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K-star Products on Dual of Lie Algebras

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Abstract. We consider a generalization $(\star_\alpha)$ of the Kontsevich family of star products $(\star_\alpha^K)$ for linear Poisson structures α. Such a family is characterized by a formal function F. We study some general properties of such families: invariance and covariance, closeness and relativity, symmetry and reality. Finally, we characterize the Kontsevich family $(\star_\alpha^K)$ among all them.

1. Introduction

The study of star products *i.e.* associative deformations of usual multiplication of functions has been introduced by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer [8] as a tool for the quantization of a symplectic or Poisson [15] manifold.

Recently, M. Kontsevich solved the problem of the existence of star products on any finite dimensional Poisson manifold [13]. He built a star product $\star_\alpha^K$ on $\mathbb{R}^d$ equipped with any Poisson bracket α. This star product is defined by using oriented graphs $\overrightarrow{\Gamma}$.

In this paper, we restrict ourselves to the case of linear Poisson structures α, *i.e.* to the case of dual of Lie algebras.

Many authors studied the Kontsevich construction in this case and its properties ([13], [1], [4],[11], [17], [2], [3], [10]). In particular, in [4], we defined a generalization $\alpha \mapsto \star_\alpha$ of the Kontsevich family of star products $\alpha \mapsto \star_\alpha^K$, as:

$$u \star_\alpha v = \sum_{n=0}^{\infty} \sum_{\Gamma \in G_{n,2}} \sum_{orientation} a_{\overrightarrow{\Gamma}} B_{\overrightarrow{\Gamma}}(\alpha)(u,v) = \sum_{n=0}^{\infty} C_n(u,v)$$

(As a matter of fact, we imposed in [4] $C_n(u,v) = (-1)^n C_n(v,u)$ and $C_n(1,u) = 0, \forall n \geq 1$). Then $\star_\alpha$ is given by an integral formula:

$$(u \star_\alpha v)(\xi) = \int_{\mathfrak{g}^2} \hat{u}(X)\hat{v}(Y) \frac{F(X)F(Y)}{F(X \times_\alpha Y)} e^{2i\pi\langle \xi, X \times_\alpha Y\rangle} dX\, dY$$

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where $X \times_\alpha Y$ is the Baker-Campbell-Hausdorff formula, viewed as a formal mapping from $\mathfrak{g}^2$ to $\mathfrak{g}$, $F(X) = J(X)H(X) = J(X)e^{h(X)}$ and

$$H(X) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{s_1,\dots,s_p \\ |s|=2n}} a_{s_1\dots s_p} Tr(ad\ 2i\pi X)^{s_1} \dots Tr(ad\ 2i\pi X)^{s_p}$$

if $J(X) = det(j(ad\ X))$ and $j(x) = \left[\frac{sh\ x/2}{x/2}\right]^{1/2}$.

In the present paper, we call such a generalization of $(\star_\alpha^K)$ a K-family and we study the properties of these K-families.

We first consider general properties for every K-star product $\star_\alpha$ in these K-families, proving that they always satisfy

$$C_n(1,u) = C_n(u,1) = 0 \quad (n \geq 1),$$

they are graded and characterized by $(X,u) \mapsto X \star_\alpha u$ and $(u,X) \mapsto u \star_\alpha X$. We know that they are equivalent to the Kontsevich family of star products $(\star_\alpha^K)$, the intertwining operator being :

$$T = Id + \sum_{n=1}^{\infty} \sum_{\substack{s_1\dots s_p \\ |s|=n}} a_{s_1\dots s_p}\ T_{s_1} \circ \dots \circ T_{s_p}$$

where

$$(T_s u)(\xi) = (2i\pi)^s \int_{\mathfrak{g}} \hat{u}(X) Tr(adX)^s\ e^{2i\pi\langle \xi, X\rangle} dX.$$

We prove that T_1 is a derivation for any $\star_\alpha$, then they are all invariant and covariant.

We study then the symmetry and reality properties for $(\star_\alpha)$, $(\star_\alpha^K)$ is symmetric and real and a $(\star_\alpha)$ is real if and only if $\overline{H(X)} = H(-X)$.

Restricting ourselves to the case of analytic star products *i.e.* to formal function H converging near 0, we suppose there exists $r > 0$ such that:

$$1 + \sum_{n \geq 1} r^n \sum_{|s|=n} |a_{s_1\dots s_p}| < \infty.$$

In this case, we look for closed and relative K-family. The definition of closed star product was introduced by A. Connes, M. Flato and D. Sternheimer. These star products were also studied by G. Felder and B. Shoikhet when they are coming from cyclic formality [11].

If the Lie algebra $\mathfrak{g}$ defined by our linear Poisson structure α is not unimodular, then a K-star product is never closed on $\mathfrak{g}^*$. Thus we shall say that a K-family $(\star_\alpha)$ is closed if, for any unimodular $\mathfrak{g}$, $\star_\alpha$ is closed on $\mathfrak{g}^*$. We prove that any closed $(\star_\alpha)$ is relative in the sense of [2] and if $(\star_\alpha)$ is strict and relative, it is closed.

Of course $(\star_\alpha^K)$ is an analytic, strict and closed K-family.

Finally we prove that the only symmetric, strict and closed $(\star_\alpha)$ is the Kontsevich family $(\star_\alpha^K)$, moreover the weights w_Γ occurring in the computation of $\star_\alpha^K$ are all rational numbers.

2. Graded star products

Let M be a smooth manifold, a star product $\star$ is a (formal) deformation of the usual point-wise product on the space of $C^\infty(M)$ ([8]). Such a deformation is a formal series:

$$(u,v) \mapsto u \star v = uv + \sum_{r=1}^{\infty} C_r(u,v) = C_0(u,v) + \sum_{r\geq 1} C_r(u,v)$$

which is associative in the following sense:

$$\sum_{r+s=n} C_r(C_s(u,v),w) = \sum_{r+s=n} C_r(u, C_s(v,w)).$$

We suppose the C_r to be bi-differential operators. In fact it is well known [13] that if $\star$ is such a deformation, then the antisymmetric part of C_1 is a Poisson bracket on M:

$$\frac{1}{2}\left(C_1(u,v) - C_1(v,u)\right) = \{u,v\}$$

i.e an antisymmetric bracket satisfying the Jacobi identity and being a derivation for each argument. Such a bracket defines uniquely an antisymmetric bi-vector α on M such that:

$$\{u,v\} = \langle \alpha, du \wedge dv \rangle.$$

This bi-vector α is the Poisson tensor associated to $\{,\}$. If $M = \mathbb{R}^d$, M. Kontsevich in [13] gives an explicit construction of a star product for any Poisson tensor α. We first generalize a little bit this construction.

In order to associate to each graph Γ an m-differential operator $C_\Gamma(\alpha)$, Kontsevich considered some oriented admissible graphs $\overrightarrow{\Gamma}$. More precisely, let $A = \{p_1, \ldots, p_n\}$ and $B = \{q_1, \ldots, q_m\}$ be finite sets . Points of A (resp. B) are the vertices of type 1 (resp. 2) of the graph Γ. The edges are arrows $\overrightarrow{px}$ starting from a vertex p of type 1 and ending at a vertex x in $A \cup B$ (in fact, Kontsevich restricts himself to the case where the edge ends at x distinct from p, but here, we shall allow this sort of edge). From each vertex p in A, there are exactly two arrows starting. Now we define an orientation of Γ by choosing a total ordering $\leq$ on A and on the set $E(\Gamma)$ of edges of Γ which are compatible in the sense that:

$$p \leq p' \Longrightarrow \overrightarrow{px} \leq \overrightarrow{p'x'} \quad \forall p,\ p' \in A, \quad \forall x, x' \in A \cup B.$$

We denote the ordered set of edges by:

$$E(\Gamma) = \{\overrightarrow{e_1} \leq \overrightarrow{e_2} \leq \ldots \leq \overrightarrow{e_{2n}}\}.$$

Let us denote by $\mathrm{End}(x)$ the set of arrows ending at the point x and by $\partial_{\mathrm{End}(x)}$ the differential operator:

$$\partial_{\mathrm{End}(x)} = \partial_{l_1 \ldots l_s} \quad \text{if} \quad \mathrm{End}(x) = \{\overrightarrow{e_{l_1}} \leq \ldots \leq \overrightarrow{e_{l_s}}\}.$$

Denote the edges starting from the vertex p_i by $e_{\ell_1^i} < e_{\ell_2^i}$. Thus, for any Poisson structure α, the m-differential operator $C_{\overrightarrow{\Gamma}}(\alpha)$ is by definition:

$$C_{\overrightarrow{\Gamma}}(\alpha)(u_1, \ldots, u_m) = \sum_{1 \leq \ell_1 \ldots \ell_{2n} \leq d} \quad \prod_{i=1}^{n} \partial_{\mathrm{End}(p_i)} \alpha^{\ell_1^i \ell_2^i} \prod_{k=1}^{m} \partial_{\mathrm{End}(q_k)} u_k$$

Let us choose any total ordering $p_1 < p_2 < \ldots < p_n$ on A, for any graph Γ, we define the particular orientation $lex(\Gamma)$ by putting lexicographic ordering on $E(\Gamma)$:

$$\begin{aligned} \overrightarrow{p_ip_j} \leq \overrightarrow{p_{i'}p_{j'}} &\ \text{ iff } \ i < i' \text{ or } i = i' \text{ and } j \leq j' \\ \overrightarrow{p_ip_j} \leq \overrightarrow{p_{i'}q_k} &\ \text{ iff } \ i \leq i' \\ \overrightarrow{p_iq_k} \leq \overrightarrow{p_{i'}q_{k'}} &\ \text{ iff } \ i < i' \text{ or } i = i' \text{ and } k \leq k' \end{aligned}$$

Finally, starting with $lex(\Gamma)$, we symmetrize Γ by the action of the group S_n of all permutations of vertices $p_1, \ldots, p_n$. Let Γ^σ the graph obtained from Γ by re-labeling $p_{\sigma^{-1}(i)}$ the vertices p_i. Then, with the lexicographic ordering, σ induces a permutation $\tilde{\sigma}$ on the edges: we extend first σ to the vertices q_k by putting $q_{\sigma^{-1}(k)} = q_k$ for all k, then each edge $e_\ell = \overrightarrow{p_ip_j}$ or $e_\ell = \overrightarrow{p_iq_k}$ of $lex(\Gamma)$ becomes the edge $e_{\tilde{\sigma}(\ell)} = \overrightarrow{p_{\sigma^{-1}(i)}p_{\sigma^{-1}(j)}}$ or $e_{\tilde{\sigma}(\ell)} = \overrightarrow{p_{\sigma^{-1}(i)}q_{\sigma^{-1}(k)}}$ of $lex(\Gamma^\sigma)$.

We put now:

$$C_\Gamma(\alpha) = \sum_{\sigma \in S_n} \frac{1}{n!} \varepsilon(\tilde{\sigma}) C_{lex(\Gamma^\sigma)}(\alpha).$$

Let $V_{n,m}$ be the space of admissible graphs. If $\gamma = \sum a_i\Gamma_i$ is a linear combination of graphs Γ_i in $V_{n,m}$, C_γ will be by definition

$$C_\gamma = \sum a_i C_{\Gamma_i}.$$

Definition 2.1. (K-family on $\mathbb{R}^d$) Let $(\gamma_n)_{n\geq 1}$ be an element of $\prod_{n\geq 1} V_{n,2}$, the map $\alpha \mapsto \star_\alpha$ defined by:

$$u_1 \star_\alpha u_2 = u_1u_2 + \sum_{n=1}^{\infty} C_{\gamma_n}(\alpha)(u_1, u_2)$$

is called a *K-family* if, for any linear Poisson structure α,

$$\sum_{r+s=n} C_{\gamma_r}(\alpha)\left(C_{\gamma_s}(\alpha)(u_1, u_2), u_3\right) = \sum_{r+s=n} C_{\gamma_r}(\alpha)\left(u_1, C_{\gamma_s}(\alpha)(u_2, u_3)\right)$$

and

$$C_{\gamma_1}(\alpha)(u_1, u_2) - C_{\gamma_1}(\alpha)(u_2, u_1) = 2\langle \alpha, du_1 \wedge du_2 \rangle.$$

An element $\star_\alpha$ of a K-family $(\star_\alpha)$ is called a K-star product.

From now on, we are looking for 'universal' star products on dual of any Lie algebra, since a linear α;

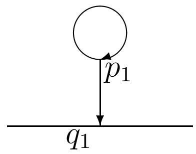
$$\alpha = \sum_{i,j,k} C_{ij}^k x_k \partial_i \wedge \partial_j$$

is a Poisson tensor if and only if the C_{ij}^k are structure constants of a Lie algebra $\mathfrak{g}$.

We consider thus only graphs Γ with $\#\operatorname{End}(p_i) \leq 1$ for all i. Let $W_{n,m}$ be the space of linear combination of such graphs.

For instance $dim(W_{1,1}) = 1$, the only graph being Γ_1:

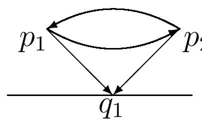
$$E(\Gamma_1) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_1}\}.$$

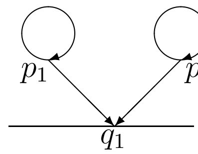


$$C_{\Gamma_1}(\alpha) = C_{lex(\Gamma)}(\alpha) = \sum_{i,j} C^i_{ij}\partial_j.$$

Similarly, $dimW_{2,1} = 2$ the only graphs being:

$$\begin{array}{rl} \Gamma_{1,1}: & E(\Gamma_{1,1}) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_1}, \overrightarrow{p_2p_2}, \overrightarrow{p_2q_1}\} \\ \Gamma_2: & E(\Gamma_2) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1q_1}, \overrightarrow{p_2p_1}, \overrightarrow{p_2q_1}\} \end{array}$$





More generally a basis of $W_{n,1}$ is given by the $\Gamma_{\mathcal{P}}$, where $\mathcal{P}$ is a partition of $\{1,\dots,n\}$ in a family of disjoint finite sequences:

$$\{1,\dots,n\} = A_1 \cup A_2 \cup \dots \cup A_\ell, \qquad A_i = (j^i_1,\dots,j^i_{s_i}), \quad s_i > 1,$$

and

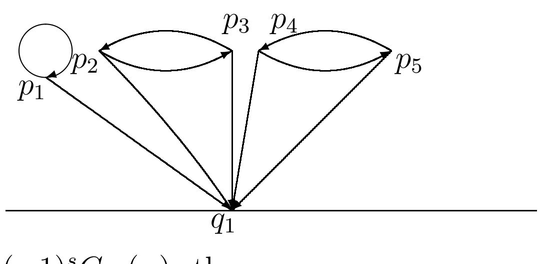
$$A_i \cap A_j = \emptyset \quad \text{if} \quad i \neq j.$$

For the C_Γ, since, up to a sign, C_Γ and C_{Γ^σ} coincide, we shall only consider the graphs $\Gamma_{s_1,\dots,s_\ell}$ $(1 \leq s_1 \leq s_2 \leq \dots \leq s_\ell,\ \sum s_j = n)$, associated to the subsets

$$A_j = \{\sum_{i<j} s_i + 1, \sum_{i<j} s_i + 2, \dots, \sum_{i<j} s_i + s_j\} \qquad (1 \leq j \leq \ell).$$

For instance the graph $\Gamma_{1,2,2}$ is defined by:

$$E(\Gamma_{1,2,2}) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_2p_3}, \overrightarrow{p_3p_2}, \overrightarrow{p_4p_5}, \overrightarrow{p_5p_4}\} \cup \{\overrightarrow{p_iq_1}, i = 1,\dots,5\}.$$



Let us put $T_s = (-1)^s C_{\Gamma_s}(\alpha)$, then:

$$(-1)^{\sum s_i} T_{s_1} \circ \dots \circ T_{s_p} = C_{\Gamma_{s_1,\dots,s_p}}(\alpha).$$

Now, in [8], the authors consider only star products vanishing on constants.

Definition 2.2. (Star product vanishing on constants) A star product on a Poisson manifold M is said to be *vanishing on constants* if, for all u in $C^\infty(M)$ and for all $r > 0$,

$$C_r(1, u) = C_r(u, 1) = 0.$$

Lemma 2.3. (K-family vanishes on constants)
If $(\star_\alpha)$ is a K-family , then $\star_\alpha$ vanishes on the constants.

Proof. First, looking at the total number of edges in admissible graphs Γ in $W_{n,2}$, we see that for each $n > 0$, there is at most n edges ending on some vertex p_i. Since they are $2n > n$ edges, some of them end on a vertex q_ℓ, thus:

$$1 \star_\alpha 1 = 1.$$

Let us set now

$$1 \star_\alpha u = L(\alpha)u = u + \sum_{n=1}^{\infty} L_n(\alpha)(u).$$

By definition, each $L_n(\alpha)$ is a differential operator defined by graphs without any edge ending on q_1, thus the preceding discussion gives:

$$L_n(\alpha) = \sum_{\substack{1\leq s_1 \leq \ldots \leq s_p \\ |s|=n}} a_{s_1 \ldots s_p}\, T_{s_1} \circ \ldots \circ T_{s_p}$$

where the numbers $a_{s_1\ldots s_p}$ do not depend on α. Now, by formal associativity:

$$1 \star_\alpha (1 \star_\alpha u) = 1 \star_\alpha (L(\alpha)u) = L(\alpha)^2 u = (1 \star_\alpha 1) \star_\alpha u = 1 \star_\alpha u = L(\alpha)u.$$

Thus we obtain $L(\alpha)^2 u = L(\alpha)u$. This implies $L(\alpha) = id$.

Indeed, suppose $L(\alpha) \neq id$ and let n be the smallest index for which $L_n(\alpha) \neq 0$, write $L_n(\alpha) = \sum_{\substack{1\leq s_1 \leq \ldots \leq s_p \\ |s|=n}} a_{s_1\ldots s_p} T_{s_1} \circ \ldots \circ T_{s_p}$, let p_0 be the largest p for which one of the $a_{s_1\ldots s_p}$ does not vanish, fix the lexicographic ordering on the set of p_0-tuples $(s_1, \ldots, s_{p_0})$ and let $(\sigma_1, \ldots, \sigma_{p_0})$ be the largest p_0-tuple for which $a_{s_1\ldots s_p}$ does not vanish. Choose finally $d = 2p_0$ and the following α_0 on $\mathbb{R}^d$:

$$\alpha_0 = \sum_{k=0}^{p_0-1} x_{2k+1}\partial_{2k+1} \wedge \partial_{2k+2}.$$

The only $(\partial_2)^{\sigma_1} \ldots (\partial_{2p_0})^{\sigma_{p_0}}$ term in $L_n(\alpha_0)$ is:

$$a_{\sigma_1\ldots\sigma_{p_0}} (\partial_2)^{\sigma_1} \ldots (\partial_{2p_0})^{\sigma_{p_0}}.$$

Now, in $L(\alpha_0)^2 = (id + L_n(\alpha_0) + \ldots)^2$ the term of order n is $2L_n(\alpha_0)$, thus for that α_0 we get an unique term:

$$2a_{\sigma_1\ldots\sigma_{p_0}} (\partial_2)^{\sigma_1} \ldots (\partial_{2p_0})^{\sigma_{p_0}}.$$

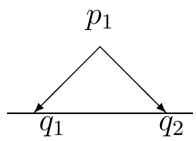
This is impossible, $L(\alpha) = id$ for any α. The same proof holds for $u \star_\alpha 1$. ∎

Corollary 2.4. (The C_1 term of a K-family)
Let $(\star_\alpha)$ be a K-family, then:

$$C_1(\alpha)(u_1,u_2) = \langle \alpha, du_1 \wedge du_2 \rangle = \{u_1,u_2\}.$$

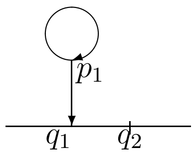
Proof. The only graphs Γ in $W_{1,2}$ are:

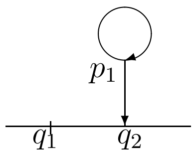
$$\text{the } \Lambda\text{- graph such that} \quad E(\Lambda) = \{\overrightarrow{p_1q_1}, \overrightarrow{p_1q_2}\}$$



and graphs which does not vanish on constants:

$$\Gamma_1 \text{ such that } E(\Gamma_1) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_1}\}; \quad \Gamma_2 \text{ such that } E(\Gamma_2) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_2}\}.$$





Since Γ_1 and Γ_2 do not occur,

$$C_1(\alpha)(u_1,u_2) = a_\Lambda C_\Lambda(\alpha)(u_1,u_2) = a_\Lambda\{u_1,u_2\}$$

and necessarily $a_\Lambda = 1$. ∎

Let $S(\mathfrak{g})$ be the algebra of all polynomial functions on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. The algebra $S(\mathfrak{g})$ is naturally graded. If u is an homogeneous element of $S(\mathfrak{g})$, we will note by $|u|$ its degree.

A multi-linear function C:

$$C : S(\mathfrak{g}) \times S(\mathfrak{g}) \ldots \times S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$$

is said to be homogeneous with degree $-n$ if for $u_1, \ldots, u_k$ homogeneous elements of $S(\mathfrak{g})$, $C(u_1, \ldots, u_k)$ is homogeneous with degree $|u_1| + \ldots + |u_k| - n$.

Definition 2.5. (Graded star products) Let $S(\mathfrak{g})$ be the algebra of polynomial functions on $\mathfrak{g}^*$ and S^p be the space of homogeneous polynomials with degree p. A star product on $S(\mathfrak{g})$ is *graded* if:

$$\forall r,p,q \in \mathbb{N}, \forall (u,v) \in S^p \times S^q,\ C_r(u,v) \in S^{p+q-r}.$$

Let us show that a graded star product is totally defined by $X \star u$ and $u \star X$ where u belongs to $S(\mathfrak{g})$ and X belongs to $\mathfrak{g}$.

Lemma 2.6. (Construction of graded star products)

Let C_n and C'_n be two sequences of bilinear maps:

$$C_n : \ \mathfrak{g} \times S(\mathfrak{g}) \longrightarrow S(\mathfrak{g}) \quad \textit{and} \quad C'_n : \ S(\mathfrak{g}) \times \mathfrak{g} \longrightarrow S(\mathfrak{g})$$

such that:

1- $C_0(X,u) = X.u$, $C'_0(u,X) = u.X$, $C_1(X,u) = \{X,u\}$, $C'_1(u,X) = \{u,X\}$.

2- If u is homogeneous then $C_n(X,u)$ and $C'_n(u,X)$ are homogeneous and their degree is $|u| + 1 - n$.

3- We set:

$$X \circ u = \sum_{n=0}^{\infty} C_n(X,u) \quad \textit{and} \quad u \circ X = \sum_{n=0}^{\infty} C'_n(u,X)$$

and we suppose that:

a) $C_2(X,Y) = C'_2(X,Y)$, $(X\,,\ Y \in \mathfrak{g})$.

b) $X \circ (u \circ Y) = (X \circ u) \circ Y$, $\forall X, Y \in \mathfrak{g}$, $\forall u \in S(\mathfrak{g})$.

c) $X \circ (Y \circ u) - Y \circ (X \circ u) = [X,Y] \circ u$, $(X\,,\ Y \in \mathfrak{g},\ u \in S(\mathfrak{g}))$.

d) $(u \circ X) \circ Y - (u \circ Y) \circ X = u \circ [X,Y]$, $(X\,, Y \in \mathfrak{g},\ u \in S(\mathfrak{g}))$.

Then there exists one and only one star product $\star$ such that:

$$X \star u = X \circ u \quad \textit{and} \quad u \star X = u \circ X, \quad \forall X \in \mathfrak{g}, \quad \forall u \in S(\mathfrak{g}).$$

This star product is graded.

Proof. Let v be in $S(\mathfrak{g})$, for any u in $S(\mathfrak{g})$, we define $u \star v$ by induction on the degree of u starting with:

$$1 \star v = v, \quad X \star v = X \circ v, \quad v \star 1 = v \quad \text{and} \quad v \star X = v \circ X.$$

If u is an homogeneous polynomial function of the form $u = Xu'$ then there exists a polynomial function u'' such that:

$$Xu' = X \circ u' + u'' \quad \text{and} \quad |u''| \leq |u| - 1.$$

We suppose now $u_1 \star v$ defined for any u_1 such that $|u_1| < |u|$, we suppose also that:

$$u_1 \star (v \star w) = (u_1 \star v) \star w \quad \text{if} \quad |u_1| + |v| < |u|.$$

Then we set:

$$(Xu') \star v = X \circ (u' \star v) + u'' \star v.$$

This formula defines without ambiguity $u \star v$. In fact if u has the form:

$$\begin{aligned} u = X_1 X_2 w &= X_1 \circ (X_2 \circ w) + u''_1 \\ &= X_2 \circ (X_1 \circ w) + u''_2 \end{aligned}$$

Then:

$$X_2 \circ (X_1 \circ w) + u''_2 = X_1 \circ (X_2 \circ w) + u''_2 + [X_2, X_1] \circ w.$$

Thus:

$$u''_1 = u''_2 + [X_2, X_1] \circ w$$

and:

$$\begin{aligned} X_1 \circ ((X_2 \circ w) \star v) + u_1'' \star v &= X_1 \circ (X_2 \circ (w \star v)) + u_2'' \star v + ([X_2, X_1] \circ w) \star v \\ &= X_1 \circ (X_2 \circ (w \star v)) + [X_2, X_1] \circ (w \star v) + u_2'' \star v \\ &= X_2 \circ (X_1 \circ (w \star v)) + u_2'' \star v \\ &= X_2 \circ ((X_1 \circ w) \star v) + u_2'' \star v \end{aligned}$$

The homogeneous term of maximum degree in $u \star v$ is $C_0(u, v) = u.v$ then we set:

$$C_n(u, v) = \text{the homogeneous term of degree } \ |u| + |v| - n.$$

A simple computation shows that $\star$ is a star product. In fact, let us first show that:

$$C_1(u, v) = \{u, v\}.$$

It is clear that the term of degree $|u| + |v| - 1$ in $u'' \star v$ is $\{-X, u'\}v$ (coming from $u''v$) and, in $X \circ (u' \star v)$, it is $X\{u', v\} + \{X, u'.v\}$ thus the term of degree $|u| + |v| - 1$ in $u \star v$ is the sum of these terms:

$$\begin{aligned} X\{u', v\} + \{X, u'v\} - \{X, u'\}v &= X.\{u', v\} + u'\{X, v\} \\ &= \{Xu', v\} = \{u, v\}. \end{aligned}$$

Using the same construction, we can then define similarly $u \star' v$ by induction 'on the right side' on the degree of v. In fact, thanks to a), $\star$ and $\star'$ coincide if $|u| = |v| \leq 1$. Now suppose that they coincide for u' and v' such that $|u'| + |v'| < |u| + |v|$, then:

$$\begin{aligned} (X \circ u') \star (v' \circ Y) &= X \circ (u' \star (v' \circ Y)) \\ &= X \circ (u' \star' (v' \circ Y)) = X \circ ((u' \star' v') \circ Y) \\ &= (X \circ (u' \star' v')) \circ Y = (X \circ u') \star' v') \circ Y \\ &= (X \circ u') \star' (v' \circ Y) \end{aligned}$$

By induction on the degree of u and w, we can moreover show that $\star$ is associative.

By definition $u \star (v \star w) = (u \star v) \star w$ if $|u| \leq 1$ and $|w| \leq 1$. Then, by induction on $|u|$, the same holds if $|w| \leq 1$ since:

$$\begin{aligned} (X \circ u' \star v) \star w &= (X \circ (u' \star v)) \star w = X \circ ((u' \star v) \star w) = X \circ (u' \star (v \star w)) \\ &= (X \circ u') \star (v \star w) \end{aligned}$$

and similarly for any w since:

$$\begin{aligned} u \star (v \star (w' \circ Y)) &= u \star ((v \star w') \circ Y) \\ &= ((u \star v) \star w') \circ Y = (u \star v) \star (w' \circ Y). \end{aligned}$$

Finally $\star$ is a graded star product by construction. ∎

Corollary 2.7. (Characterization of graded star product) (see also [14]) *If $\star$ is a graded star product, then $\star$ is entirely determined by the mappings:*

$$\begin{aligned} C:\ \mathfrak{g}\times S(\mathfrak{g}) &\longrightarrow S(\mathfrak{g}) \\ (X,u) &\mapsto X\star u \end{aligned}$$

and

$$\begin{aligned} C':\ S(\mathfrak{g})\times \mathfrak{g} &\longrightarrow S(\mathfrak{g}) \\ (u,X) &\mapsto u\star X. \end{aligned}$$

In particular all K-star products are graded star products, each of them being totally determined by the giving of $X\star u$ and $u\star X$.

3. Universal derivation

In [4] we showed that every K-star product is given by an integral formula of the form:

$$(u\star_\alpha v)(\xi)=\int_{\mathfrak{g}^2}\hat{u}(X)\hat{v}(Y)\frac{F(X)F(Y)}{F(X\times_\alpha Y)}\ e^{2i\pi\langle\xi,X\times_\alpha Y\rangle}\ dXdY$$

for all u, v in $S(\mathfrak{g})$.

These star products are all equivalent to the fundamental star product $\star_\alpha^K$ built by Kontsevich.

From results of Kontsevich and Shoikhet, we can deduce that $\star_\alpha^K$ can be written as:

$$(u\star_\alpha^K v)(\xi)=\int_{\mathfrak{g}^2}\hat{u}(X)\hat{v}(Y)\frac{J(X)J(Y)}{J(X\times_\alpha Y)}\ e^{2i\pi\langle\xi,X\times_\alpha Y\rangle}\ dXdY$$

for all u, v in $S(\mathfrak{g})$, (or u, v smooth functions such that $\hat{u}$ and $\hat{v}$ are compactly supported with a sufficiently small support) where:

$$J(X)=det\left(\frac{sh\ ad\frac{X}{2}}{ad\ \frac{X}{2}}\right)^{\frac{1}{2}}$$

(see [3] for instance). An equivalence operator T between $\star_\alpha^K$ and $\star_\alpha$:

$$u\star_\alpha v=T^{-1}(Tu\star_\alpha^K Tv)$$

is given by:

$$(Tu)(\xi)=\int_{\mathfrak{g}}\hat{u}(X)H(X)\ e^{2i\pi\langle\xi,X\rangle}\ dX$$

if:

$$F(X)=J(X)H(X).$$

The operator T is a formal series of differential operators, each of them being a linear combination of products of operators T_s as defined in the first section. Indeed for each positive integer s,

$$(T_s u)(\xi) = (2i\pi)^s \int_{\mathfrak{g}} \hat{u}(X) Tr(ad\ X)^s\ e^{2i\pi\langle \xi, X\rangle}\ dX.$$

Thus T_s has the following expression:

$$(T_s u) = \sum_{i_1 \dots i_s} \sum_{j_1 \dots j_s} C^{j_2}_{i_1 j_1} C^{j_3}_{i_2 j_2} \dots C^{j_s}_{i_{s-1} j_{s-1}} C^{j_1}_{i_s j_s} \partial_{i_1 \dots i_s} u,$$

with that expression, we see that T_s is the operator associated to the graph Γ_s, called a “wheel” by Kontsevich ([13]). Now we can write:

$$T = Id + \sum_{n=1}^{\infty} \sum_{|s|=n} a_{s_1 \dots s_p} T_{s_1} \circ \dots \circ T_{s_p}$$

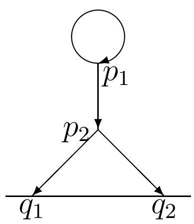
if

$$H(X) = 1 + \sum_{n=1}^{\infty} \sum_{|s|=n} a_{s_1 \dots s_p} Tr(2i\pi\ ad\ X)^{s_1} \dots Tr(2i\pi\ ad\ X)^{s_p}.$$

Proposition 3.1. (Annulation of a C_Γ)
Let P be the operator $C_\Gamma(\alpha)$ associated to the graph Γ in $W_{2,2}$ with edges:

$$E(\Gamma) = \{\overrightarrow{p_1 p_1}, \overrightarrow{p_1 p_2}, \overrightarrow{p_2 q_1}, \overrightarrow{p_2 q_2}\}.$$

Then P is null.



$$P = C_\Gamma(\alpha) = 0$$

Proof. The operator P associated to Γ can be written:

$$P(u, v) = -\sum_{k,\ell} \sum_{i,j} C^i_{k\ell} C^j_{ij} \partial_k u \partial_\ell v.$$

Let $(E_i)_{1 \le i \le d}$ be the basis of $\mathfrak{g}$ dual to the canonical basis of $\mathbb{R}^d$, then one has:

$$ad\ [E_k, E_\ell] = \sum_i C^i_{k\ell}\ ad\ E_i$$

Thus:

$$Tr(ad\ [E_k, E_\ell]) = \sum_i C^i_{k\ell} Tr(ad\ E_i) = \sum_{i,j} C^i_{k\ell} C^j_{ij}.$$

But:

$$Tr(ad\ [E_k, E_\ell]) = Tr(ad\ E_k \circ ad\ E_\ell - ad\ E_\ell \circ ad\ E_k) = 0.$$

Finally, for all u and v:

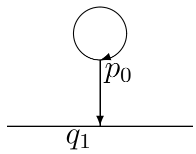
$$P(u,v) = -\sum_{k,\ell} Tr(ad\ [E_k, E_\ell])\ \partial_k u \partial_\ell v = 0$$

and $P = 0$. ∎

Theorem 3.2. (An universal derivation) *The 'wheel' operator T_1 associated to the wheel graph Γ_1 having one vertex p_0 and one vertex q_1 and the edges of Γ_1 are $\{\overrightarrow{p_0p_0}, \overrightarrow{p_0q_1}\}$ is an 'universal' derivation for K-star products. We have:*

$$T_1(u \star_\alpha v) = T_1 u \star_\alpha v + u \star_\alpha T_1 v,$$

for all u, v and for all linear α, all K-star products $\star_\alpha$.



Proof. By definition, T_1 is a vector field. Now, for any admissible graph Γ, we say that a vertex p_i of Γ is 'free' if $\mathrm{End}(p_i) = \emptyset$. Then, for the corresponding Kontsevich operator $C_\Gamma(\alpha)$, if $\mathcal{L}_{T_1}$ is the Lie derivative, one has:

$$\begin{aligned}\mathcal{L}_{T_1} C_\Gamma(\alpha)(u_1, \dots, u_m) &= T_1(C_\Gamma(\alpha)(u_1, \dots, u_m)) - C_\Gamma(\alpha)(T_1 u_1, \dots, u_m) + \dots \\ &\qquad \dots - C_\Gamma(\alpha)(u_1, \dots, T_1 u_m) \\ &= C_\gamma(\alpha)(u_1, \dots, u_m)\end{aligned}$$

where γ is the sum of all the graphs $\sum_{free\ p_i} \Gamma_1 \cup_{p_i} \Gamma$. The vertices of $\Gamma_1 \cup_{p_i} \Gamma$ are $\{p_0, p_1, \dots, p_n, q_1, \dots, q_m\}$ and its edges are $\{\overrightarrow{p_0p_0}, \overrightarrow{p_0p_i}\} \cup E(\Gamma)$. For each $\Gamma_1 \cup_{p_i} \Gamma$, there is a subgraph P:

$$E(P) = \{\overrightarrow{p_0p_0}, \overrightarrow{p_0p_i}, \overrightarrow{p_ix_1}, \overrightarrow{p_ix_2}\}$$

if the edges starting from p_i are $\overrightarrow{p_ix_1}$ and $\overrightarrow{p_ix_2}$.

Then:

$$\begin{aligned}C_{\Gamma_1 \cup_{p_i} \Gamma}(\alpha)(u_1, \dots, u_m) &= -\sum_{\ell_1 \dots \ell_{2n}} \left(\sum_{j,k} C^j_{kj} C^k_{\ell_{2i+1}\ell_{2i+2}} \right) \prod_{s \neq i} \partial_{\mathrm{End}(p_s)} \\ &\qquad (C^r_{\ell_{2s+1}\ell_{2s+2}} x_r) \prod_{k=1}^m \partial_{\mathrm{End}\,(q_k)} u_k \\ &= 0,\end{aligned}$$

since $\sum_{j,k} C^j_{kj} C^k_{\ell_{2i+1}\ell_{2i+2}} = 0$. ∎

We will frequently use the following corollary.

Corollary 3.3. (Normalized equivalence) *Each K-star product is equivalent to the Kontsevich star product $\star^K_\alpha$ through an operator T associated to a formal function F (or H) such that:*

$$F(X) = 1 + \sum_{n\geq 2} \sum_{|s|=n} a_{s_1\ldots s_\ell} Tr(adX)^{s_1} \ldots Tr(adX)^{s_\ell}.$$

Proof. Indeed, T_1 being a derivation of our star products, $e^{-a_1 T_1}$ is an automorphism and:

$$u \star_\alpha v = T^{-1}(Tu \star^K_\alpha Tv) = (T \circ e^{-a_1T_1})^{-1}(T \circ e^{-a_1T_1} u \star^K_\alpha T \circ e^{-a_1T_1} v)$$

and $T \circ e^{-a_1T_1}$ is a formal series without any order 1 term. ∎

4. Invariance and Covariance

We recall that all K-star products are equivalent to the Gutt star product $\star^G_\alpha$ [12] associated to the complete symmetrization mapping between the space of polynomial functions $S(\mathfrak{g})$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

$$(u \star^G_\alpha v)(\xi) = \int_{\mathfrak{g}^2} \hat{u}(X)\hat{v}(Y) e^{2i\pi\langle \xi, X\times_\alpha Y\rangle}\, dXdY.$$

This equivalence being through an intertwining operator T of the form:

$$(Tu)(\xi) = \int_{\mathfrak{g}} \hat{u}(X) F(X) e^{2i\pi\langle \xi, X\rangle}\, dX.$$

This operator is also composed of ‘wheel’ operators as in the previous section.

Gutt proved in [12] that $\star^G_\alpha$ is invariant under the coadjoint action. The same is true for the Kontsevich star product $\star^K_\alpha$ (see [13]). Let us first recall the definition of the invariance under the coadjoint action Ad^*.

Definition 4.1. (Invariant star product) A star product $\star$ is said to be *invariant* under the coadjoint action if, for all X in $\mathfrak{g}$:

$$X^-(u \star v) = (X^- u) \star v + u \star (X^- v)$$

where:

$$(X^- u)(\xi) = \frac{d}{dt}|_{t=0} u(Ad^*(\exp\ tX)\xi).$$

Now using the equivalence operator T, we show that all K-star products are invariant under the coadjoint action.

Theorem 4.2. (Invariance of K-star products) *All the K-star products are invariant under the coadjoint action.*

Proof. S. Gutt showed in [12] that:

$$\frac{1}{2}(X \star_\alpha^G u - u \star_\alpha^G X) = \{X, u\} = X^- u$$

for each element X of $\mathfrak{g}$ and every polynomial function u. This implies invariance for $\star_\alpha^G$. Now since $TX = X + T_1(X)$ and $T_1(X)$ is a constant, we obtain:

$$\begin{aligned}\{X, Tu\} = \{TX, Tu\} &= \frac{1}{2}(TX \star_\alpha^G Tu - Tu \star_\alpha^G TX)\\ &= \frac{1}{2}T(X \star_\alpha u - u \star_\alpha X).\end{aligned}$$

Thus to show the invariance of $\star_\alpha$ it is sufficient to prove $X^- Tu = TX^- u$ or, for all 'wheel' operators T_n,

$$X^-(T_n u) = T_n(X^- u).$$

But a direct computation gives:

$$X^- u(\xi) = \int_{\mathfrak{g}} ((\mathcal{L}_X \hat{u})(Z) - Tr(ad\ X)\hat{u}(Z))e^{2i\pi\langle \xi, Z\rangle}\, dZ$$

where

$$(\mathcal{L}_X \varphi)(Y) = \frac{d}{dt}|_{t=0}\varphi(Ad(e^{-tX})(Y)).$$

Then:

$$(T_n(X^- u))(\xi) = \int_{\mathfrak{g}} (\mathcal{L}_X \hat{u}(Z) Tr(ad\ Z)^n - Tr(ad\ X)Tr(ad\ Z)^n \hat{u}(Z))e^{2i\pi\langle \xi, Z\rangle}\, dZ.$$

Now let us compute $X^-(T_n u)$:

$$X^-(T_n u)(\xi) = \int_{\mathfrak{g}} (\mathcal{L}_X(\hat{u}(Z) Tr(ad\ Z)^n) - Tr(ad\ X)Tr(ad\ Z)^n \hat{u}(Z))\, e^{2i\pi\langle \xi, Z\rangle}\, dZ.$$

But we have:

$$(\mathcal{L}_X [\hat{u} Tr(ad.)^n])(Z) = (\mathcal{L}_X \hat{u})(Z) Tr(ad\ Z)^n + \hat{u}(Z)(\mathcal{L}_X Tr(ad.)^n)(Z)$$

and

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} Tr\left(ad(Ad\ e^{-tX}(Z))^n\right) &= nTr((ad\ Z)^{n-1} ad([Z, X]))\\ &= nTr(ad\ Z^n ad\ X - ad\ Z^{n-1} ad\ X ad\ Z)\\ &= 0.\end{aligned}$$

Then

$$\int_{\mathfrak{g}} \hat{u}(Z)\,(\mathcal{L}_X Tr(ad.)^n)\,(Z)\;dZ = 0.$$

And finally:

$$\begin{aligned} X^-(T_n u)(\xi) &= \int_{\mathfrak{g}} ((\mathcal{L}_X \hat{u})(Z) Tr(ad\ Z)^n - Tr(ad\ X) Tr(ad\ Z)^n \hat{u}(Z))\, e^{2i\pi\langle \xi, Z\rangle} dZ \\ &= T_n(X^- u)(\xi). \end{aligned}$$

■

Now let us recall the definition of a covariant star product.

Definition 4.3. (Covariant star product) A star product on $S(\mathfrak{g})$ is called *covariant* if, for all X and Y in $\mathfrak{g}$,

$$\frac{1}{2}(X \star Y - Y \star X) = \{X, Y\} = [X, Y].$$

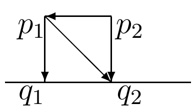
In the following theorem we shall show that each K-star product is covariant by considering all the possible graphs.

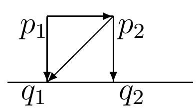
Theorem 4.4. (Covariance of K-star products) *Each K-star product $\star_\alpha$ is covariant and so is strongly invariant in the sense of [6].*

Proof. Since a K-star product $\star_\alpha$ is graded, in order to show its covariance, it is sufficient to prove that for all α,

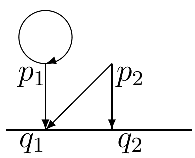
$$C_2(\alpha)(X, Y) = C_2(\alpha)(Y, X) \qquad \text{for all } X \text{ and } Y \text{ in } \mathfrak{g}.$$

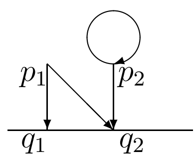
But the coefficients $C_2(\alpha)(u, v)$ for any u, v in $S(\mathfrak{g})$ are given by graphs having two vertices of first kind p_1, p_2 and two vertices of second kind q_1, q_2. We remark that there are, up to the ordering of the vertices p_i, only eight graphs of the above type non vanishing on constants. These graphs are respectively:



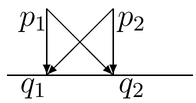


$E(\Gamma_1) = \{\overrightarrow{p_1q_1}, \overrightarrow{p_1q_2}, \overrightarrow{p_2p_1}, \overrightarrow{p_2q_2}\}$, $\qquad E(\Gamma_2) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1q_1}, \overrightarrow{p_2q_1}, \overrightarrow{p_2q_2}\}$,

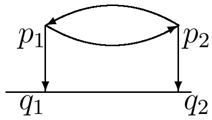




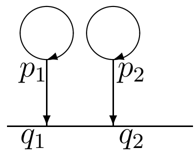
$E(\Gamma_3) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_1}, \overrightarrow{p_2q_1}, \overrightarrow{p_2q_2}\}$, $\qquad E(\Gamma_4) = \{\overrightarrow{p_1q_1}, \overrightarrow{p_1q_2}, \overrightarrow{p_2p_2}, \overrightarrow{p_2q_2}\}$,



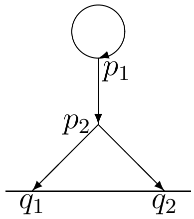
$$E(\Gamma_5) = \{\overrightarrow{p_1q_1}, \overrightarrow{p_1q_2}, \overrightarrow{p_2q_1}, \overrightarrow{p_2q_2}\},$$



$$E(\Gamma_6) = \{\overrightarrow{p_1p_2}, \overrightarrow{p_1q_1}, \overrightarrow{p_2p_1}, \overrightarrow{p_2q_2}\},$$



$$E(\Gamma_7) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1q_1}, \overrightarrow{p_2p_2}, \overrightarrow{p_2q_2}\},$$



$$E(\Gamma_8) = \{\overrightarrow{p_1p_1}, \overrightarrow{p_1p_2}, \overrightarrow{p_2q_1}, \overrightarrow{p_2q_2}\}.$$

Among these graphs, only the last three are possibly non vanishing in the computation of $C_2(\alpha)(X,Y)$ and $C_2(\alpha)(Y,X)$ since X and Y are linear functions. The last graph corresponds to the operator P of proposition 2-1 which is null. The two other graphs give symmetric $C_\Gamma(\alpha)$ and thus $C_2(\alpha)(X,Y) = C_2(\alpha)(Y,X)$. ■

5. Symmetry and reality

The star products defined by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer [8] had symmetry properties. Thus we put:

Definition 5.1. (Symmetric star products) A star product on a Poisson manifold M is said to be *symmetric* if C_r is symmetric (resp skew symmetric) if r is even (resp odd) that is:

$$C_r(u,v) = (-1)^r C_r(v,u) \quad \text{for all } u,\ v \in C^\infty(M).$$

For instance, the Gutt star product $\star_\alpha^G$ is symmetric for any α and the star product built by Kontsevich $\star_\alpha^K$ is also symmetric for any α. In fact this is a consequence of the action of the symmetry $z \mapsto -\overline{z}$ on the graphs and the orientation of the configuration space. But in the linear case, it is also a consequence of the parity of the function J since we have:

Proposition 5.2. (Symmetric K-star products) *Let $(\star_\alpha)$ be a K-family given by a function F then each $\star_\alpha$ is symmetric if and only if:*

$$F(X) = e^{2i\pi a_1 Tr(ad\ X)} G(X)$$

where G is an even function, or:

$$F(X) = e^{2i\pi a_1 Tr(ad\ X)} \left(1 + \sum_{n=1}^{\infty} \sum_{\substack{s_1 \dots s_p \\ |s|=2n}} b_{s_1\dots s_p} Tr(2i\pi\ ad\ X)^{s_1} \dots Tr(2i\pi\ ad\ X)^{s_p} \right).$$

Moreover, $\star_\alpha$ is thus associated to the even function G itself.

Proof. First the Gutt star product $\star_\alpha^G$ is symmetric, since, for the Campbell-Hausdorff formula,

$$(-X)\times_\alpha(-Y) = -(Y\times_\alpha X).$$

Thus, if $CH_r(X,Y)$ is the degree r term in this formula:

$$X\times_\alpha Y = \sum_r CH_r(X,Y) \quad \text{with} \quad CH_r(Y,X) = (-1)^{r+1}CH_r(X,Y).$$

The symmetry of $\star_\alpha^G$ is a consequence of this ([7]). Now let $(\star_\alpha)$ be a K-family characterized by:

$$\begin{aligned}
F(X) &= 1+\sum_{n=1}^{\infty}(2i\pi)^n \sum_{\substack{s_1\dots s_p\\|s|=n}} a_{s_1\dots s_p} Tr(ad\ X)^{s_1}\dots Tr(ad\ X)^{s_p}\\
&= e^{2i\pi\ a_1 Tr(ad\ X)}\left(1+\sum_{n=2}^{\infty}(2i\pi)^n \sum_{\substack{s_1\dots s_p\\|s|=n}} b_{s_1\dots s_p} Tr(ad\ X)^{s_1}\dots Tr(ad\ X)^{s_p}\right)\\
&= e^{2i\pi a_1 Tr(ad\ X)}G(X).
\end{aligned}$$

Let us set:

$$u\star_\alpha v = \sum_{r=0}^{\infty} C_r(u,v) \quad \text{and} \quad u\star_\alpha^G v = \sum_{r=0}^{\infty} C_r^G(u,v).$$

$\star_\alpha$ is equivalent to $\star_\alpha^G$ through the equivalence operator:

$$\begin{aligned}
T &= e^{a_1T_1}\circ\left(Id+\sum_{n=2}^{\infty}\sum_{|s|=n} b_{s_1\dots s_p}T_{s_1}\circ\dots\circ T_{s_p}\right)\\
&= e^{a_1T_1}\circ T'.
\end{aligned}$$

Since T_1 is a derivation for any $\star_\alpha$, $e^{a_1T_1}$ is an automorphism and $\star_\alpha$ is equivalent to $\star_\alpha^G$ through the equivalence operator T'.

Let us put:

$$T' = Id+\sum_{m\geq 2} T'_m.$$

Now if G is not even and $\star_\alpha$ is symmetric for each α , let n_0 be the first odd n such that there exists $b_{s_1\dots s_p}\neq 0$ with $|s| = n_0$. We select the largest element $(s_1, s_2, \dots , s_{p_0})$ for the lexicographic ordering in the set:

$$\left\{(s_1,\dots,s_p),\ \text{such that } |s| = n_0 \text{ and } b_{s_1\dots s_p}\neq 0\right\}.$$

Then we choose $\mathfrak{g}$ as in the proof of lemma 2-3, then T'_{n_0} contains an unique term

$$b_{s_1\dots s_{p_0}}(\partial_2)^{s_1}\dots(\partial_{2p_0})^{s_{p_0}}.$$

We compute now the n_0 order term in $T'(u\star_\alpha v) = (T'u)\star_\alpha^G(T'v)$:

$$\sum_{r+s=n_0} T'_r C_s(\alpha)(u,v) = \sum_{m_1+m_2+m_3=n_0} C^G_{m_1}(\alpha)(T'_{m_2}u, T'_{m_3}v)$$

or

$$T'_{n_0}(uv)+\sum_{\substack{r+s=n_0\\ r<n_0}} T'_r C_s(\alpha)(u,v) = (T'_{n_0}u)v+u(T'_{n_0}v)+\sum_{\substack{m_1+m_2+m_3=n_0\\ m_2<n_0,m_3<n_0}} C^G_{m_1}(\alpha)(T'_{m_2}u,T'_{m_3}v).$$

In these expression s and m_1 are necessarily odd, thus we can separate the symmetric and the skew symmetric part and get:

$$T'_{n_0}(uv)-(T'_{n_0}u)v-u(T'_{n_0}v)\equiv 0$$

and this is impossible since T'_{n_0} contains an unique term $b_{s_1\dots s_{p_0}}(\partial_2)^{s_1}\dots(\partial_{2p_0})^{s_{p_0}}$ of order n_0 which is not a vector field. Then G is necessarily an even formal function.

Conversely if G is an even formal function, we have:

$$F(X)=e^{aTr(ad\ X)}G(X).$$

Thus $a=(2i\pi)a_1$ and

$$G(X)=1+\sum_{n=1}^{\infty}(2i\pi)^{2n}\sum_{|s|=2n} b_{s_1\dots s_p}Tr(ad\ X)^{s_1}\dots Tr(ad\ X)^{s_p}.$$

Then $\star_\alpha$ is equivalent to $\star^G_\alpha$ through T':

$$\begin{aligned} T' &= Id+\sum_{n=1}^{\infty}\sum_{|s|=2n} b_{s_1\dots s_p}T_{s_1}\circ\dots\circ T_{s_p}\\ &= Id+\sum_{m\geq 1}T'_{2m}. \end{aligned}$$

By induction, we suppose now:

$$C_s(\alpha)(u,v)=(-1)^sC_s(\alpha)(v,u)\quad\text{for any}\quad s<n$$

and compute

$$T'(u\star_\alpha v)=(T'u)\star^G_\alpha(T'v).$$

We get:

$$\begin{aligned} \sum_{r+s=n}T'_rC_s(\alpha)(u,v) &= \sum_{m_1+m_2+m_3=n} C^G_{m_1}(\alpha)(T'_{m_2}u,T'_{m_3}v)\\ &= C_n(\alpha)(u,v)+\sum_{\substack{r+s=n\\ s<n}}T'_rC_s(\alpha)(u,v). \end{aligned}$$

Thus:

$$\begin{aligned} C_n(\alpha)(v,u) &= \sum_{m_1+m_2+m_3=n} C^G_{m_1}(\alpha)(T'_{m_2}v,T'_{m_3}u)-\sum_{\substack{r+s=n\\ s<n}}T'_rC_s(\alpha)(v,u)\\ &= \sum_{m_1+m_2+m_3=n}(-1)^{m_1}C^G_{m_1}(\alpha)(T'_{m_3}u,T'_{m_2}v)-\sum_{\substack{r+s=n\\ s<n}}(-1)^sT'_rC_s(\alpha)(u,v) \end{aligned}$$

but the parities of m_1 and s coincide with the parity of n, then:

$$\begin{aligned} C_n(\alpha)(v,u) &= (-1)^n \left(\sum_{m_1+m_2+m_3=n} C^G_{m_1}(\alpha)(T'_{m_3}u, T'_{m_2}v) - \sum_{\substack{r+s=n \\ s<n}} T'_r C_s(\alpha)(u,v) \right) \\ &= (-1)^n C_n(\alpha)(u,v). \end{aligned}$$

And $\star_\alpha$ is symmetric. ■

The symmetry is a property related in some way to the reality of star products, this notion was considered by A. Lichnerowicz.

Definition 5.3. (Real star products) Let $(\star_\alpha)$ be a K-family. Then $(\star_\alpha)$ is said to be *real* if for any α and for any smooth functions φ_1 and φ_2 with sufficiently small supported Fourier transform $\hat{\varphi}_1$, $\hat{\varphi}_2$:

$$\overline{\varphi_1 \star_\alpha \varphi_2} = \overline{\varphi}_2 \star_\alpha \overline{\varphi}_1.$$

Each element $\star_\alpha$ of a real K-family $(\star_\alpha)$ is said to be a real K-star product.

Now accordingly to this definition we can show the following proposition.

Proposition 5.4. (Symmetric and real K-star products) *Let $(\star_\alpha)$ be a symmetric K-family associated to a function F. As the term $e^{a_1 Tr(adX)}$ does not play any role, we suppose F even. Then $(\star_\alpha)$ is real if and only if F is a real function.*

Proof. Let us compute:

$$\begin{aligned} (\overline{\varphi}_2 \star_\alpha \overline{\varphi}_1)(\xi) &= \int_{\mathfrak{g}^2} \widehat{\overline{\varphi}_2}(X)\widehat{\overline{\varphi}_1}(Y)\frac{F(X)F(Y)}{F(X \times_\alpha Y)} e^{2i\pi\langle \xi, X \times_\alpha Y\rangle}\, dXdY \\ &= \int_{\mathfrak{g}^2} \overline{\hat{\varphi}_2}(X)\overline{\hat{\varphi}_1}(Y)\frac{F(-X)F(-Y)}{F((-X) \times_\alpha (-Y))} e^{2i\pi\langle \xi, (-X) \times_\alpha (-Y)\rangle}\, dXdY \\ &= \int_{\mathfrak{g}^2} \overline{\hat{\varphi}_2}(X)\overline{\hat{\varphi}_1}(Y)\frac{F(X)F(Y)}{F(-(Y \times_\alpha X))} e^{2i\pi\langle \xi, -(Y \times_\alpha X)\rangle}\, dXdY \\ &= \int_{\mathfrak{g}^2} \overline{\hat{\varphi}_2}(X)\overline{\hat{\varphi}_1}(Y)\frac{F(X)F(Y)}{F(Y \times_\alpha X)} \overline{e^{2i\pi\langle \xi, (Y \times_\alpha X)\rangle}}\, dXdY \end{aligned}$$

and

$$\overline{\varphi_1 \star_\alpha \varphi_2}(\xi) = \overline{\int_{\mathfrak{g}^2} \hat{\varphi}_2(X)\hat{\varphi}_1(Y)\frac{F(X)F(Y)}{F(Y \times_\alpha X)} e^{2i\pi\langle \xi, (Y \times_\alpha X)\rangle}\, dXdY}$$

Thus $(\overline{\varphi}_2 \star_\alpha \overline{\varphi}_1)(\xi) = \overline{\varphi_1 \star_\alpha \varphi_2}(\xi)$ if and only if $\overline{F} = F$, if and only if

$$\frac{F(X)F(Y)}{F(Y \times_\alpha X)}$$

is real. If $Y = -X$, this becomes $F(X)^2$ real but then each coefficient in the formal series defining $F(X)$ should be real. Conversely, if F is real, $(\star_\alpha)$ is also real. ■

However this condition is no more true if $(\star_\alpha)$ is not symmetric.

Proposition 5.5. (Characterization of real K-family)
Let $(\star_\alpha)$ be a K-family (perhaps not symmetric) defined by a function F:

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{s_1 \dots s_p \\ |s|=n}} a_{s_1 \dots s_p} Tr(2i\pi \ ad \ X)^{s_1} \dots Tr(2i\pi \ ad \ X)^{s_p},$$

then the following are equivalent:
i) $(\star_\alpha)$ is real.
ii) $\overline{F(X)} = F(-X)$.
iii) $a_{s_1 \dots s_p}$ is real for all $s_1, \dots, s_p$.

Proof. It is not difficult to show that (i) $\Longrightarrow$ (iii). In fact, $(\star_\alpha)$ is real if and only if for any $\mathfrak{g}$ and for all φ_1, φ_2 smooth such that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are sufficiently small supported:

$$\begin{aligned}\int_{\mathfrak{g}^2} \overline{\hat{\varphi}_2}(X)\overline{\hat{\varphi}_1}(Y) \frac{F(-X)F(-Y)}{F(-(Y \times_\alpha X))} e^{2i\pi\langle \xi, -(Y \times_\alpha X)\rangle} \, dX dY = \\ = \int_{\mathfrak{g}^2} \overline{\hat{\varphi}_2(X)\hat{\varphi}_1(Y) \frac{F(X)F(Y)}{F(Y \times_\alpha X)} e^{2i\pi\langle \xi, (Y \times_\alpha X)\rangle} \, dX dY}\end{aligned}$$

but this equality happens if and only if

$$\overline{\frac{F(X)F(Y)}{F(Y \times_\alpha X)}} = \frac{F(-X)F(-Y)}{F(-Y \times_\alpha -X)}.$$

Taking $Y = -X$, we get $\overline{F(X)^2} = F(-X)^2$. This implies that the first perhaps not real coefficient $a_{s_1 \dots s_p}$ satisfies $2\overline{a_{s_1 \dots s_p}} = 2a_{s_1 \dots s_p}$, thus it is in fact real.

Now, (iii) implies (ii) and (ii) implies (i) are obvious since $F = F_e + F_o$ where:

$$F_e = 1 + \sum_{n=1}^{\infty} \sum_{|s|=2n} a_{s_1 \dots s_p} Tr(2i\pi \ ad \ X)^{s_1} \dots Tr(2i\pi \ ad \ X)^{s_p}$$

and

$$F_o(X) = \sum_{n=0}^{\infty} \sum_{|s|=2n+1} a_{s_1 \dots s_p} Tr(2i\pi \ ad \ X)^{s_1} \dots Tr(2i\pi \ ad \ X)^{s_p}.$$

Then $\overline{F(X)} = \overline{F_e(X)} + \overline{F_o(X)}$ and $F(-X) = F_e(X) - F_o(X)$. Thus $a_{s_1 \dots s_p}$ is real for all $s_1, \dots, s_p$ implies that $\overline{F(X)} = F(-X)$ and $(\star_\alpha)$ is real. ∎

6. Closed and relative star products

Definition 6.1. (Analytic star products) A K-family defined by a function $F(X) = J(X)e^{h(X)}$ is said to be *analytic* if the series

$$h(x) = \sum_{s_1 \dots s_p} a_{s_1 \dots s_p} Tr(2i\pi ad X)^{s_1} \dots Tr(2i\pi ad X)^{s_p}$$

has a strictly positive convergence radius: there exists $r > 0$ such that:

$$\sum_{s_1 \dots s_p} |a_{s_1 \dots s_p}| r^{s_1 + \dots + s_p} < \infty.$$

An element $\star_\alpha$ of an analytic K-family $(\star_\alpha)$ is said to be an analytic K-star product.

Accordingly to this definition we have the following lemma.

Lemma 6.2. (F is holomorphic) *Let $\star_\alpha$ be an analytic K-star product then F is holomorphic on a neighborhood of zero for any Lie algebra $\mathfrak{g}$.*

Proof. Let us fix a norm $\|.\|$ on $\mathfrak{g}$. As the function trace Tr from the space $L(\mathfrak{g})$ (of linear mapping from $\mathfrak{g}$ to $\mathfrak{g}$) to $\mathbb{C}$ is linear,then there exists a constant K such that:

$$|Tr(A)| \leq K\|A\| \quad \text{for all} \quad A \in L(\mathfrak{g}).$$

Then

$$|Tr(ad\ X)^{s_i}| \leq K\|(ad\ X)^{s_i}\| \leq K\|ad\ X\|^{s_i} \quad \forall s_i \in \mathbb{N}.$$

Thus we obtain:

$$\sum_{s_1 \dots s_p} \left| a_{s_1 \dots s_p} Tr(2i\pi\ ad\ X)^{s_1} Tr(2i\pi\ ad\ X)^{s_2} \dots Tr(2i\pi\ ad\ X)^{s_p} \right|$$
$$\leq \sum_{s_1 \dots s_p} |a_{s_1 \dots s_p}|\ K^p\ \|ad\ X\|^{s_1 + \dots + s_p} (2\pi)^{s_1 + \dots + s_p}.$$

If $K \leq 1$, then:

$$\sum_{s_1 \dots s_p} \left| a_{s_1 \dots s_p} Tr(2i\pi\ ad\ X)^{s_1} \dots Tr(2i\pi\ ad\ X)^{s_p} \right|$$

$$\leq \sum_{s_1 \dots s_p} |a_{s_1 \dots s_p}|\ \|ad\ X\|^{s_1 + \dots + s_p} (2\pi)^{s_1 + \dots + s_p}.$$

Now there exists $\rho > 0$ such that

$$\|ad\ X\| \leq \frac{r}{2\pi} \quad \text{if} \quad \|X\| < \rho,$$

thus F is holomorphic on the ball $B(0, \rho)$.

If $K > 1$, then $K^p \leq K^{s_1 + \dots + s_p}$. But one has

$$\|adX\| < \frac{r}{2\pi K} \quad \text{if} \quad \|X\| < \frac{\rho}{K}$$

and F is holomorphic on the ball $B(0, \frac{\rho}{K})$. ∎

Proposition 6.3. (An integral relation) *Let $\star_\alpha$ be an analytic K-star product associated to a function F and let φ_1, φ_2 be two smooth functions on $\mathfrak{g}^*$ such that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ have sufficiently small support then:*

$$\begin{aligned}\int_{\mathfrak{g}^*} (\varphi_1 \star_\alpha \varphi_2)(\xi)\ d\xi &= \int_{\mathfrak{g}^*} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)e^{-1/2TradX}\ dX \\ &= \int_{\mathfrak{g}^*} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)(\Delta(X))^{1/2}\ dX,\end{aligned}$$

where $H(X) = F(X)J^{-1}(X)$ and $\Delta(X)$ is the modular function for $\mathfrak{g}$.

In particular if $\mathfrak{g}$ is unimodular we get:

$$\int_{\mathfrak{g}^*} (\varphi_1 \star_\alpha \varphi_2)(\xi)\ d\xi = \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)\ dX.$$

Proof. We can write:

$$\begin{aligned}(\varphi_1 \star_\alpha \varphi_2)(\xi) &= \int_{\mathfrak{g}^2} \hat{\varphi}_1(X)\hat{\varphi}_2(Y)\frac{F(X)F(Y)}{F(X\times_\alpha Y)}e^{2i\pi\langle\xi, X\times_\alpha Y\rangle}\ dXdY \\ &= \int_{\mathfrak{g}^2} \hat{\varphi}_1(X)\hat{\varphi}_2((-X)\times_\alpha Z)\frac{F(X)F((-X)\times_\alpha Z)}{F(Z)} \\ &\qquad \times \omega((-X)\times_\alpha Z)^{-1}\omega(Z)e^{2i\pi\langle\xi,Z\rangle}\ dXdZ\end{aligned}$$

where

$$\omega(X) = det(d\ exp\ X) = det\left(\frac{1-e^{-ad\ X}}{ad\ X}\right).$$

Then:

$$\begin{aligned}\int_{\mathfrak{g}} (\varphi_1 \star_\alpha \varphi_2)(\xi)\ d\xi &= \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)F(X)F(-X)\omega(-X)^{-1}\ dX \\ &= \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)J^2(X)H(X)H(-X)J^{-2}(X)det\left(e^{-ad\ X/2}\right)\ dX \\ &= \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)e^{-Tr(ad\ X/2)}\ dX \\ &= \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)(\Delta(X))^{1/2}\ dX.\end{aligned}$$

■

A. Connes, M. Flato and D. Sternheimer studied closed star products in [9]. Let us recall their definition.

Definition 6.4. (Closed star products) A star product on $\mathfrak{g}^*$ is said to be *closed* if, for all smooth compactly supported φ_1, φ_2, one has:

$$\int_{\mathfrak{g}^*} (\varphi_1 \star \varphi_2)(\xi)\ d\xi = \int_{\mathfrak{g}^*} \varphi_1(\xi)\varphi_2(\xi)\ d\xi.$$

Proposition 6.5. ($\mathfrak{g}$ has to be unimodular) *Let $\mathfrak{g}$ be a not unimodular Lie algebra and let $\star_\alpha$ be an analytic K-star product on $\mathfrak{g}^*$ then $\star_\alpha$ is not closed.*

Proof. Suppose that $\star_\alpha$ is closed. We can write:

$$H(X) = e^{h(X)} \quad \text{where} \quad h(X) = \sum_{s_1 \dots s_p} a_{s_1 \dots s_p} Tr(2i\pi \ ad \ X)^{s_1} \dots Tr(2i\pi \ ad \ X)^{s_p}.$$

Since the function $e^{h(X)+h(-X)-1/2TradX}$ is holomorphic and equal to 1 on some neighborhood of zero then it is identically equal to 1 that is:

$$h(X) + h(-X) - \frac{1}{2} TradX = 0.$$

But one has:

$$h(X) + h(-X) = 2 \sum_{\substack{s_1 \dots s_p \\ |s|=2n}} a_{s_1 \dots s_p} Tr(2i\pi \ ad \ X)^{s_1} \dots Tr(2i\pi \ ad \ X)^{s_p}$$

is an even function. Thus we cannot obtain $h(X) + h(-X) - 1/2Tr(ad \ X) = 0$, since $Tr(ad \ X)$ is a non-everywhere vanishing odd function. ■

Accordingly to the above proposition, from now on, we shall define a closed K-star product as follows:

Definition 6.6. (Closed K-family) An analytic K-family $(\star_\alpha)$ will be called *closed* if and only if $\star_\alpha$ is closed for all unimodular Lie algebra $\mathfrak{g}$.

Now let us recall the definitions of relative star products and strict Kontsevich star products.

Definition 6.7. (Relative K-star-products) A K-family $(\star_\alpha)$ is said to be *relative* (to the algebra of invariant polynomial functions) if:

$$u_1 \star_\alpha u_2 = u_1.u_2$$

for every α and every invariant polynomial functions u_1, u_2 $(u_i \in S(\mathfrak{g})^G)$.

An element $\star_\alpha$ of a relative K-family $(\star_\alpha)$ is said to be a relative K-star product.

Definition 6.8. (Strict Kontsevich star products) ([3]) A K-family associated to a function F is said to be *strict* if there exists a function f holomorphic on a neighborhood of 0 such that:

$$f(0) = 1 \quad \text{and} \quad F(X) = det(f(ad \ X)).$$

An element $\star_\alpha$ of a strict K- family $(\star_\alpha)$ is said to be a strict K-star product.

An easy corollary of the result of [3] is:

Corollary 6.9. (Strict relative K-star products) *A K-star product $\star_\alpha$ is a strict relative star product if and only if:*

$$F(X) = J(X) e^{\sum_{s=0}^{\infty} a_{2s+1} Tr(ad \ 2i\pi \ X)^{2s+1}}.$$

Proposition 6.10. (The unimodular case) *Let $\star_\alpha$ be an analytic K-star product then:*

1) If $\star_\alpha$ is closed then $\star_\alpha$ is relative.

2) If $\star_\alpha$ is strict and relative then $\star_\alpha$ is closed.

Proof. Let $\mathfrak{g}$ be unimodular, thus $Tr(ad\ X) = 0$ and for any φ_1, φ_2 with sufficiently small supported Fourier transform we have:

$$\int_{\mathfrak{g}^*} (\varphi_1 \star_\alpha \varphi_2)(\xi)\ d\xi = \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)H(X)H(-X)\ dX.$$

Since $\star_\alpha$ is an analytic K-star product then we choose $H(X) = e^{h(X)}$ where h is an holomorphic function on a neighborhood of zero.

Suppose that $\star_\alpha$ is closed then $e^{h(X)+h(-X)}$ is an holomorphic function equal to 1 on the neighborhood of zero and so is identically equal to 1. This means that:

$$h(-X) = -h(X).$$

Thus $\star_\alpha$ is a relative star product.

Now suppose that $\star_\alpha$ is strict and relative then we choose

$$H(X) = e^{\sum_{s=0}^{\infty} a_{2s+1} Tr(ad\ 2i\pi\ X)^{2s+1}}$$

and so $H(X)H(-X) = 1$. Then we obtain, if $\times$ is the usual convolution of functions,

$$\begin{aligned} \int_{\mathfrak{g}^*} (\varphi_1 \star_\alpha \varphi_2)(\xi)\ d\xi &= \int_{\mathfrak{g}} \hat{\varphi}_1(X)\hat{\varphi}_2(-X)\ dX \\ &= (\hat{\varphi}_1 \times \hat{\varphi}_2)(0) = \int_{\mathfrak{g}^*} \varphi_1(\xi)\varphi_2(\xi)\ d\xi. \end{aligned}$$

This concludes the proof of the proposition. ■

Let us remark that the reality of a K-family $(\star_\alpha)$ implies that:

$$(\varphi_1, \varphi_2) \mapsto \int_{\mathfrak{g}^*} (\overline{\varphi_1} \star_\alpha \varphi_2)(\xi)\ d\xi$$

is a scalar product. This useful property is a consequence of the closeness of $\star_\alpha$ if $\mathfrak{g}$ is unimodular, but it is still holding for a general $\mathfrak{g}$. In our opinion, it is the good generalization of closeness for any $\mathfrak{g}$.

7. Characterization of $\star_\alpha^K$

Proposition 7.1. (Properties of $\star_\alpha^K$) *The star product $\star_\alpha^K$ built by Kontsevich is analytic strict relative, closed symmetric and real.*

Proof. First $\star_\alpha^K$ is clearly an analytic star product since $h(X) = 0$. We showed in [3] that $\star_\alpha^K$ is strict and relative. Since

$$J(X) = det\left(\frac{sh\ ad(X/2)}{ad(X/2)}\right)^{1/2}$$

is an even real function then $\star_\alpha^K$ is symmetric and real. Now by proposition 6-10 we conclude that $\star_\alpha^K$ is closed. ■

Our goal is now to characterize $(\star_\alpha^K)$ among all K-families.

Proposition 7.2. (Characterization of $(\star_\alpha^K)$) *If $(\star_\alpha)$ is a symmetric strict and closed K-family then $(\star_\alpha) = (\star_\alpha^K)$. Or, equivalently, the only K-family strict relative and symmetric is $(\star_\alpha^K)$.*

Proof. Let $(\star_\alpha)$ be a strict relative symmetric K-family defined by a function F then there exists a function f holomorphic on a neighborhood of 0 such that $f(0) = 1$ and

$$F(X) = det(f(ad\ X)).$$

Accordingly to the corollary 6-9, we can write:

$$F(X) = J(X)\ det\ e^{h(2i\pi\ ad\ X)} \quad \text{where} \quad h(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}$$

is holomorphic near 0.

Now, remark we can replace $h(x)$ by $h(x) - a_1 x$ since T_1 is a derivation. Thus we can suppose that:

$$h(x) = \sum_{n=1}^{\infty} a_{2n+1}x^{2n+1}.$$

Let us suppose $a_3 = a_5 = \ldots = a_{2n-1} = 0$. Thus, for each α, $\star_\alpha$ is equivalent to $\star_\alpha^K$ through an intertwining operator T given by:

$$T = Id + b_{2n+1}T_{2n+1} + \sum_{\ell \geq 2n+2} b_\ell T_\ell.$$

We have:

$$T(u \star_\alpha v) = Tu \star_\alpha^K Tv.$$

Now let us set:

$$u \star_\alpha v = \sum_{\ell \geq 0} C_\ell(u,v) \quad \text{and} \quad u \star_\alpha^K v = \sum_{\ell \geq 0} C_\ell^K(u,v).$$

Since $\star_\alpha$ and $\star_\alpha^K$ are symmetric, one has:

$$C_\ell(u,v) = (-1)^\ell C_\ell(v,u) \quad \text{and} \quad C_\ell^K(u,v) = (-1)^\ell C_\ell^K(v,u).$$

Thus we have:

$$C_{2n+1}(u,v) + b_{2n+1}T_{2n+1}(uv) = b_{2n+1}(T_{2n+1}u)v + b_{2n+1}u(T_{2n+1}v) + C_{2n+1}^K(u,v).$$

Using the antisymmetry of C_{2n+1} and C_{2n+1}^K, we obtain:

$$2b_{2n+1}(T_{2n+1}(uv) - (T_{2n+1}u)v - u(T_{2n+1}v)) = -2b_{2n+1}(\delta T_{2n+1})(u,v) = 0,$$

where δ is the Hochschild cobundary operator.But $\delta T_{2n+1} \neq 0$ for all $n \geq 1$ (just consider, as usual, $\mathbb{R}^2$ with $\alpha = x_1\partial_{x_1} \wedge \partial_{x_2}$) thus we can conclude

$$b_{2n+1} = 0 \quad \text{then} \quad a_{2n+1} = 0 \quad \forall n \geq 1 \quad \text{and} \quad (\star_\alpha) = (\star_\alpha^K).$$

■

Now since $(\star_\alpha^K)$ is graded and symmetric, it is entirely determined by the values of $X \star_\alpha u$ for X in $\mathfrak{g}$ and u in $S(\mathfrak{g})$.

Theorem 7.3. (Explicit formula for the Duflo-Kontsevich star product) *Let (E_i) be a basis of the Lie algebra $\mathfrak{g}$ and C_{ij}^k the structure constants for $\mathfrak{g}$. For any u in $S(\mathfrak{g})$ (or u with $\hat{u}$ smooth and compactly supported),*

$$(E_i \star_\alpha^K u)(\xi) = \sum_{k\geq 0} \frac{1}{(2i\pi)^k}\Bigg(\frac{B_k}{k!} C_{i_1 i}^{j_1} C_{i_2 j_1}^{j_2} \dots C_{i_k j_{k-1}}^{j_k} \xi_{j_k} \partial_{i_1\dots i_k} u$$
$$- \sum_{\ell=1}^{[k/2]} \frac{B_{k-2\ell}}{2(k-2\ell)!}\frac{B_{2\ell}}{(2\ell)!} C_{j_k j_1}^{j_{2\ell}} C_{i_2 j_2}^{j_1} \dots C_{i_{2\ell} j_{2\ell}}^{j_{2\ell-1}} C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}} \dots C_{i_k j_{k-1}}^{j_k} \partial_{i_2\dots i_k} u\Bigg).$$

here B_k is the k^{th} Bernouilli number;

Or with a 'deformation parameter' $\hbar$ like in [1] *and* [12]*:*

$$E_i \star_\hbar u(\xi) = \sum_{k\geq 0} 2^k \hbar^k \Bigg(\frac{B_k}{k!} C_{i_1 i}^{j_1} C_{i_2 j_1}^{j_2} \dots C_{i_k j_{k-1}}^{j_k} \xi_{j_k} \partial_{i_1\dots i_k} u$$
$$- \sum_{l=1}^{[k/2]} \frac{B_{k-2\ell}}{2(k-2\ell)!}\frac{B_{2\ell}}{(2\ell)!} C_{j_k j_1}^{j_{2\ell}} C_{i_2 j_2}^{j_1} \dots C_{i_{2\ell} j_{2\ell}}^{j_{2\ell-1}} C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}} \dots C_{i_k j_{k-1}}^{j_k} \partial_{i_2\dots i_k} u\Bigg)$$

Proof. Let ϕ be a smooth function such that $\hat{\phi}$ is smooth with sufficiently small support near 0, we compute $E_i \star_\alpha^K u$ as the Fourier transform of a distribution with $\{0\}$ support. If $X = \sum x_i E_i$,

$$\begin{aligned}\langle E_i \star_\alpha^K u, \phi\rangle &= \langle u(Y)J(Y), \langle E_i(X)J(X), \hat{\phi}(X\times_\alpha Y)J^{-1}(X\times_\alpha Y)\rangle\rangle \\ &= \langle \hat{u}(Y), \frac{1}{2i\pi}\frac{\partial}{\partial x_i}\left(\hat{\phi}(X\times_\alpha Y)\frac{J(X)J(Y)}{J(X\times_\alpha Y)}\right)\Big|_{X=0}\rangle\end{aligned}$$

The functions $(X,Y)\mapsto X\times_\alpha Y$ and $X\mapsto J(X)$ being holomorphic near 0, this expression holds if the support of $\hat{\phi}$ is sufficiently small. Moreover let us recall that :

$$\frac{\partial}{\partial x_i}(X\times_\alpha Y)|_{X=0} = \sum_{k\geq 0}\frac{B_k}{k!}(ad\ Y)^k E_i$$

(see [7] for instance) and that:

$$\frac{sh\ x/2}{x/2} = e^{\sum_{k\geq 1}\frac{B_{2k}}{2k(2k)!}x^{2k}}.$$

(see [16]). Thus:

$$J(X) = \prod_{k=1}^{\infty} e^{\frac{B_{2k}}{4k(2k)!}Tr(ad\ X)^{2k}} = \prod_{k=1}^{\infty} e^{\frac{B_{2k}}{4k(2k)!}\sum_{i_1\dots i_{2k}} x_{i_1}\dots x_{i_{2k}}\sum_{j_1\dots j_{2k}} C_{i_1 j_1}^{j_{2k}} C_{i_2 j_2}^{j_1}\dots C_{i_{2k} j_{2k}}^{j_{2k-1}}}.$$

Then $\frac{\partial}{\partial x_i}J(X)|_{X=0} = 0$ and

$$\frac{\partial}{\partial x_i}J(X\times_\alpha Y)|_{X=0} = J(Y)\sum_{k=2}^{\infty}\sum_{\ell=1}^{[k/2]}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{2(2\ell)!}\sum_{i_2\dots i_k} y_{i_2}\dots y_{i_k}$$
$$\sum_{j_1\dots j_k} C_{j_k j_1}^{j_{2\ell}} C_{i_2 j_2}^{j_1}\dots C_{i_{2\ell} j_{2\ell}}^{j_{2\ell-1}} C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}}\dots C_{i_k j_{k-1}}^{j_k}.$$

Or

$$\frac{\partial}{\partial x_i}\left(\frac{J(X)J(Y)}{J(X\times_\alpha Y)}\right)\Big|_{X=0} = -\sum_{k=2}^{\infty}\sum_{\ell=1}^{[k/2]}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{2(2\ell)!}\sum_{i_2\dots i_k} y_{i_2}\dots y_{i_k}$$
$$\sum_{j_1\cdots j_k} C^{j_{2\ell}}_{j_k j_1} C^{j_1}_{i_2 j_2}\dots C^{j_{2\ell-1}}_{i_{2\ell}j_{2\ell}} C^{j_{2\ell+1}}_{i_{2\ell+1}i} C^{j_{2\ell+2}}_{i_{2\ell+2}j_{2\ell+1}}\dots C^{j_k}_{i_k j_{k-1}}.$$

Thus we get:

$$E_i \star^K_\alpha u = \sum_{k\geq 0}\frac{1}{(2i\pi)^k}\Bigg(\sum_{\substack{i_2\cdots i_k\\ j_1\cdots j_k}}\frac{B_k}{k!}C^{j_1}_{i_1 i}C^{j_2}_{i_2 j_1}\dots C^{j_k}_{i_k j_{k-1}}E_{j_k}\partial_{i_1\dots i_k}u$$
$$-\sum_{\ell=1}^{[k/2]}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{2(2\ell)!}\sum_{\substack{i_2\cdots i_k\\ j_1\cdots j_k}} C^{j_{2\ell}}_{j_k j_1}C^{j_1}_{i_2 j_2}\dots C^{j_{2\ell-1}}_{i_{2\ell}j_{2\ell}}C^{j_{2\ell+1}}_{i_{2\ell+1}i}C^{j_{2\ell+2}}_{i_{2\ell+2}j_{2\ell+1}}\dots C^{j_k}_{i_k j_{k-1}}\partial_{i_2\dots i_k}u\Bigg).$$

This is a special case of the formula with deformation parameter $\hbar$, if we put $\hbar = \frac{1}{4i\pi}$. ■

Remark 7.4. 1) For each k, the only graphs occurring in the last expression are one graph Γ_k without any wheel and graphs $\Gamma_{k,2\ell}$, with $1\leq \ell \leq [k/2]$ with exactly one wheel of size 2ℓ.

Γ_k has k vertices of first kind $p_1, \dots, p_k$, 2 vertices of second kind q_1, q_2 and its edges are:

$$\{\overrightarrow{p_1q_1}, \overrightarrow{p_1q_2}, \overrightarrow{p_2p_1}, \overrightarrow{p_2q_2}, \dots, \overrightarrow{p_kp_{k-1}}, \overrightarrow{p_kq_2}\}.$$

$\Gamma_{k,2\ell}$ has the same vertices of Γ_k and its edges are:

$$\{\overrightarrow{p_1p_k}, \overrightarrow{p_1p_2}, \overrightarrow{p_2p_3}, \overrightarrow{p_2q_2}, \dots, \overrightarrow{p_{2\ell}p_1}, \overrightarrow{p_{2\ell}q_2}, \overrightarrow{p_{2\ell+1}q_1}, \overrightarrow{p_{2\ell+1}q_2}, \overrightarrow{p_{2\ell+2}p_{2\ell+1}}, \overrightarrow{p_{2\ell+2}q_2}, \dots,$$
$$\overrightarrow{p_kp_{k-1}}, \overrightarrow{p_kq_2}\}.$$

2) For each k, there is $k!2^k$ graphs having the same weight. These graphs are obtained from Γ_k or $\Gamma_{k,2\ell}$ $(1\leq\ell\leq[k/2])$ by using permutations of the vertices of first kind or the edges coming from these vertices.

Corollary 7.5. (Weights of graphs)
a) The weight of the graph Γ_k is $w_k = \frac{B_k}{(k!)^2}$ (see also [1] and [14]).
b) The weight of the graph $\Gamma_{k,2\ell}$ is:

$$w_{k,2\ell} = -\frac{1}{k!}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{2(2\ell)!} = -\frac{1}{\binom{k}{2\ell}}\frac{B_{k-2\ell}}{((k-2\ell)!)^2}\frac{B_{2\ell}}{2((2\ell)!)^2}$$
$$= -\frac{1}{\binom{k}{2\ell}}w_{k-2\ell}\frac{w_{2\ell}}{2}.$$

Especially, for the wheel graph $\Gamma_{2\ell,2\ell}$, we get:

$$w_{2\ell,2\ell} = -\frac{B_{2\ell}}{2((2\ell)!)^2} = -\frac{1}{2}w_{2\ell}.$$

c) The weight of any graph Γ having at least a 'parachute' i.e a wheel linked only with the point q_2 vanishes.

Remark 7.6. 1) Since the edges of $\Gamma_{2\ell,2\ell}$ are $\{\overrightarrow{p_1q_1}, \overrightarrow{p_1p_2}, \overrightarrow{p_2p_3}, \overrightarrow{p_2q_2}, \dots, \overrightarrow{p_{2\ell-1}p_{2\ell}}, \overrightarrow{p_{2\ell-1}q_2}, \overrightarrow{p_{2\ell}p_1}, \overrightarrow{p_{2\ell}q_2}\}$, then passing from $\Gamma_{2\ell,2\ell}$ to $\Gamma_{2\ell}$ consists only to move the edge $\overrightarrow{p_1p_2}$ to $\overrightarrow{p_1q_2}$. Thus probably the formula

$$w_{2\ell,2\ell} = -\frac{1}{2}w_{2\ell},$$

could be deduced directly from this operation.

2) It is easy to prove directly (see [4],[14]) that the only graphs Γ which can occur in $E_i \star_\alpha^K u$ are those of our corollary.

Moreover we see directly that the weight of a graph Γ having at least a parachute vanishes. Indeed, let W be a parachute inside Γ with vertices $p_1, \dots, p_\ell$, we set the parameters of the configuration space $C_{k,2}$ defined by Kontsevich ([13]) by putting $q_2 = 0$, $|p_1| = 1$:

$$C_{k,2} = \{(p_1, p_2, \dots, p_k, q_1) \quad \mathfrak{Im}(p_i) > 0, \quad p_i \neq p_j \quad \text{if} \quad i \neq j, \quad |p_1| = 1, \quad q_1 < 0\}.$$

Then $C_{k,2}$ is a dense subset in the product $U \times V$, where:

$$U = \{(p_1, p_2, \dots, p_\ell), \quad p_i \neq p_j \quad \text{if } i \neq j\} \subset \mathbb{T} \times \mathcal{H}^{\ell-1}$$

if $\mathcal{H}$ is the standard upper half-plane $\{z, \ \mathfrak{Im}(z) > 0\}$ and $\mathbb{T}$ the one dimensional torus. And

$$V = \{(p_{\ell+1}, \dots, p_k, q_1)\} \subset \mathcal{H}^{k-\ell} \times \mathbb{R}^-.$$

Thus the form ω_Γ associated by Kontsevich to the graph Γ can be written as:

$$\omega_\Gamma = \omega_W \wedge \omega_{\Gamma\backslash W},$$

ω_W depending only of the points $p_1, \dots, p_\ell$ and so ω_W can be viewed as a form on U. Now:

$$w_\Gamma = \int_{C_{k,2}} \omega_\Gamma = \int_U \omega_W \int_V \omega_{\Gamma\backslash W} = 0$$

since the dimension of U is $2\ell - 1$ and ω_W is an 2ℓ-form.

Corollary 7.7. (Rationality of weights) *Let u and v be two polynomial functions on $\mathfrak{g}^*$, then all the weights of graphs occurring in the expression of $u \star_\alpha v$ are rational numbers.*

Proof. We first remark that if X is in $\mathfrak{g}$ and u is in $S(\mathfrak{g})$ then an easy consequence of corollary 7.4 is that weights of all graphs which occur in the expression of $X \star_\alpha u$ are rational numbers.

Now we can show the corollary by induction on the degree of u. If u is an homogeneous polynomial function of the form $u = Xu'$ then there exists a polynomial function u'' such that:

$$Xu' = X \star_\alpha u' + u'' \quad \text{with degree of} \quad u'' \leq \text{ degree of } u - 1.$$

Then:

$$u \star_\alpha v = (Xu') \star_\alpha v = X \star_\alpha (u' \star_\alpha v) + u'' \star_\alpha v.$$

But the weights of graphs in $X \star_\alpha (u' \star_\alpha v)$ and in $u'' \star_\alpha v$ are all rational numbers. Thus, we conclude the proof of the corollary. ■

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References

[1] Arnal, D., *Le produit star de Kontsevich sur le dual d'une algèbre de Lie nilpotente*, C. R. Acad. Sc. **327** (1998), 823–826.

[2] Arnal, D. and N. Ben Amar, *Kontsevich's wheels and Invariant Polynomial functions on the dual of Lie algebras*, Lett. Math. Phys. **52** (2000), 291–300.

[3] —, *$\mathfrak{g}$-relative star products*, Lett. Math. Phys. **55** (2001), 33–42.

[4] Arnal, D., N. Ben Amar and M. Masmoudi, *Cohomology of good graphs and Kontsevich linear star products*, Lett. Math. Phys. **48** (1999), 291–306.

[5] Arnal, D., M. Cahen and S. Gutt, *Deformation on coadjoint orbits*, J. Geom. Phys. **3** (1986), 327–351.

[6] Arnal, D., J. C. Cortet, P. Molin, and G. Pinczon, *Covariance and geometrical invariance in $\star$ quantization*, J. Math. Phys. **24** (1983), 276–283.

[7] Ben Amar, N., *Tangential deformations on the dual of nilpotent special Lie algebras*, Pacific J. of Math. **170**, 2 (1995), 297–318.

[8] Bayen, F., M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization I, II*, Ann. of Phys. **111** (1978), 61–151.

[9] Connes, A., M. Flato, and D. Sternheimer, *Closed star products and cyclic cohomology*, Lett. Math. Phys. **24** (1992), 1–12.

[10] Dito, G., *Kontsevich star product on the dual of a Lie Algebra*, Lett. Math. Phys **48** (1999), 307–322.

[11] Felder, G., and B. Shoikhet, *Deformation quantization with traces*, QA/0002057 (2000).

[12] Gutt, S., *An explicit $\star$-product on the cotangent bundle of a Lie group*, Lett. Math. Phys. **7** (1983), 249–259.

[13] Kontsevich, M., *Deformation quantization of Poisson manifolds I.*, QA/9709040 (1997).

[14] Kathotia, V., *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula, I*, QA/9811174.

[15] Lichnerowicz, A., *Variété de Poisson et feuilletage*, Ann. Fac. Sci. Toulouse **4** (1982) 15–262.

[16] Magnus, W., F. Oberhettinger, and R. P. Soni, "Formulas and theorems for the special functions of mathematical physics," Springer-Verlag Berlin, Heidelberg, New York, 1966.

[17] Shoikhet, B., *Vanishing of the Kontsevich integral of the wheels*, QA/0007080 (2000).

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