Journal of Lie Theory
Volume **13** (2003) 189–191

A Note on the Linear Cycle Space for Groups of Hermitian Type

Joseph A. Wolf[1] **& Roger Zierau**[2]

Communicated by K.-H. Neeb

Abstract. Let G_0 be a simple Lie group of hermitian type and let B denote the corresponding hermitian symmetric space. The linear cycle space for any nonholomorphic type flag domain of G_0 is biholomorphic to $B \times \overline{B}$. When G_0 is a classical group this was proved by the authors in a paper published several years ago. Here we show that the result follows for arbitrary groups of hermitian type. This is done without case by case arguments by combining results from that paper with recent results of A. T. Huckleberry and the first author.

1. Introduction

Let G be a complex semisimple Lie group with noncompact real form G_0 of hermitian type. Let K_0 be a maximal compact subgroup of G_0 and K its complexification. Denote by Q a parabolic subgroup of G and set $Z = G/Q$, the corresponding flag variety. There is a one to one correspondence between K-orbits and G_0-orbits on Z ([2], [8], [9]). Under this correspondence open G_0-orbits correspond to closed K-orbits. Following [13, page 529] we let D be an open G_0-orbit in Z and Y the corresponding closed K-orbit. The closed K-orbit Y is a maximal compact subvariety of D. The *linear cycle space* is the connected component of

$$\{g \cdot Y : g \in G \text{ and } g \cdot Y \subset D\}$$

containing Y and is denoted by M_D. Since G_0 is noncompact and of hermitian type, it is not transitive on Z [12, Corollary 1.7], and M_D is not reduced to a point. Now there are two types of open orbits D, holomorphic type where it is known that M_D is biholomorphic to B or to $\overline{B}$, and nonholomorphic type where we conjectured (and proved in many cases) that M_D is biholomorphic to $B \times \overline{B}$. See [13, Definition 1.2] for the definitions and [13, Theorem 1.3] for a number of equivalent conditions.

As M_D sits in a holomorphic double fibration with D, it plays a role in the study of representations occurring in cohomology on D. Therefore it is of

[1]Partially supported by NSF Grant DMS 99–88643 and by hospitality from the MSRI.
[2]Partially supported by NSF Grant DMS 98-01605 and by hospitality from the MSRI.

ISSN 0949–5932 / $2.50

interest to understand the precise structure of M_D. Theorem 1 gives an explicit description of M_D.

2. The structure theorem

Let B denote the hermitian symmetric space G_0/K_0. In the above setting the complete theorem on the structure of the linear cycle space is the following.

Theorem 1. *If G_0 is an arbitrary simple Lie group of hermitian type then*

(1) *if D is of nonholomorphic type then M_D is biholomorphic to $B \times \overline{B}$,*

(2) *if D is of holomorphic type then M_D is biholomorphic to B or $\overline{B}$.*

Proof. Statement (2) is known by two different general arguments, [11] and [13].

Statement (1) was proved in [13] for classical groups as follows. The inclusion $B \times \overline{B} \subset M_D$ is proved in [13] for the classical groups by a case by case argument. The other inclusion is proved in [13] (see Section 3 below) for arbitrary hermitian type groups. Now we observe that the other inclusion, $B \times \overline{B} \subset M_D$ for arbitrary hermitian type groups, is essentially contained in the literature.

The containment is fairly direct in [7]: A certain domain $\Omega_S(D)$, based on Schubert incidence theory, was introduced in [6], and it was recently proved that $M_D = \Omega_S(D)$ [7, Corollary 3.4].

The containment is slightly less direct in [4]. A. T. Huckleberry observed that one can obtain the desired inclusion $B \times \overline{B} \subset M_D$ from a careful look at [4, Proposition 2.4], though it is not actually stated in [4]. □

Corollary 2. *For G_0 a simple Lie group of hermitian type and D any nonholomorphic type orbit $\mathcal{A} \simeq M_D$.*

In this note, D is measurable in the sense of [10] and [11] because G_0 is of hermitian type; it has been known for some time [11] that if D is measurable then M_D is Stein. The Schubert domain considerations [7], used in our proof of Theorem 1, show that M_D is Stein without any measurability condition on D.

3. Remark

It is announced in [4] that [13] contains a gap in the proof of $M_D \subset B \times \overline{B}$ (Theorem 3.8) for general hermitian type groups. At the very end of our proof of Case 1 in the proof of that Theorem 3.8, we made a few inaccurate statements which tend to obscure the argument. Referring to [13], the following adjustments should clarify that argument.

1. Omit '$c_\Gamma =$' in line 2 of page 537.

2. Omit the sentence 'In particular $c_\Gamma(z) \notin D$.' in line 3, page 537. This is true by Lemma 3.6, but is irrelevant in view of (1) above.

3. In view of (1) above, we must change 'c_Γ' to '$c_{\Gamma \cap \Delta(\mathfrak{r}_+)}$' in line 4, page 537. The corresponding change of g_0 in the same line to $g_0' = \prod_{\psi \in \Delta(\mathfrak{r}_+) \setminus \Gamma} g_{0,\psi}$ is also necessary.

Now make the following simple observation. Let $\psi \in \Psi^{\mathfrak{g}} \subset \Delta(\mathfrak{s}_+)$, and suppose that $g \in G[\psi]$ has decomposition $g = \exp(\xi_+)k\exp(\xi_-)$, $\xi_\pm \in \mathfrak{g}_{\pm\psi}$ and $k \in K \cap G[\psi]$. Then $k \in H \subset Q$ and
(1) if $\psi \in \Delta(\mathfrak{q}) = \Delta(\mathfrak{l} + \mathfrak{r}_-)$ then $\xi_+ \in \mathfrak{g}_\psi \subset \mathfrak{q}$ so $\exp(\xi_+)(z) = z$, and

(2) if $\psi \in \Delta(\mathfrak{r}_+)$ then $\xi_- \in \mathfrak{g}_{-\psi} \subset \mathfrak{r}_-$ so $\exp(\xi_+)(z) = \exp(\xi_+)k\exp(\xi_-)(z) = g(z)$.

References

[1] Akhiezer, D. N., & S. G. Gindikin, *On the Stein extensions of real symmetric spaces*, Math. Annalen **286** (1990), 1–12.

[2] Bremigan, R. J. & J. D. Lorch, *Matsuki duality for flag manifolds*, to appear.

[3] Burns, D, S. Halverscheid & R. Hind, *The geometry of Grauert tubes and complexification of symmetric spaces*, to appear.

[4] Gindikin, S. & T. Matsuki, *Stein extensions of riemannian symmetric spaces and dualities of orbits on flag manifolds*, MSRI Preprint 2001–028.

[5] Halverscheid, S., *Maximal domains of definition of adapted complex structures for symmetric spaces of non-compact type*, Thesis, Ruhr–Universität Bochum, 2001.

[6] Huckleberry, A. T., *On certain domains in cycle spaces of flag manifolds*, Math. Annalen, to appear.

[7] Huckleberry, A. T., & J. A. Wolf, *Schubert varieties and cycle spaces*, to appear.

[8] Matsuki, T., *Orbits of affine symmetric spaces under the action of parabolic subgroups*, Hiroshima Math. J. **12** (1982), 307–320.

[9] Mirkovič, I., K. Uzawa, & K. Vilonen, *Matsuki correspondence for sheaves*, Invent. math. **109** (1992), 231–245.

[10] Wolf, J. A., *The action of a real semisimple group on a complex flag manifold, I: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237.

[11] Wolf, J. A., *The Stein condition for cycle spaces of open orbits on complex flag manifolds*, Annals of Math. **136** (1992), 541–555.

[12] Wolf, J. A., *Real groups transitive on complex flag manifolds*, Proc. Amer. Math. Soc., **129** (2001), 2483–2487.

[13] Wolf, J. A., & R. Zierau, *Linear cycle spaces in flag domains*, Math. Annalen **316** (2000), 529–545.

Joseph A. Wolf
Department of Mathematics
University of California
Berkeley, CA 94720–3840, USA
jawolf@math.berkeley.edu

Roger Zierau
Mathematics Department
Oklahoma State University
Stillwater, OK 7407, USA
zierau@math.okstate.edu

Received January 4, 2002
and in final form June 4, 2002

Editorial Note. Journal of Lie Theory publishes "Short Notes" on communications containing short or elegant proofs of known facts, relevant comments on ongoing research within the scope of Journal of Lie Theory, remarks on the history of the subject, or other items deemed constructive for the clarification of pending issues in the area of Lie Theory. Short Notes are not instruments for setting priorities but for furthering a scholarly discourse. The length of a Short Note should not exceed four pages.