q, t)$, because of

$$\eta_k(\delta_\ell(b_\ell(p, q, t))) = \eta_k\left(\ell^2 p, q, \ell^2 t\right) = \eta_h(p, q, t).$$

It holds, that $b_\ell \in \mathrm{Sp}(n, K)$, as $b_\ell = s(\ell I)$, and δ_ℓ commutes with all elements in $\mathrm{Sp}(n, K)$. So for $g \in \mathrm{Sp}(n, K)$

$$\begin{aligned}
& \eta_h(g(p, q, t)) \\
= \; & (\eta_k \circ b_\ell g)(\delta_\ell(p, q, t)) \\
= \; & \omega_k(b_\ell g)\,\omega_k(b_\ell)^{-1}\,\eta_k(b_\ell \circ \delta_\ell(p, q, t))\,\omega_k(b_\ell)\,\omega_k(b_\ell g)^{-1} \\
= \; & \omega_k(b_\ell)\,\omega_k(g)\,\omega_k(b_\ell)^{-1}\,\eta_h(p, q, t)\left(\omega_k(b_\ell)\,\omega_k(g)\,\omega_k(b_\ell)^{-1}\right)^{-1}.
\end{aligned}$$

With the uniqueness of ω_k from the above lemma $\omega_h(g) = \omega_k(b_\ell)\,\omega_k(g)\,\omega_k(b_\ell)^{-1}$ for all $g \in \mathrm{Sp}(n, K)$, so ω_k and ω_h are equivalent, if $n \neq 1$ or $K \not\cong \mathbb{Z}/3\mathbb{Z}$.

Let ω_k and ω_h be equivalent, then $\mathrm{tr}\,\omega_k = \mathrm{tr}\,\omega_h$. But this is only possible, if hk is a square. Otherwise a consideration of $\mathrm{tr}(\omega_k(u(b)))$ and $\mathrm{tr}(\omega_h(u(b)))$ shows, that they cannot be equal for all $b = b^*$. ■

References

[1] Artin, E., "Geometric algebra", Interscience Publishers, 1957.

[2] Auslander, L., and R. Tolimeri, *Ring structure and the Fourier transform*, The Mathematical Intelligencer **7:3** (1985), 49–52, 54.

[3] Carlitz, L., *Evaluation of some exponential sums over a finite field*, Mathematische Nachrichten **96** (1980), 319–339.

[4] Dickson, L. E., "Linear groups," B. G. Teubner, Leipzig, 1901.

[5] Folland, G. B., "Harmonic analysis in phase space," Princeton University Press, 1989.

[6] Gérardin, P., *Weil representations associated to finite fields*, J. Algebra, **46** (1977), 54–101.

[7] Howe, R., *Invariant theory and duality for classical groups over finite fields, with applications to their singular representation theory*, unpublished manuscript.

[8] —, *On the character of Weil's representation*, Trans. Amer. Math. Soc. **177** (1973), 287–298.

[9] Lang, S., "Algebra," Addison-Wesley, 1967.

[10] —, "$\mathrm{SL}_2(\mathbb{R})$," Addison-Wesley, 1975.

[11] Leutbecher, A., „Zahlentheorie", Springer-Verlag, 1996.

[12] Mœglin, C., M.-F. Vignéras, and J.-L. Waldspurger, "Correspondances de Howe sur un corps p-adic", Springer-Verlag, 1987.

[13] Shinoda, K.-I., *Sur le caractère de la représentation de Weil pour un corps fini*, C. R. Acad. Sc. Paris Série A **288** (1979), 485–488.

[14] —, *The characters of Weil representations associated to finite fields*, J. Algebra **66** (1980), 251–280.

[15] Šťovíček, P., and J. Tolar, *Quantum mechanics in a discrete space time*, Reports on Mathematical Physics **20** No. 2 (1984), 157–170.

[16] Varadarajan, V. S., *Variations on a theme of Schwinger and Weyl*, Letters in Mathematical Physics **34** (1995), 319–326.

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