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Root Systems Extended by an Abelian Group and their Lie Algebras

Yoji Yoshii *

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Abstract. We introduce the notion of a root system extended by an abelian group G. This concept generalizes extended affine root systems. We classify them in terms of (translated) reflection spaces of G. Then we see that division (Δ, G)-graded Lie algebras have such root systems. Finally, division (B_l, G)-graded Lie algebras and as a special case, Lie G-tori of type B_l, are classified for $l \geq 3$.
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1. Introduction

We have two purposes in the paper. The first one is to introduce the notion of a root system extended by an abelian group G. This concept generalizes the extended affine root systems in the sense of both [A-P] and [Sa]. More precisely, the root systems extended by $\mathbb{Z}^n$ correspond to Saito's systems in [Sa] and the 'reduced' root systems extended by $\mathbb{Z}^n$ correspond to the systems in [A-P]. The classification of such root systems is essentially the same as in [A-P]. But the lattice in a vector space can be any abelian group in their classification, and it seems that our definition and classification are simpler and easier to understand. Moreover, it turns out that the set of certain support sets of a division (Δ, G)-graded Lie algebra becomes a root system extended by G. Thus the purpose here is to acquaint people with a nice and natural class of Lie algebras and their root systems.

The second purpose is to classify division (B_l, G)-graded Lie algebras for the root system B_l. Division (Δ, G)-graded Lie algebras generalize the core of extended affine Lie algebras (EALAs) when $G = \mathbb{Z}^n$ [Y4] and the finite-dimensional isotropic simple Lie algebras when G is trivial (see Example 4.3(3)). Those Lie algebras were classified for $\Delta = \mathrm{A}_l$ $(l \geq 3)$, D_l, E_l in [Y2], and A_2 when $G = \mathbb{Z}^n$ in [Y3] (see Example 4.3(3)). So the cases of type A_1, B_l, C_l, F_4,

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G_2, and BC_l were open. Using the recognition theorem of B_l-graded Lie algebras by Benkart and Zelmanov [BZ], and the approach in [AG] to the classification of the cores of EALAs of type B_l, we have classified division (Δ, G)-graded Lie algebras in the case of type B_l for $l \geq 3$. For type B_2, it is more difficult to classify.

As a special case of division (Δ, G)-graded Lie algebras, we define Lie G-tori. This notion is more concrete and there is more hope to obtain a complete classification. Lie G-tori again generalize the core of EALAs when $G = \mathbb{Z}^n$. We state the classification of Lie G-tori of type B_l for $l \geq 3$, using the notion of a Clifford G-torus (which appears in several papers [A-P], [AG], [T], [Y1] and [NY]), as a corollary of the result above.

The organization of the paper is as follows. In §2 we define a root system extended by an abelian group, and in §3 they are classified. In §4 a division (Δ, G)-graded Lie algebra and a Lie G-torus are defined, and in §5 we show that these Lie algebras have root systems extended by G. In §6 we obtain the classification of division (B_l, G)-graded Lie algebras for $l \geq 3$. In the final section we specialize the result in §6 to the case of Lie G-tori.

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2. Definition of a root system extended by an abelian group

Let Δ be a finite irreducible root system, i.e., $\Delta = A_l$ $(l \geq 1)$, B_l $(l \geq 1$, $B_1 = A_1)$, C_l $(l \geq 2$, $C_2 = B_2)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4, G_2 or BC_l $(l \geq 1)$. Let

$$\Delta^{\text{red}} := \begin{cases} \Delta & \text{if } \Delta \text{ is reduced} \\ \{\mu \in \Delta \mid \mu \text{ is reduced, i.e., } \frac{1}{2}\mu \notin \Delta\} & \text{otherwise, i.e., if } \Delta = BC_l. \end{cases}$$

Note that $\Delta^{\text{red}} = B_l$ if $\Delta = BC_l$. Recall the Weyl group W acting on Δ, i.e.,

$$W = \langle \sigma_\mu \mid \mu \in \Delta \rangle,$$

where

$$\sigma_\mu(\nu) = \nu - < \nu, \mu > \mu$$

is the reflection on Δ with Cartan integer $< \nu, \mu >$ for $\nu \in \Delta$. Let G be an abelian group and

$$R = \{S_\mu\}_{\mu \in \Delta}$$

a family of nonempty subsets S_μ of G indexed by $\mu \in \Delta$. We define a subset $S_\mu \cdot S_\nu$ of G by

$$S_\mu \cdot S_\nu : = S_\nu - < \nu, \mu > S_\mu = \{s - < \nu, \mu > s' \mid s \in S_\nu, s' \in S_\mu\}.$$

Then R is called a *root system of type* Δ *extended by* G if it satisfies the following 3 axioms:

(R1) $\bigcup_{\mu\in\Delta} S_\mu$ generates G;

(R2) $0 \in S_\mu$ for all $\mu \in \Delta^{\mathrm{red}}$;

(R3) $S_\mu \cdot S_\nu \subset S_{\sigma_\mu(\nu)}$ for all $\mu, \nu \in \Delta$.

Furthermore, when Δ is nonreduced,

(R4) $\mu, 2\mu \in \Delta \Rightarrow S_{2\mu} \cap 2S_\mu = \emptyset$, then R is called *reduced.*

Remark 2.1. If G is trivial, i.e., $G = \{0\}$, then the root system $\{S_\mu\}_{\mu\in\Delta}$ extended by $\{0\}$ is given by the sets $S_\mu = \{0\}$ for all $\mu \in \Delta$, which can be viewed as Δ itself. If Δ is nonreduced, there does not exist a reduced root system extended by $\{0\}$ since (R4) cannot be satisfied.

Remark 2.2. When $G = \mathbb{Z}^n$, the root system $\{S_\mu\}_{\mu\in\Delta}$ naturally corresponds to an extended affine root system (see the last part of §3), and each S_μ is an isotropic part of it.

3. Classification of root systems extended by G

Let $R = \{S_\mu\}_{\mu\in\Delta}$ be a root system extended by G. By (R2) and (R3), we have

$$S_\nu \subset S_\mu \cdot S_\nu \subset S_{\sigma_\mu(\nu)}$$

for all $\mu \in \Delta^{\mathrm{red}}$ and $\nu \in \Delta$. Since W is generated by σ_μ for $\mu \in \Delta^{\mathrm{red}}$,

$$S_{w(\nu)} = S_\nu \quad \text{for all } w \in W \text{ and } \nu \in \Delta. \tag{1}$$

In particular, for $\mu, \nu \in \Delta$,

$$S_\mu = S_\nu \quad \text{if } \mu \text{ and } \nu \text{ have the same length,} \tag{2}$$

and $S_{-\mu} = S_\mu$ for all $\mu \in \Delta$. Also, by (R3) and (1), we have

$$S_\nu - <\nu, \mu> S_\mu \subset S_\nu \quad \text{for all } \mu, \nu \in \Delta. \tag{3}$$

In particular, taking $\nu = \mu$,

$$S_\mu - 2S_\mu \subset S_\mu \quad \text{for all } \mu \in \Delta, \tag{4}$$

which implies $-S_\mu = S_\mu$ for all $\mu \in \Delta$.

We next partition the root system Δ according to length. Roots of Δ of minimal length are called *short.* Roots of Δ which are two times a short root of Δ are called *extra long.* Finally, roots of Δ which are neither short nor extra long are are called *long.* We denote the roots of short, long and extra long roots of Δ by Δ_{sh}, Δ_{lg} and Δ_{ex} respectively. Thus

$$\Delta = \Delta_{sh} \sqcup \Delta_{lg} \sqcup \Delta_{ex}.$$

Of course the last two terms in this union may be empty. Indeed,

$$\Delta_{lg} = \emptyset \quad \Longleftrightarrow \quad \Delta \text{ has simply laced type or type } \mathrm{BC}_1,$$

and

$$\Delta_{ex} = \emptyset \quad \Longleftrightarrow \quad \Delta = \Delta^{\mathrm{red}}.$$

In view of (2), we can simplify the notation. We define

$$\begin{aligned} S &= S_\mu \quad \text{for } \mu \in \Delta_{sh}, \\ L &= S_\mu \quad \text{for } \mu \in \Delta_{lg}, \text{ provided that } \Delta_{lg} \neq \emptyset, \text{ and} \\ E &= S_\mu \quad \text{for } \mu \in \Delta_{ex}, \text{ provided that } \Delta_{ex} \neq \emptyset. \end{aligned}$$

Thus our root system $R = \{S_\mu\}_{\mu \in \Delta}$ can be written as

$$R = R(S, L, E)_\Delta.$$

Definition 3.1. Let G be an abelian group.

(i) A subset E of G is called a *translated reflection space* if $E - 2E \subset E$.

(ii) A translated reflection space E of G is called *full* if E generates G.

(iii) A translated reflection space E of G is called a *reflection space* if $0 \in E$.

Remark 3.2. (1) The notion of a reflection space was introduced in [L], which is a more general concept than ours. If $G = \mathbb{Z}^n$, it was called a semilattice in [A-P].

(2) If S is a reflection space of G, then so is $S + S$. In fact, we have $0 \in S \subset S + S$, and $S + S - 2(S + S) = (S - 2S) + (S - 2S) \subset S + S$.

By (R2) and (4), S and L in our root system R are reflection spaces of G, and E is a translated reflection space of G. If R is reduced, i.e., assuming (R4), then

$$E \cap 2S = \emptyset,$$

and in particular, $0 \notin E$.

We will see that L and E are always contained in S. Moreover, we can prove certain properties of S, L and E, depending on the type of Δ. If $\Delta_{lg} \neq \emptyset$, we use the notation k for the ratio of the long square root length to the short square root length in Δ. Hence,

$$k = \begin{cases} 2 & \text{if } \Delta \text{ has type } \mathrm{B}_l, \mathrm{C}_l, \mathrm{F}_4 \text{ or } \mathrm{BC}_l \text{ for } l \geq 2. \\ 3 & \text{if } \Delta \text{ has type } \mathrm{G}_2. \end{cases}$$

Proposition 3.3. *Let $R = R(S, L, E)_\Delta$ be a root system extended by G.*

(a) *If $\Delta_{lg} \neq \emptyset$, then*

$$L + kS \subset L \quad \textit{and} \quad S + L \subset S.$$

(b) *If $\Delta_{ex} \neq \emptyset$, i.e., Δ has type BC_l, then*

$$E + 4S \subset E \quad \textit{and} \quad S + E \subset S.$$

Moreover, if $\Delta_{lg} \neq \emptyset$, *i.e., if* $l \geq 2$, *then*

$$E + 2L \subset E \quad \textit{and} \quad L + E \subset L.$$

(c) $E \subset L \subset S$, *and* S *is a full reflection space of* G. *(*S *is the biggest!)*

(d) *If* Δ *does not have type* A_1, B_l *or* BC_l, *then* $S = G$.

(e) *If* Δ *has* B_l, F_4, G_2 *or* BC_l *for* $l \geq 3$, *then* L *is a subgroup of* G.

Proof. If the Dynkin diagram of Δ^{red} contains a subgraph $\circ = \circ$ or $\circ \equiv \circ$, then there exist $\mu, \nu \in \Delta$ with μ short such that $< \mu, \nu >= -2$ and $< \nu, \mu >= -1$, or $< \mu, \nu >= -3$ and $< \nu, \mu >= -1$, respectively. So (a) follows from (3).

If Δ has type BC_l, then there exists $\mu \in \Delta$ such that $-2\mu \in \Delta$, and so $< -2\mu, \mu >= -4$ and $< \mu, -2\mu >= -1$. So the first assertion of (b) follows from (3). The second assertion also follows from (3) since there exist $\mu \in \Delta_{lg}$ and $\nu \in \Delta_{ex}$ such that $< \mu, \nu >= -2$ and $< \mu, \nu >= -1$.

The fourth inclusion of (b) and the second inclusion of (a) show that $E \subset L \subset S$ since $0 \in L$ and $0 \in S$. Hence, by (R1), we obtain (c).

If the Dynkin diagram of Δ^{red} contains a subgraph $\circ - \circ$, then there exist $\mu, \nu \in \Delta$ of the same length such that $< \mu, \nu >= -1$. In this case, by (3), we get $S + S \subset S$ if μ and ν are short, and $L + L \subset L$ if μ and ν are long. If Δ does not have type A_1, B_l, BC_l or G_2, you can find a subgraph $\circ - \circ$ in short roots, and so S becomes a subgroup of G. Hence $S = G$ since S generates G. Similarly, for the types in (e) except type G_2, you can find a subgraph $\circ - \circ$ in long roots, and so L becomes a subgroup of G. So (d) and (e) hold except type G_2. But if Δ has type G_2, both Δ_{sh} and Δ_{lg} have type A_2 whose Dynkin diagram is $\circ - \circ$. Hence (d) and (e) also hold for type G_2. ∎

Theorem 3.4. *Let* $\{S_\mu\}_{\mu \in \Delta}$ *be a root (resp. a reduced root) system extended by* G *of type* Δ. *Then* $S_\mu = S$ *for all* $\mu \in \Delta_{sh}$, $S_\mu = L$ *for all* $\mu \in \Delta_{lg}$ *and* $S_\mu = E$ *for all* $\mu \in \Delta_{ex}$, *where* S *is a full reflection space,* L *is a reflection space and* E *is a translated reflection space satisfying*

$$\begin{aligned}
&L + kS \subset L \quad \textit{and} \quad S + L \subset S,\\
&E + 4S \subset E \quad \textit{and} \quad S + E \subset S\\
&E + 2L \subset E \quad \textit{and} \quad L + E \subset L\\
&S = G \ \textit{if}\ \Delta \neq A_1,\ B_l,\ BC_l,\\
&L \textit{ is a subgroup if } \Delta = B_l\ (l \geq 3),\ F_4,\ G_2,\ BC_l\ (l \geq 3)\\
&(\textit{resp. and furthermore,} \quad E \cap 2S = \emptyset).
\end{aligned}$$

Conversely, let S, L *and* E *be as above, and define* $S_\mu = S$ *for all* $\mu \in \Delta_{sh}$, $S_\mu = L$ *for all* $\mu \in \Delta_{lg}$ *and* $S_\mu = E$ *for all* $\mu \in \Delta_{ex}$. *Then* $R(S, L, E)_\Delta$ *is a root (resp. a reduced root) system extended by* G.

Proof. We only need to show the second statement. So we show that the set $R(S, L, E)_\Delta$ is a (reduced) root system. Thus, we must check the axioms (R1)-(R4). But all except (R3) are clear. Considering the possible Cartan integers $< \mu, \nu >$ in each type Δ, (R3) also follows from the relations among S, L and E. ∎

Example 3.5. (1) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. By Theorem 3.4, a root system extended by G of type B_l for $l \geq 3$ is determined by a subgroup L and a full reflection space S of G satisfying $S + L \subset S$ and $L + 2S \subset L$. Note that the second inclusion is equivalent to $2S \subset L$ since L is a subgroup. This is not the case for type B_2. Let $L = 2G$ and

$$S = L \cup (L + (1,0)) \cup (L + (0,1)).$$

Then $R(S, L)_{\mathrm{B}_l}$ is an example of a root system extended by G.

(2) Let R be a root system extended by $\mathbb{Q}$ of type B_l for $l \geq 3$. Then since $\mathbb{Q} = 2\mathbb{Q} = 2\langle S\rangle \subset L$, where $\langle S\rangle$ is the subgroup of $\mathbb{Q}$ generated by S, we have $L = S = \mathbb{Q}$. Hence $R = R(\mathbb{Q}, \mathbb{Q})_{\mathrm{B}_l}$.

(3) Let R be a root system extended by $\mathbb{Z}$ of type B_l for $l \geq 2$. Then any reflection space of $\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Hence $R = R(\mathbb{Z}, \mathbb{Z})_{\mathrm{B}_l}$ or $R(\mathbb{Z}, 2\mathbb{Z})_{\mathrm{B}_l}$.

A reduced root system $\{S_\mu\}_{\mu\in\Delta}$ extended by $\mathbb{Z}^n$ can be considered as an extended affine root system (EARS): Let V_1 be a euclidean space with root system Δ, V_2 an n-dimensional real vector space and $V = V_1 \oplus V_2$ the vector space with positive semidefinite symmetric bilinear form $(\cdot,\cdot)$ which is the natural extension of the form on V_1 with radical V_2. Identify $\mathbb{Z}^n$ with the lattice spanned by a basis of V_2. Let

$$S_0 = S_\mu + S_\mu$$

for any $\mu \in \Delta_{sh}$ (see Remark 3.2(2)) and

$$R = \bigcup_{\mu\in\Delta\cup\{0\}} (\mu + S_\mu).$$

Then the descriptions of S_μ for $\mu \in \Delta \cup \{0\}$ in Theorem 3.4 are exactly the same as in the classification of EARS (see [A-P, 2.32 and 2.37]), and so (V, R) is an EARS and any EARS can be considered as a root system extended by $\mathbb{Z}^n$. In particular, a reduced root system extended by $\mathbb{Z}$ can be thought as an affine root system.

A root system $\{S_\mu\}_{\mu\in\Delta}$ extended by $\mathbb{Z}^n$ can be considered as a Saito's extended affine root system (SEARS) (see [Sa]): In the same setting as above, let

$$R_S = \bigcup_{\mu\in\Delta} (\mu + S_\mu).$$

Then by the same argument as Azam did (he showed a one-to-one correspondence between EARS and reduced SEARs in [A]), (V, R_S) is a SEARS, and any SEARS can be considered as a root system extended by $\mathbb{Z}^n$. We summarize these as a corollary.

Corollary 3.6. *We have the following one-to-one correspondences:*

$$\begin{array}{ccc} \{\textit{root systems extended by } \mathbb{Z}^n\} & \leftrightarrow & \{\textit{SEARS of nullity } n\} \\ \cup & & \cup \\ \{\textit{reduced root systems extended by } \mathbb{Z}^n\} & \leftrightarrow & \{\textit{reduced SEARS of nullity } n\} \\ & & \leftrightarrow \{\textit{EARS of nullity } n\} \end{array}$$

■

4. Division (Δ, G)-graded Lie algebras

Throughout the paper the base field F has characteristic 0. We first recall Δ-graded Lie algebras and division (Δ, G)-graded Lie algebras, and as a special case, we define Lie G-tori. Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra over F with a split Cartan subalgebra $\mathfrak{h}$ and the root system Δ^{red} so that $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\mu \in \Delta^{\mathrm{red}}} \ \mathfrak{g}_\mu \right)$ with $\mathfrak{h} = \mathfrak{g}_0$. A Δ-graded Lie algebra L over F with grading subalgebra $\mathfrak{g}$ or grading pair $(\mathfrak{g}, \mathfrak{h})$ is defined as

(i) L contains $\mathfrak{g}$ as a subalgebra;
(ii) $L = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$, where $L_\mu = \{x \in L \mid [h, x] = \mu(h)x \text{ for all } h \in \mathfrak{h}\}$; and
(iii) $L_0 = \sum_{\mu \in \Delta} [L_\mu, L_{-\mu}]$.

Note that the Jacobi identity implies $[L_\mu, L_\nu] \subset L_{\mu+\nu}$ for all $\mu, \nu \in \Delta \cup \{0\}$, defining $L_{\mu+\nu} = 0$ if $\mu + \nu \notin \Delta \cup \{0\}$. Also, the centre $Z = Z(L)$ of L is contained in L_0, and L/Z is again a Δ-graded Lie algebra with $Z(L/Z) = 0$. A Δ-graded Lie algebra having trivial centre is called *centreless*.

Remark 4.1. The Δ-graded Lie algebras for a reduced Δ were introduced by Berman and Moody [BM], and have been recently generalized for a nonreduced case by Allison, Benkart and Gao [ABG]. They classified more general BC_l-graded Lie algebras in [ABG] (and [BS] for $l = 1$) than our concept above. We only consider a special class of BC_l-Lie algebras in their sense. The reason comes from the theory of EALAs. Namely, the core of an EALA is a Δ-graded Lie algebra in our sense, which was shown in [AG, Proposition 1.16].

Let G be an abelian group. We will consider a G-graded (Lie) algebra $L = \oplus_{g \in G} L^g$, which is a G-graded vector space satisfying $L^g L^h \subset L^{g+h}$ ($[L^g, L^h] \subset L^{g+h}$ if L is Lie) for all $g, h \in G$. For convenience, we always assume that

$$\operatorname{supp} L := \{g \in G \mid L^g \neq 0\} \text{ generates } G.$$

Definition 4.2.

(1) A Δ-graded Lie algebra $L = \oplus_{\mu \in \Delta \cup \{0\}} L_\mu$ with grading pair $(\mathfrak{g}, \mathfrak{h})$ is called (Δ, G)*-graded* if $L = \oplus_{g \in G} L^g$ is a G-graded Lie algebra such that $\mathfrak{g} \subset L^0$. Then we have

$$L = \bigoplus_{\mu \in \Delta \cup \{0\}} \bigoplus_{g \in G} L_\mu^g,$$

where $L_\mu^g = L_\mu \cap L^g$ since L^g is an $\mathfrak{h}$-submodule of L. Note that if G is trivial, L is just a Δ-graded Lie algebra.

(2) Let $Z(L)$ be the centre of L and let $\mu^\vee \in \mathfrak{h}$ for $\mu \in \Delta$ be the coroot of μ. Then L is called a *division* (Δ, G)*-graded Lie algebra* if for any $\mu \in \Delta$ and any $0 \neq x \in L_\mu^g$, there exists

$$y \in L_{-\mu}^{-g} \text{ such that } [x, y] \equiv \mu^\vee \text{ modulo } Z(L). \quad \text{(division property)}$$

(These Lie algebras for a reduced Δ were introduced in [Y2].)

If G is trivial, L is called a *division Δ-graded Lie algebra.*

(3) A division (Δ, G)-graded Lie algebra $L = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{g\in G} L_\mu^g$ is called a *Lie G-torus of type* Δ if

$$\dim_F L_\mu^g \leq 1 \text{ for all } g \in G \text{ and } \mu \in \Delta. \quad \text{(1-dimensionality)}$$

If $G = \mathbb{Z}^n$, we call it a *Lie n-torus* or simply a *Lie torus.*

Example 4.3.

(1) A finite-dimensional simple isotropic Lie algebra L is a centreless division Δ-graded Lie algebra. In fact, let $\mathfrak{h}$ be a maximal abelian diagonalizable subalgebra of L (a maximal split torus in [Se]), which is nonzero since L is isotropic. Then L has the root space decomposition relative to $\mathfrak{h}$, and the root system Δ is a finite irreducible root system (see [Se, Ch.I, §2]). Thus $L = \oplus_{\mu\in\Delta\cup\{0\}} L_\mu$ which is the condition (ii) of Δ-graded Lie algebras. Since $\sum_{\mu\in\Delta} ([L_\mu, L_{-\mu}] + L_\mu)$ is a nonzero ideal of L and L is simple, the condition (iii) holds. The division property relative to this $\mathfrak{h}$ is one of the most important and starting properties to develop the theory, which is shown in [Se, Ch.I, Lemma 3]. Thus we need to show the condition (i) of Δ-graded Lie algebras, i.e., the existence of a grading subalgebra $\mathfrak{g}$ so that the grading pair is $(\mathfrak{g}, \mathfrak{h})$. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots of Δ^{red}. As Seligman did in [Se, Ch.III, §1], using the division property, one can choose an sl_2-triplet $0 \neq e_i \in L_{\alpha_i}$, $0 \neq f_i \in L_{-\alpha_i}$ and $\alpha_i^\vee \in \mathfrak{h}$ for each α_i. These e_i, f_i, $\alpha_i^\vee$ for $1 \leq i \leq l$ satisfy the Serre relations, and by Serre's Theorem, one can show that the Lie algebra $\mathfrak{g}$ generated by those elements is a split simple Lie algebra of type Δ^{red} and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Note that for the nonreduced case, he constructed a split simple Lie algebra of type C_l instead of type $\Delta^{\mathrm{red}} = \mathrm{B}_l$. See also the remark [ABG, 1.15].

(2) Loop (or twisted loop) algebras, toroidal Lie algebras and the cores of EALAs are all examples of Lie tori.

(3) Let $F^t[G]$ be a twisted group algebra. Then $sl_{l+1}(F^t[G])$ is a centreless Lie G-torus of type A_l. Moreover, any Lie G-torus of type A_l for $l \geq 3$ is isomorphic to a central extension of some $sl_{l+1}(F^t[G])$. More generally, division (Δ, G)-graded Lie algebras of type A_l ($l \geq 3$), D_l and E_l were, up to central extensions, classified in terms of crossed product algebras, and if $G = \mathbb{Z}^n$, they were classified in more precise description (see [Y2]). Also, division $(\mathrm{A}_2, \mathbb{Z}^n)$-graded Lie algebras were, up to central extensions, classified in [Y3].

Centreless division Δ-graded Lie algebras are (possibly infinite-dimensional) isotropic simple Lie algebras by the following:

Lemma 4.4. *A centreless division (Δ, G)-graded Lie algebra is G-graded simple. Hence, in particular, a centreless division Δ-graded Lie algebra is simple.*

Proof. Suppose $L = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{g\in G} L_\mu^g$ is a centreless division (Δ, G)-graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$, and $I = \oplus_{g\in G} I^g$ is a nonzero graded ideal. Since I^g is an $\mathfrak{h}$-submodule, I^g is graded for the Δ-grading, say $I^g = \oplus_{\mu\in\Delta\cup\{0\}} I^g_\mu$ for all $g \in G$. Since $I \neq 0$, there exists some $k \in G$ such that $I^k \neq 0$. If $I^k = I^k_0$, then $[I^k, L^l_\mu] \subset I^{k+l}_\mu$ for $\mu \in \Delta$ and $l \in G$. If $I^{k+l}_\mu = 0$ for all $\mu \in \Delta$ and $l \in G$, then $I^k \subset Z(L) = 0$, contradiction. Hence there exist some $\mu \in \Delta$ and $l \in G$ such that $I^{k+l}_\mu \neq 0$. Therefore, without loss of generality, we may assume that $I^g_\mu \neq 0$ for some $\mu \in \Delta$ and $g \in G$. Then $\mu^\vee \in I$ by the division property. If $<\nu, \mu> \neq 0$, then $\mu^\vee \in I$ implies $L_\nu \subset I$. Hence, by the division property, $\nu^\vee \in I$. Repeating this process, we get $L_\nu \subset I$ for all $\nu \in \Delta$, by the irreducibility of Δ. So $I = L$. Thus L is graded simple. ■

5. The root systems of division (Δ, G)-graded Lie algebras

We show that a division (Δ, G)-graded Lie algebra has a root system extended by G of type Δ in the following sense:

Theorem 5.1. *Let $\mathcal{L} = \oplus_{\mu\in\Delta\cup\{0\}} \oplus_{g\in G} \mathcal{L}^g_\mu$ be a division (Δ, G)-graded Lie algebra with centre $\mathcal{Z}$. For each $\mu \in \Delta \cup \{0\}$, let*

$$S_\mu := \{g \in G \mid \mathcal{L}^g_\mu \neq 0\} \quad \textit{and} \quad R = R(\mathcal{L}) := \{S_\mu\}_{\mu\in\Delta}.$$

Then $R = R(\mathcal{L}/\mathcal{Z})$, R is a root system extended by G, and $S_0 = S_\mu + S_\mu$ for $\mu \in \Delta_{sh}$.

Moreover, if $\mathcal{L}$ is a Lie G-torus, then R is reduced.

Proof. Let $S'_0 := \{g \in G \mid (\mathcal{L}/\mathcal{Z})^g_0 \neq 0\}$. Clearly $S'_0 \subset S_0$. We claim that

$$(*) \qquad S_0 = \bigcup_{\mu\in\Delta} (S_\mu + S_{-\mu}) = S'_0.$$

Since $\mathcal{L}$ is Δ-graded, $S_0 \subset \cup_{\mu\in\Delta} (S_\mu + S_{-\mu})$. So we need to show that $\cup_{\mu\in\Delta} (S_\mu + S_{-\mu}) \subset S'_0$. Let $g \in S_\mu$ and $k \in S_{-\mu}$ and let $0 \neq x \in \mathcal{L}^g_\mu$ and $0 \neq v \in \mathcal{L}^k_{-\mu}$. Suppose $[x, v] = 0$ in $\mathcal{L}/\mathcal{Z}$. By the division property, there exists $y \in \mathcal{L}^{-g}_{-\mu}$ so that $sl_2 := \langle x, y, \mu^\vee\rangle$ in $\mathcal{L}/\mathcal{Z}$ is isomorphic to $sl_2(F)$. Consider $\mathcal{L}$ as an sl_2-module and the submodule generated by v. Note that $[\mu^\vee, v] = -2v$, and so v has the negative weight. Hence $0 \neq (\operatorname{ad} y)^2(v) \in \mathcal{L}_{-3\mu}$, which cannot happen. Hence $0 \neq [x, v] \in \mathcal{L}^{g+k}_0$, i.e., $g + k \in S'_0$, and our claim is settled. Since $\mathcal{Z} \subset \mathcal{L}_0$, $\mathcal{L}_\mu = (\mathcal{L}/\mathcal{Z})_\mu$ for $\mu \neq 0$, and so $R(\mathcal{L}) = R(\mathcal{L}/\mathcal{Z})$.

For the second statement, we need to check the axioms (R1)-(R3). (R1) and (R2) follow from the definition of a division (Δ, G)-graded Lie algebra. Thus we show (R3). Notice that $\operatorname{ad} x$, for any $g \in G$, $\mu \neq 0$ and $0 \neq x \in \mathcal{L}^g_\mu$, is ad-nilpotent since Δ is finite. Let $y \in \mathcal{L}^{-g}_{-\mu}$ be such that $[x, y] \equiv \mu^\vee \mod \mathcal{Z}$. Then one can define the automorphism

$$\theta^g_\mu := \exp \operatorname{ad} x \exp \operatorname{ad}(-y) \exp \operatorname{ad} x$$

of $\mathcal{L}$. Then by the same way as in [A-P, Prop. 1.27], one can see that

$$\theta^g_\mu(\mathcal{L}^k_\nu) = \mathcal{L}^{k-<\nu,\mu>g}_{\sigma_\mu(\nu)}$$

for all $\nu \in \Delta$ and $k \in G$. So $S_\nu - < \nu, \mu > S_\mu \subset S_{\sigma_\mu(\nu)}$, which is (R3). Therefore, R is a root system extended by G. So in particular, $S_\nu \subset S_\mu = S_{-\mu}$ for all $\nu \in \Delta$ and $\mu \in \Delta_{sh}$ (see §3, the equation (2) and Proposition 3.3(c)). Hence by $(*)$, we get $S_0 = S_\mu + S_\mu$ for $\mu \in \Delta_{sh}$.

For the last statement, we need to check (R4). Let $\mu, 2\mu \in \Delta$ and $g \in S_\mu$. Since $R = R(\mathcal{L}/\mathcal{Z})$, we may assume that $\mathcal{L}$ is centreless. Thus again let $x \in \mathcal{L}_\mu^g$ and $y \in \mathcal{L}_{-\mu}^{-g}$ be so that $sl_2 := \langle x, y, \mu^\vee \rangle \cong sl_2(F)$. By 1-dimensionality, we have $[\mathcal{L}_\mu^g, \mathcal{L}_{-\mu}^{-g}] + [\mathcal{L}_{2\mu}^{2g}, \mathcal{L}_{-2\mu}^{-2g}] \subset F\mu^\vee$. Hence,

$$M := \mathcal{L}_{2\mu}^{2g} \oplus \mathcal{L}_\mu^g \oplus F\mu^\vee \oplus \mathcal{L}_{-\mu}^{-g} \oplus \mathcal{L}_{-2\mu}^{-2g}$$

is a finite-dimensional sl_2-module. Since the 0-weight subspace of M which is $F\mu^\vee$ is 1-dimensional, M is irreducible. Since M contain sl_2 as a submodule, $M = sl_2$. So in particular, $\mathcal{L}_{2\mu}^{2g} = 0$. Hence $S_{2\mu} \cap 2S_\mu = \emptyset$. ■

The 1-dimensionality forces the root system to be reduced. The following shows that 2-dimensionality is enough to have a nonreduced example.

Example 5.2. (An example of a Lie algebra which has a nonreduced root system extended by $\mathbb{Z}^n$, or a nonreduced SEARS of nullity n.) Let $\theta = (\theta_{ij})$ be an $n \times n$ skew symmetric matrix over $\mathbb{R}$, and let $T := \mathbb{C}_\theta[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the noncommutative torus determined by θ, i.e., the associative algebra over $\mathbb{C}$ generated by $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$ with relations

$$t_j t_i = e^{\sqrt{-1}\theta_{ij}} t_i t_j \quad \text{for all } 1 \le i, j \le n.$$

Let $^-$ be the involution of T over $\mathbb{R}$ determined by

$$\bar{t}_1 = t_1, \quad \ldots, \quad \bar{t}_n = t_n,$$

which extends the complex conjugation of $\mathbb{C}$. So $(T, ^-)$ is an associative algebra with involution over $\mathbb{R}$. Also, T has the natural $\mathbb{Z}^n$-grading, i.e., $T = \oplus_{\alpha \in \mathbb{Z}^n} \mathbb{C}t_\alpha$, where $t_\alpha := t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. (All the homogeneous spaces are 2-dimensional over $\mathbb{R}$.) Thus it is easy to show that the Kantor construction $\mathcal{K} := K(T, ^-)$, which is a BC_1-graded Lie algebra over $\mathbb{R}$, is a division $(\mathrm{BC}_1, \mathbb{Z}^n)$-graded Lie algebra over $\mathbb{R}$. So $R(\mathcal{K}) = R(S, E)_{\mathrm{BC}_1}$ is a root system extended by $\mathbb{Z}^n$, $S = \operatorname{supp} T = \mathbb{Z}^n$, and

$$E = \{\alpha \in \mathbb{Z}^n \mid \mathbb{C}t_\alpha \cap T_- \neq 0\},$$

where T_- is the set of skew elements of $(T, ^-)$. It is easy to see that $E = \mathbb{Z}^n$. Thus $R(\mathcal{K})$ is nonreduced.

6. Division (Δ, G)-graded Lie algebras of type B

In this section we will classify division (B_l, G)-graded Lie algebras for $l \ge 3$, and see that there exist such Lie algebras for each root system extended by G of type B_l for $l \ge 3$.

First we recall a Jordan algebra of a symmetric bilinear form. Let $\mathcal{Z}$ be a commutative associative algebra, $\mathcal{W}$ a $\mathcal{Z}$-module, and f a symmetric $\mathcal{Z}$-bilinear form on $\mathcal{W}$. Then

$$\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$$

is called a *Jordan algebra of* f if the multiplication is defined as

$$(z+w)\cdot(z'+w') = zz' + f(w,w') + zw' + z'w$$

for $z, z' \in \mathcal{Z}$ and $w, w' \in \mathcal{W}$, which is a Jordan algebra over $\mathcal{Z}$.

Next we need a concrete realization of a finite-dimensional split simple Lie algebra $\mathfrak{g}$ over F of type B_l $(l \geq 2)$. Let V be a $(2l+1)$-dimensional space over F with basis $v_1, \ldots, v_{2l+1}$. Let u be the symmetric bilinear form on V so that

$$u(v_i, v_{l+i}) = 1 \text{ for } 1 \leq i \leq l, \quad u(v_{2l+1}, v_{2l+1}) = 1 \quad \text{and} \quad u(v_i, v_j) = 0 \text{ otherwise.}$$

We may identify $\mathfrak{g}$ as the Lie algebra of endomorphisms of V which are skew relative to u. We may also identify the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the Lie algebra of elements of $\mathfrak{g}$ which are diagonal relative to the basis $v_1, \ldots, v_{2l+1}$. Let $\{e_1, \ldots, e_l\}$ be a basis for $\mathfrak{h}$ defined by

$$e_i\Big(\sum_{j=1}^{2l+1} a_j v_j\Big) = a_i v_i - a_{l+i} v_{l+i}$$

for $a_j \in F$. Let $\{\epsilon_1, \ldots, \epsilon_l\}$ be the dual basis of $\{e_1, \ldots, e_l\}$ in $\mathfrak{h}^*$. Then

$$V_{\epsilon_i} = Fv_i, \quad V_{-\epsilon_i} = Fv_{l+i}, \quad \text{for } i = 1, \ldots, l, \quad \text{and} \quad V_0 = Fv_{2l+1}.$$

In this case,

$$\text{(1)} \qquad \Delta_{sh} = \{\pm\epsilon_i \mid 1 \leq i \leq l\} \quad \text{and} \quad \Delta_{lg} = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i \neq j \leq l\}.$$

In particular, the weights of the module V are the elements of $\Delta_{sh} \cup \{0\}$. Let

$$J = F \oplus V$$

be the Jordan algebra of u. Then we may identify

$$\mathfrak{g} = D_{V,V} := \operatorname{span}\{D_{v,v'} \mid v, v' \in V\},$$

where $D_{v,v'} = [L_v, L_{v'}]$ and L is the left multiplication operator, by extending the action of elements of $\mathfrak{g}$ to J so that they kill F. We use the following recognition theorem by Benkart and Zelmanov [BZ]:

Theorem 6.1. *Let $\mathcal{L}$ be a centreless B_l-graded Lie algebra for $l \geq 3$ with grading pair $(\mathfrak{g}, \mathfrak{h})$. Then there exists a Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$ of a symmetric bilinear form f over a commutative associative algebra $\mathcal{Z}$ so that $\mathcal{L} = \mathfrak{B}(\mathcal{J})$, where*

$$\mathfrak{B}(\mathcal{J}) = (\mathfrak{g} \otimes_F \mathcal{Z}) \oplus (V \otimes_F \mathcal{W}) \oplus D_{\mathcal{W},\mathcal{W}} \quad \textit{with} \quad \mathfrak{g} \otimes 1 = \mathfrak{g} = D_{V,V},$$

$$D_{\mathcal{W},\mathcal{W}} := \operatorname{span}\{D_{w,w'} \mid w, w' \in \mathcal{W}\}$$

with $D_{w,w'} = [L_w, L_{w'}]$ as above, and the Lie bracket is defined by

$$\begin{aligned}
[x \otimes z, x' \otimes z'] &= [x, x'] \otimes zz', \\
[x \otimes z, v \otimes w] &= xv \otimes zw, \\
[x \otimes z, D] &= 0, \\
[v \otimes w, v' \otimes w'] &= D_{v,v'} \otimes f(w, w') + u(v, v') D_{w,w'}, \\
[D, v \otimes w] &= v \otimes Dw
\end{aligned} \tag{2}$$

for $x, x' \in \mathfrak{g}$, $z, z' \in \mathcal{Z}$, $v, v' \in V$, $w, w' \in \mathcal{W}$ and $D \in D_{\mathcal{W},\mathcal{W}}$, containing $D_{\mathcal{W},\mathcal{W}}$ as a subalgebra.

Conversely, for any such Jordan algebra $\mathcal{J}$, $\mathfrak{B}(\mathcal{J})$ is a centreless B_l-graded Lie algebra for $l \geq 2$.

Remark 6.2. If $\mathcal{W} = 0$, i.e., $\mathcal{J} = \mathcal{Z}$ is a commutative associative algebra, then $\mathfrak{B}(\mathcal{J}) = \mathfrak{g} \otimes_F \mathcal{Z}$ is called *untwisted*.

Remark 6.3. In [BZ] or [AG], $D_{v,v'} = -[L_v, L_{v'}]$ and $D_{w,w'} = -[L_w, L_{w'}]$ were chosen. Let $\mathfrak{B}^-(\mathcal{J})$ be the Lie algebra constructed by a Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$ of f as above, using their choice. Let $\mathcal{J}^- = \mathcal{Z} \oplus \mathcal{W}$ be the Jordan algebra of $-f$. Then our Lie algebra $\mathfrak{B}(\mathcal{J})$ is isomorphic to $\mathfrak{B}^-(\mathcal{J}^-)$ via the identity map. We will explain in Remark 6.5 that our choice is more natural.

Also, Seligman [Se] originally defined the Lie bracket on the same space

$$\mathfrak{B}_S(\mathcal{J}) = (\mathfrak{g} \otimes_F \mathcal{Z}) \oplus (V \otimes_F \mathcal{W}) \oplus D_{\mathcal{W},\mathcal{W}}$$

as $\mathfrak{B}(\mathcal{J})$, using the natural right action of $\mathfrak{g}$ on V and identifying

$$\mathfrak{g} = \operatorname{span}\{[R_v, R_{v'}] \mid v, v' \in V\},$$

where R is the right multiplication operator, as

$$\begin{aligned}
[x \otimes z, x' \otimes z'] &= [x, x'] \otimes zz', \\
[v \otimes w, x \otimes z] &= v.x \otimes wz, \\
[x \otimes z, D] &= 0, \\
[v \otimes w, v' \otimes w'] &= [R_v, R_{v'}] \otimes f(w, w') + u(v, v')[R_w, R_{w'}], \\
[v \otimes w, D] &= v \otimes wD
\end{aligned}$$

for $x, x' \in \mathfrak{g}$, $z, z' \in \mathcal{Z}$, $v, v' \in V$, $w, w' \in \mathcal{W}$ and

$$D \in D_{\mathcal{W},\mathcal{W}} = \mathrm{span}\{[R_w, R_{w'}] \mid w, w' \in \mathcal{W}\},$$

containing $D_{\mathcal{W},\mathcal{W}}$ as a subalgebra. One can see that the natural right action is equivalent to the action defined by

$$v.x := -xv$$

for $x \in \mathfrak{g}$ and $v \in V$ (the right-hand side is the natural left action defined above). Then since $v''[R_v, R_{v'}] = -[L_v, L_{v'}]v''$ for $v'' \in V$, we have $v''.([L_v, L_{v'}]) = -[L_v, L_{v'}]v'' = v''[R_v, R_{v'}]$. Thus, the map from $\mathfrak{B}(\mathcal{J})$ onto $\mathfrak{B}_S(\mathcal{J})$ defined by

$$x \otimes z + v \otimes w + [L_{w'}, L_{w''}] \mapsto x \otimes z + v \otimes w + [R_{w'}, R_{w''}]$$

is an isomorphism of Lie algebras.

A G-graded Jordan algebra is called *division graded* if all nonzero homogeneous elements are invertible. Now we are ready to show one of our main results.

Theorem 6.4. *Let $\mathcal{L}$ be a centreless (B_l, G)-graded Lie algebra for $l \geq 3$. Then there exists a G-graded Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W} = \oplus_{g\in G} (\mathcal{Z}_g \oplus \mathcal{W}_g)$ of a symmetric bilinear form f over a commutative associative algebra $\mathcal{Z}$ so that $\mathcal{L} = \mathfrak{B}(\mathcal{J})$.*

Moreover, if $\mathcal{L}$ is division graded, then so is $\mathcal{J}$. In this case $\mathrm{supp}\,\mathcal{J} = S$ *and $\mathcal{Z}$ is division L-graded with* $\mathrm{supp}\,\mathcal{Z} = L$, *where S and L are defined in the system $R(\mathcal{L}) = R(S, L)_{\mathrm{B}_l}$.*

Conversely, for a root system $R(S, L)_{\mathrm{B}_l}$ for $l \geq 2$ so that a division G-graded Jordan algebra $\mathcal{J} = \oplus_{g\in G} (\mathcal{Z}_g \oplus \mathcal{W}_g)$ of a symmetric bilinear form satisfies $\mathrm{supp}\,\mathcal{J} = S$ *and* $\mathrm{supp}\,\mathcal{Z} = L$, *$\mathfrak{B}(\mathcal{J})$ is a centreless division (B_l, G)-graded Lie algebra whose root system is $R(S, L)_{\mathrm{B}_l}$.*

Proof. Let

$$\mathcal{L} = \bigoplus_{\mu\in\Delta\cup\{0\}} \bigoplus_{g\in G} \mathcal{L}_\mu^g$$

be a centreless division (B_l, G)-graded Lie algebra for $l \geq 3$ with grading pair $(\mathfrak{g}, \mathfrak{h})$. Then, by Theorem 6.1, we have

$$\mathcal{L}_\mu = \begin{cases} \mathfrak{g}_\mu \otimes \mathcal{Z} & \text{if } \mu \in \Delta_{lg} \\ (\mathfrak{g}_\mu \otimes \mathcal{Z}) \oplus (V_\mu \otimes \mathcal{W}) & \text{if } \mu \in \Delta_{sh}. \end{cases} \tag{3}$$

Let $L = \{g \in G \mid \mathcal{L}_\mu^g \neq 0, \mu \in \Delta_{lg}\}$. For all $\mu \in \Delta_{lg}$ and $g \in G$, we define $\mathcal{Z}_\mu^g$ as

$$\mathcal{L}_\mu^g = \mathfrak{g}_\mu \otimes \mathcal{Z}_\mu^g,$$

and so $\mathcal{Z} = \oplus_{g\in G} \mathcal{Z}_\mu^g$, and in particular, $\mathcal{Z}_\mu^g = 0$ if $g \notin L$. If $\mu, \nu \in \Delta_{lg}$ and $\mu - \nu \in \Delta$, then

$$\mathfrak{g}_\mu \otimes \mathcal{Z}_\mu^g = [\mathfrak{g}_\nu \otimes \mathcal{Z}_\nu^g, \mathfrak{g}_{\mu-\nu} \otimes 1] = \mathfrak{g}_\mu \otimes \mathcal{Z}_\nu^g.$$

Thus by the same argument in [AG, (5.11)], we get $\mathcal{Z}^g_\mu = \mathcal{Z}^g_\nu$ for any $\mu, \nu \in \Delta_{lg}$. So for $g \in G$ we put

$$\mathcal{Z}_g := \mathcal{Z}^g_\mu \quad \text{for any choice of } \mu \in \Delta_{lg}.$$

Then

$$(4) \quad \mathcal{Z} = \bigoplus_{g \in G} \mathcal{Z}_g = \bigoplus_{l \in L} \mathcal{Z}_l, \quad \text{with} \quad \mathcal{L}^l_\mu = \mathfrak{g}_\mu \otimes \mathcal{Z}_l \quad \text{for all } \mu \in \Delta_{lg} \text{ and } l \in L,$$

is an L-graded space. Moreover, there exist $\mu, \nu, \mu+\nu \in \Delta_{lg}$ from (1) ($l \geq 3$), and so for $l, k \in L$, we have

$$[\mathfrak{g}_\mu \otimes \mathcal{Z}_l, \mathfrak{g}_\nu \otimes \mathcal{Z}_k] = \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_l \mathcal{Z}_k \subset \mathcal{L}^{l+k}_{\mu+\nu} = \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_{l+k}.$$

Hence $\mathcal{Z}_l \mathcal{Z}_k \subset \mathcal{Z}_{l+k}$ Thus $\mathcal{Z}$ is a $\langle L \rangle$-graded algebra with $1 \in \mathcal{Z}_0$, where $\langle L \rangle$ is the subgroup of G generated by L. ($L = \operatorname{supp} Z$ is not necessarily a subgroup here.)

Let $\mu \in \Delta_{sh}$ and $g \in G$. We define $\mathcal{Z}^g_\mu$ and $\mathcal{W}^g_\mu$ as

$$\mathfrak{g}_\mu \otimes \mathcal{Z}^g_\mu = (\mathfrak{g}_\mu \otimes \mathcal{Z}) \cap \mathcal{L}^g_\mu \quad \text{and} \quad \mathfrak{g}_\mu \otimes \mathcal{W}^g_\mu = (V_\mu \otimes \mathcal{W}) \cap \mathcal{L}^g_\mu.$$

We claim that

$$\mathcal{L}^g_\mu = (\mathfrak{g}_\mu \otimes \mathcal{Z}^g_\mu) \oplus (V_\mu \otimes \mathcal{W}^g_\mu) \quad \text{and} \quad \mathcal{Z}^g_\mu = \mathcal{Z}_g.$$

Let $x \in \mathcal{L}^g_\mu$. Then by (3), $x = t + v$ for some $t \in \mathfrak{g}_\mu \otimes \mathcal{Z}$ and $v \in V_\mu \otimes \mathcal{W}$. So we need to show that $t, v \in \mathcal{L}^g_\mu$. By (4), we have $t = \sum_{l \in L} e_\mu \otimes z_l$ for some $0 \neq e_\mu \in \mathfrak{g}_\mu$ and $z_l \in \mathcal{Z}_l$. From (1), one can see that there exists $\nu \in \Delta$ such that $\mu + \nu \in \Delta_{lg}$. Let $0 \neq e_\nu \in \mathfrak{g}_\nu = \mathfrak{g}_\nu \otimes 1 \subset \mathcal{L}^0_\nu$. Then

$$(5) \qquad [e_\nu, x] \in \mathcal{L}^g_{\mu+\nu} = \begin{cases} 0 & \text{if } g \notin L \\ \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_g & \text{if } g \in L. \end{cases}$$

Also, $[e_\nu, v] \in [\mathfrak{g}_\nu \otimes \mathcal{Z}_0, V_\mu \otimes \mathcal{W}] \subset \mathfrak{g}_\nu . V_\mu \otimes \mathcal{W}$, but $\mathfrak{g}_\nu . V_\mu = 0$ since $\mu + \nu \in \Delta_{lg}$. Hence $[e_\nu, v] = 0$. Thus we obtain $[e_\nu, t] = [e_\nu, x]$.

If $g \notin L$, then $\sum_{l \in L} [e_\nu, e_\mu \otimes z_l] = 0$ by (5), and so $[e_\nu, e_\mu \otimes z_l] = 0$ for all $l \in L$. Since $[\mathfrak{g}_\nu, \mathfrak{g}_\mu] \neq 0$, we get $z_l = 0$ for all $l \in L$, i.e., $t = 0$. Therefore, $x = v \in \mathcal{L}^g_\mu$.

If $g \in L$, then $[e_\nu, t] \in \mathcal{L}^g_{\mu+\nu}$, and so $t = e_\mu \otimes z_g \in \mathfrak{g}_\mu \otimes \mathcal{Z}_g$. ¿From (1), there exists $\xi \in \Delta_{lg}$ such that $\mu - \xi \in \Delta$. So by (4), we have

$$\mathfrak{g}_\mu \otimes \mathcal{Z}_g = [\mathfrak{g}_\xi \otimes \mathcal{Z}_g, \mathfrak{g}_{\mu-\xi} \otimes 1] \subset [\mathcal{L}^g_\xi, \mathcal{L}^0_{\mu-\xi}] \subset \mathcal{L}^g_\mu.$$

Therefore, $t \in \mathcal{L}^g_\mu$ and $v = x - t \in \mathcal{L}^g_\mu$. Finally, since $\mu + \nu \in \Delta_{lg}$, we get

$$[\mathfrak{g}_\mu \otimes \mathcal{Z}^g_\mu, \mathfrak{g}_\nu \otimes 1] = \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}^g_\mu \subset \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_g.$$

Hence $\mathcal{Z}_\mu^g \subset \mathcal{Z}^g$. Also,

$$[\mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}^g, \mathfrak{g}_{-\nu} \otimes 1] = \mathfrak{g}_\mu \otimes \mathcal{Z}^g \subset \mathfrak{g}_\mu \otimes \mathcal{Z}_\mu^g,$$

and hence $\mathcal{Z}^g \subset \mathcal{Z}_\mu^g$. Thus our claim is settled.

Now, $\mathcal{W} = \oplus_{g\in G}\, \mathcal{W}_\mu^g$, and $V_\mu \otimes \mathcal{W}_\mu^g = \mathcal{L}_\mu^g$ if $g \notin L$ since $\mathcal{Z}_g = 0$. If $\mu, \nu \in \Delta_{sh}$ and $\mu - \nu \in \Delta$, then

$$V_\mu \otimes \mathcal{W}_\mu^g = [V_\nu \otimes \mathcal{W}_\nu^g, \mathfrak{g}_{\mu-\nu} \otimes 1] = V_\mu \otimes \mathcal{W}_\nu^g \quad \text{for all } g \in G.$$

Therefore, $\mathcal{W}_\mu^g = \mathcal{W}_\nu^g$. Thus by the same argument in [AG, (5.11)], we get $\mathcal{W}_\mu^g = \mathcal{W}_\nu^g$ for any $\mu, \nu \in \Delta_{sh}$ and all $g \in G$. So for $g \in G$ we put

$$\mathcal{W}_g := \mathcal{W}_\mu^g \quad \text{for any choice of } \mu \in \Delta_{sh}.$$

Consequently,

$$\mathcal{W} = \bigoplus_{g\in G} \mathcal{W}_g, \quad \text{with}$$

$$\mathcal{L}_\mu^g = V_\mu \otimes \mathcal{W}_g \quad \text{for all } \mu \in \Delta_{sh} \text{ and } g \notin L,$$

and

$$\mathcal{L}_\mu^g = (\mathfrak{g}_\mu \otimes \mathcal{Z}_g) \oplus (V_\mu \otimes \mathcal{W}_g) \quad \text{for all } \mu \in \Delta_{sh} \text{ and } g \in L. \tag{6}$$

Our next claim is that

$$\mathcal{J} = \bigoplus_{g\in G} \mathcal{J}_g, \quad \text{where} \quad \mathcal{J}_g := \begin{cases} \mathcal{Z}_g \oplus \mathcal{W}_g & \text{if } g \in L \\ \mathcal{W}_g & \text{if } g \notin L, \end{cases}$$

satisfies for all $g, k \in G$,

$$\mathcal{Z}_g\mathcal{W}_k \subset \mathcal{W}_{g+k}, \quad \mathcal{W}_g\mathcal{W}_k \subset \mathcal{Z}_{g+k} \quad \text{and} \quad \mathcal{W}_g\mathcal{W}_k = 0 \quad \text{if } g+k \notin L.$$

In fact, there exist $\mu \in \Delta_{lg}$ and $\nu \in \Delta_{sh}$ such that $\mu + \nu \in \Delta_{sh}$ by (1). Since $[\mathfrak{g}_\mu, V_\nu] = V_{\mu+\nu}$, we have

$$[\mathfrak{g}_\mu \otimes \mathcal{Z}_g, V_\nu \otimes \mathcal{W}_k] = V_{\mu+\nu} \otimes \mathcal{Z}_g\mathcal{W}_k \subset \mathcal{L}_{\mu+\nu}^{g+k} = (\mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_{g+k}) \oplus (V_{\mu+\nu} \otimes \mathcal{W}_{g+k}).$$

Hence $\mathcal{Z}_g\mathcal{W}_k \subset \mathcal{W}_{g+k}$. Next, one can choose $\mu, \nu \in \Delta_{sh}$ such that $\mu + \nu \in \Delta_{lg}$. Let $0 \neq v_1 \in V_\mu$ and $0 \neq v_2 \in V_\nu$. Then we have

$$[v_1 \otimes \mathcal{W}_g, v_2 \otimes \mathcal{W}_k] = D_{v_1,v_2} \otimes f(\mathcal{W}_g, \mathcal{W}_k) \subset \mathcal{L}_{\mu+\nu}^{g+k} = \mathfrak{g}_{\mu+\nu} \otimes \mathcal{Z}_{g+k}$$

since $u(v_1, v_2) = 0$. Since $D_{v_1,v_2} \neq 0$ (e.g. $D_{v_1,v_2}V_{-\mu} \neq 0$), we get $f(\mathcal{W}_g, \mathcal{W}_k) = \mathcal{W}_g\mathcal{W}_k \subset \mathcal{Z}_{g+k}$. The last equality is clear since $\mathcal{Z}_{g+k} = 0$ for $g + k \notin L$. Hence our claim is settled.

Let $S := \operatorname{supp} \mathcal{J} = \operatorname{supp} \mathcal{L}_\mu$ for $\mu \in \Delta_{sh}$. By (6), we have $L \subset S$, and so $S + S \supset \operatorname{supp} \mathcal{L}$, which generates G. Hence S generates G, and so $\mathcal{J}$ is a G-graded Jordan algebra.

Suppose that $\mathcal{L}$ is division graded. Let $0 \neq z + w \in \mathcal{Z}_g \oplus \mathcal{W}_g = \mathcal{J}_g$ for $z \in \mathcal{Z}_g$, $w \in \mathcal{W}_g$ and $g \in S$ ($z = 0$ if $g \notin L$). To show the invertibility of $z+w$, we identify $\mathfrak{g}$ with the matrices relative to the basis $\{v_1, \dots, v_{2l+1}\}$. Let $e := e_{i,2l+1} - e_{2l+1,l+i} \in \mathfrak{g}_{\epsilon_i}$ (see (1)) for any fixed $1 \leq i \leq l$, where e_{ij} is the matrix unit, and let $e' := 2e_{2l+1,i} - 2e_{l+i,2l+1} \in \mathfrak{g}_{-\epsilon_i}$. Then $[e, e'] = 2e_{ii} - 2e_{l+i,l+i} = \epsilon_i^\vee$. Also, let $v := v_i$ and $v' := 2v_{l+i}$. Then $u(v, v') = 2$, $D_{v,v'} = [L_v, L_{v'}] = \epsilon_i^\vee$ and $ev' + e'v = -2v_{2l+1} + 2v_{2l+1} = 0$.

Now, $e \otimes z + v \otimes w \in \mathcal{L}^g_{\epsilon_i}$, and by the division property of $\mathcal{L}$, there exists $y \in \mathcal{L}^{-g}_{-\epsilon_i}$ such that $[e \otimes z + v \otimes w, y] = \epsilon_i^\vee$. Since $\mathcal{L}^{-g}_{-\epsilon_i} = (\mathfrak{g}_{-\epsilon_i} \otimes \mathcal{Z}_{-g}) \oplus (V_{-\epsilon_i} \otimes \mathcal{W}_{-g})$, we have $y = e' \otimes z' + v' \otimes w'$ for some $z' \in \mathcal{Z}_{-g}$ and $w' \in \mathcal{W}_{-g}$. So

$$\begin{aligned} \epsilon_i^\vee \otimes 1 = \epsilon_i^\vee &= [e \otimes z + v \otimes w, e' \otimes z' + v' \otimes w'] \\ &= [e, e'] \otimes zz' + D_{v,v'} \otimes f(w, w') + e.v' \otimes zw' - e'.v \otimes z'w + u(v, v')D_{w,w'} \\ &= \epsilon_i^\vee \otimes (zz' + ww') + e.v' \otimes (zw' + z'w) + 2D_{w,w'}. \end{aligned}$$

Hence $zz' + ww' = 1$ and $zw' + z'w = 0$. So $(z + w)(z' + w') = 1$. Also, we have $D_{z+w,z'+w'} = D_{w,w'} = 0$. Therefore, $z + w$ is invertible. Thus $\mathcal{J}$ is a division G-graded Jordan algebra, and in particular $L = \operatorname{supp} \mathcal{L}_\mu$ for $\mu \in \Delta_{lg}$ is a subgroup of G, and $\mathcal{Z}$ is a division L-graded commutative associative algebra. Also, the root system of $\mathcal{L}$ is determined by these S and L.

We show the last statement. For $g \in G$, let $\mathcal{L}^g_\mu := (\mathfrak{g}_\mu \otimes \mathcal{Z}_g) \oplus (V_\mu \otimes \mathcal{W}_g)$ if $\mu \in \Delta_{sh}$, and $\mathcal{L}^g_\mu := \mathfrak{g}_\mu \otimes \mathcal{Z}_g$ if $\mu \in \Delta_{lg}$. Then a centreless B_l-graded Lie algebra $\mathfrak{B}(\mathcal{J})$ admits the compatible G-grading, say $\mathfrak{B}(\mathcal{J}) = \oplus_{\mu \in \Delta \cup \{0\}} \oplus_{g \in G} \mathcal{L}^g_\mu$, where $\mathcal{L}^g_0 = \sum_{\mu \in \Delta} \sum_{g=p+q} [\mathcal{L}^p_\mu, \mathcal{L}^q_{-\mu}]$, is a centreless (B_l, G)-graded Lie algebra.

For $\mu \in \Delta_{lg}$ and $g \in L$, let $e \in \mathfrak{g}_\mu$ and $e' \in \mathfrak{g}_{-\mu}$ such that $[e, e'] = \mu^\vee$. Then for $0 \neq x \in \mathcal{L}^g_\mu$, there exists $0 \neq z \in \mathcal{Z}_g$ such that $x = e \otimes z$. Taking $y = e' \otimes z^{-1} \in \mathcal{L}^{-g}_{-\mu}$, we get $[x, y] = \mu^\vee$.

For $\epsilon_i \in \Delta_{sh}$ and $g \in S$, let $e \in \mathfrak{g}_{\epsilon_i}$, $e' \in \mathfrak{g}_{-\epsilon_i}$, $v = v_i$ and $v' = 2v_{l+i}$ as before so that $[e, e'] = \epsilon_i^\vee = D_{v,v'}$ and $ev' + e'v = 0$. Then for $0 \neq x \in \mathcal{L}^g_\mu$, there exist $z \in \mathcal{Z}_g$ and $w \in \mathcal{W}_g$ such that $x = e \otimes z + v \otimes w$. Taking $y = e' \otimes z' + v' \otimes w' \in \mathcal{L}^{-g}_{-\mu}$, where z' and w' are defined as $(z + w)^{-1} = z' + w'$, we get $[x, y] = \epsilon_i^\vee$. Hence $\mathfrak{B}(\mathcal{J})$ is division graded. ∎

Remark 6.5. In a similar way, one can show that

$$\mathfrak{B}^-(\mathcal{J}) \text{ is division graded} \iff \mathcal{J}^- \text{ is division graded},$$

but this does not imply that $\mathcal{J}$ is division graded. For example, $\mathcal{J}^- = \mathbb{R} \oplus \mathbb{R}w$ with $w^2 = -1$ is a division algebra over $\mathbb{R}$, which is $\mathbb{C}$, but $\mathcal{J} = \mathbb{R} \oplus \mathbb{R}w$ with $w^2 = 1$ is not even simple. Thus if one considers the division property of the coordinate algebras $\mathcal{J}$ for division B_l-graded Lie algebras, our $\mathfrak{B}(\mathcal{J})$ seems more natural.

However, for the case of Lie G-tori and Clifford G-tori defined in the next section, we have

$$\begin{aligned} \mathfrak{B}^-(\mathcal{J}) \text{ is a Lie } G\text{-torus} &\iff \mathcal{J} \text{ is a Clifford } G\text{-torus} \\ &\iff \mathfrak{B}(\mathcal{J}) \text{ is a Lie } G\text{-torus} \end{aligned}$$

since there is no homogeneous element $z+w \in \mathcal{J}$ with $z \neq 0$ and $w \neq 0$. So the choice of either $\mathfrak{B}^-(\mathcal{J})$ or $\mathfrak{B}(\mathcal{J})$ does not effect the division property of $\mathcal{J}$.

Now we show the existence of a division G-graded Jordan algebra in Theorem 6.4 for any root system $R(S,L)_{\mathrm{B}_l}$ extended by G for $l \geq 3$. By Theorem 3.4, S is a full reflection space of G and L is a subgroup of G such that

$$2S \subset L \quad \text{and} \quad S+L \subset S$$

(see Example 3.5(1)). We choose a collection of coset representatives $\{u_t\}_{t\in T}$ in G/L so that

$$G = \bigsqcup_{t\in T} (u_t + L).$$

By the second inclusion, S is a union of cosets of G/L containing L, say

$$S = \bigsqcup_{t\in I\sqcup\{0\}\subset T} (u_t + L) \quad \text{with } u_0 = 0.$$

Also, by the first inclusion, we have

$$l_i := 2u_i \in L \quad \text{for all } i \in I \sqcup \{0\}.$$

Let

$$\mathcal{Z} = \bigoplus_{l\in L} \mathcal{Z}_l$$

be a division L-graded commutative associative algebra over F, i.e., a commutative associative L-graded algebra so that all nonzero homogeneous elements are invertible. So in particular $\operatorname{supp}\mathcal{Z} = L$. Let $K = \mathcal{Z}_0$ which is a field extension of F. For each $i \in I \sqcup \{0\}$,

(i) let $(\mathcal{W}_i, \varphi_i, z_i)$ be a triple consisting of a vector space $\mathcal{W}_i$ over K, a symmetric *anisotropic* bilinear form φ_i of $\mathcal{W}_i$, i.e., $\varphi_i(w,w) \neq 0$ for all $0 \neq w \in \mathcal{W}_i$, and $0 \neq z_i \in \mathcal{Z}_{l_i}$ with $z_0 = 1$.

(ii) assume that $\mathcal{W}_i \neq 0$ if $i \neq 0$,

(iii) assume that φ_0 does not represent 1 (so $K \oplus \mathcal{W}_0$ is a Jordan division algebra).

We denote the family of triples by $\mathcal{W}_I$:

$$\mathcal{W}_I = \{(\mathcal{W}_i, \varphi_i, z_i)\}_{i\in I\sqcup\{0\}}.$$

For each $i \in I \sqcup \{0\}$, we extend φ_i to $\mathcal{Z} \otimes_K \mathcal{W}_i$ as a $\mathcal{Z}$-bilinear map, denoted f_i, as follows:

$$f_i(z\otimes w, z'\otimes w') = zz'\varphi_i(w,w')z_i$$

for $z, z' \in \mathcal{Z}$, $w, w' \in \mathcal{W}_i$. Obviously f_i is symmetric and the image is $\mathcal{Z}z_i$ for each $i \in I \sqcup \{0\}$. Let

$$f := \bigoplus_{i\in I\sqcup\{0\}} f_i$$

be the symmetric bilinear form over $\mathcal{Z}$ on

$$\mathcal{W} := \bigoplus_{i\in I\sqcup\{0\}} (\mathcal{Z}\otimes_K \mathcal{W}_i).$$

We claim that the Jordan algebra

$$\mathcal{J} := \mathcal{Z}\oplus\mathcal{W}$$

of f is a division G-graded Jordan algebra over F for some G-grading. First we put

$$\mathcal{W}_t := 0 \quad\text{and}\quad \varphi_t = 0 \quad \text{for } t\in T\setminus(I\sqcup\{0\}).$$

For $g\in G$, there exist unique $u_t\in G$ and $l\in L$ such that $g=u_t+l$. We set

$$\mathcal{J}_g = \mathcal{J}_{u_t+l} := \begin{cases} \mathcal{Z}_l\otimes_K(K\oplus\mathcal{W}_0) & \text{if } t=0,\\ \mathcal{Z}_l\otimes_K\mathcal{W}_t & \text{otherwise.}\end{cases}$$

Since $\mathcal{Z}\otimes_K\mathcal{W}_t = \oplus_{l\in L}\,(\mathcal{Z}_l\otimes_K\mathcal{W}_t)$ and

$$\mathcal{Z}\oplus(\mathcal{Z}\otimes_K\mathcal{W}_0) = \mathcal{Z}\otimes_K(K\oplus\mathcal{W}_0) = \bigoplus_{l\in L}\big(\mathcal{Z}_l\otimes_K(K\oplus\mathcal{W}_0)\big)$$

(identifying $\mathcal{Z}$ with $\mathcal{Z}\otimes_K K$), we get

$$\begin{aligned}\mathcal{J} &= \mathcal{Z}\oplus\Bigg(\bigoplus_{i\in I\sqcup\{0\}}(\mathcal{Z}\otimes_K\mathcal{W}_i)\Bigg) = \big(\mathcal{Z}\oplus(\mathcal{Z}\otimes_K\mathcal{W}_0)\big)\oplus\Bigg(\bigoplus_{i\in I}\mathcal{Z}\otimes_K\mathcal{W}_i\Bigg)\\ &= \Bigg(\bigoplus_{l\in L}\big(\mathcal{Z}_l\otimes_K(K\oplus\mathcal{W}_0)\big)\Bigg)\oplus\Bigg(\bigoplus_{(i,l)\in I\times L}(\mathcal{Z}_l\otimes_K\mathcal{W}_i)\Bigg) = \bigoplus_{g\in G}\mathcal{J}_g\end{aligned}$$

as K-vector spaces. Let $g=l+u_t$, $g'=l'+u_{t'}\in G$ $(l,l'\in L)$, $z\in\mathcal{Z}_l$ and $z'\in\mathcal{Z}_{l'}$. If $t=t'=0$, then for $a,b\in K$ and $x,y\in\mathcal{W}_0$,

$$\big(z\otimes(a+x)\big)\big(z'\otimes(b+y)\big) = zz'\otimes\big(ab+\varphi_0(x,y)+bx+ay\big)\in\mathcal{J}_{g+g'}.$$

If $t=0$ and $t'\neq 0$, then for $a\in K$, $x\in\mathcal{W}_0$ and $y\in\mathcal{W}_{t'}$,

$$\big(z\otimes(a+x)\big)(z'\otimes y) = azz'\otimes y\in\mathcal{J}_{g+g'}$$

since $\mathcal{W}_0\mathcal{W}_{t'}=0$. Finally, if $t\neq 0$ and $t'\neq 0$, then for $x\in\mathcal{W}_t$ and $y\in\mathcal{W}_{t'}$,

$$(z\otimes x)(z'\otimes y) = \begin{cases}\varphi_t(x,y)zz'z_t\otimes 1 & \text{if } t=t'\in I\\ 0 & \text{otherwise (since } \mathcal{W}_t\mathcal{W}_{t'}=0),\end{cases}$$

which is in $\mathcal{J}_{g+g'}$. Therefore, we obtain $\mathcal{J}_g\mathcal{J}_{g'}\subset\mathcal{J}_{g+g'}$ for all $g,g'\in G$. Since $\operatorname{supp}\mathcal{J}=S$ which generates G, $\mathcal{J}$ is a G-graded algebra over K. Any $0\neq az\otimes u\in\mathcal{J}_l$ for $a\in K$, $l\in L$, $z\in\mathcal{Z}_l$ and $u\in K\oplus\mathcal{W}_0$ has the inverse $a^{-1}z^{-1}\otimes u^{-1}$ since $K\oplus\mathcal{W}_0$ is a Jordan division algebra. (Note that $\dim_K\mathcal{Z}_l=1$ for all $l\in L$.) Also, for any $0\neq az\otimes x\in\mathcal{J}_{l+u_i}$, $i\in I$, $a\in K$, $l\in L$, $z\in\mathcal{Z}_l$ and $x\in\mathcal{W}_i$, we have $(az\otimes x)^2 = a^2\varphi_i(x,x)z^2z_i\otimes 1$, which is invertible since φ_i is anisotropic. Hence $az\otimes x$ is invertible. So we have shown that

$$\mathcal{J} = \mathcal{J}(S,L,K,\mathcal{Z},\mathcal{W}_I)$$

is a division G-graded Jordan algebra over K, and hence over F. Thus:

Theorem 6.6. *For any root system $R(S,L)_{\mathrm{B}_l}$ extended by G for $l \geq 3$, there exists a division G-graded Jordan algebra $\mathcal{J} = \mathcal{J}(S,L,K,\mathcal{Z},\mathcal{W}_I)$ of a symmetric bilinear form over an L-graded commutative associative algebra $\mathcal{Z}$ with* $\operatorname{supp}\mathcal{J} = S$ *and* $\operatorname{supp}\mathcal{Z} = L$.

Conversely, for any division G-graded Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W} = \oplus_{g\in G} (\mathcal{Z}_g \oplus \mathcal{W}_g)$ of a symmetric bilinear form over a commutative associative algebra $\mathcal{Z}$, the pair of $S := \operatorname{supp}\mathcal{J}$ and $L := \operatorname{supp}\mathcal{Z}$ satisfies the conditions of a root system $R(S,L)_{\mathrm{B}_l}$ extended by G for $l \geq 2$, and $\mathcal{J}$ is graded isomorphic to some $\mathcal{J}(S,L,K,\mathcal{Z},\mathcal{W}_I)$.

Proof. We only need to show the second statement. Note first that $L = \operatorname{supp}\mathcal{Z}$ is a subgroup of G since Z is associative, and that $S = \operatorname{supp}\mathcal{J}$ is a full reflection space of G (see [NY, 2.3(b)]). Also, for $g \in S$ and $0 \neq w \in \mathcal{W}_g$, we have $0 \neq w^2 \in \mathcal{W}_g\mathcal{W}_g = f(\mathcal{W}_g,\mathcal{W}_g) \in \mathcal{Z}_{2g}$ and so $2g \in L$. Hence $2S \subset L$, and so $L + 2S \subset L$. Moreover, since $\mathcal{W}$ is a graded $\mathcal{Z}$-module, we have $L + S \subset S$. Thus the pair of S and L satisfies the conditions of a root system extended by G of type B_l $(l \geq 2)$.

Let $S' := \operatorname{supp}\mathcal{W}$, which is a union of cosets of G/L. So, letting $u_0 = 0$,

$$S' = \bigsqcup_{i\in\tilde{I}} (u_i + L)$$

for some index set $\tilde{I}$ which may contain 0. Let $K := \mathcal{Z}_0$ which is a field extension of F, $I := \tilde{I} \setminus \{0\}$, $\mathcal{W}_i = \mathcal{W}_{u_i}$ and $\mathcal{W}_0 = 0$ if $0 \notin \tilde{I}$. Then

$$\mathcal{W} = \bigoplus_{s\in S'} \mathcal{W}_s = \bigoplus_{i\in\tilde{I}} \mathcal{Z}\mathcal{W}_{u_i} = \bigoplus_{i\in I\sqcup\{0\}} \mathcal{Z}\mathcal{W}_i.$$

Let f be the symmetric bilinear form of the Jordan algebra $\mathcal{J}$, and let

$$f_i = f\,|_{\mathcal{W}_i\times\mathcal{W}_i}$$

for all $i \in I \sqcup \{0\}$. For $0 \neq w_i \in \mathcal{W}_{u_i}$, we have $0 \neq w_i^2 \in \mathcal{Z}_{2u_i}$, and so $l_i := 2u_i \in L$ for all $i \in I$. We fix $0 \neq z_i \in \mathcal{Z}_{l_i}$ for all $i \in I$. Also, let $l_0 = 0$ and $z_0 = 1$. Then, since

$$f_i(\mathcal{W}_i,\mathcal{W}_i) \subset \mathcal{Z}_{l_i} = Kz_i,$$

one can define symmetric bilinear forms φ_i on $\mathcal{W}_i$ over K as

$$f_i(w,w') = \varphi_i(w,w')z_i$$

for $w,w' \in \mathcal{W}_i$ and all $i \in I \sqcup \{0\}$. Thus we get a family of triples of K-vector spaces $\mathcal{W}_i$, symmetric anisotropic bilinear forms φ_i and nonzero $z_i \in \mathcal{Z}_{l_i}$ indexed by $i \in I \sqcup \{0\}$, satisfying (i), (ii) and (iii) above.

Let $u_i \neq u_j$. If $u_i + u_j + L = L$, then $u_i + u_j - 2u_j + L = L$, and so $u_i + L = u_j + L$, which contradicts the fact that $\{u_i\}_{i\in\tilde{I}}$ is a collection of coset representatives of S' in G/L. Hence $u_i + u_j + L \neq L$. So we have

$$\mathcal{W}_i\mathcal{W}_j \subset f(\mathcal{W}_i,\mathcal{W}_j) \subset \mathcal{J}_{u_i+u_j} \cap \mathcal{Z} = 0$$

for all $i \in I \sqcup \{0\}$. Hence we have

$$f = \bigoplus_{i \in I \sqcup \{0\}} f_i.$$

Finally, $\mathcal{Z}\mathcal{W}_i$ is a free $\mathcal{Z}$-module with $\operatorname{rank}_{\mathcal{Z}} \mathcal{Z}\mathcal{W}_i = \dim_K \mathcal{W}_i$, and so one can easily show that there is a natural $\mathcal{Z}$-module isomorphism

$$\mathcal{Z}\mathcal{W}_i \cong \mathcal{Z} \otimes_K \mathcal{W}_i$$

via $zx \leftrightarrow z \otimes x$ for $z \in \mathcal{Z}$ and $x \in \mathcal{W}_i$. Identifying them, we get $\mathcal{J} = \mathcal{J}(S, L, K, \mathcal{W}_I)$. ■

Combining Theorem 6.4 with Theorem 6.6, we get the complete classification of division (B_l, G)-graded Lie algebra for $l \geq 3$, up to central extensions.

Theorem 6.7. *Let $\mathcal{L}$ be a centreless division (B_l, G)-graded Lie algebra for $l \geq 3$ with root system $R(S, L)_{\mathrm{B}_l}$. Then $\mathcal{L} = \mathfrak{B}(\mathcal{J})$ for some $\mathcal{J} = \mathcal{J}(S, L, K, \mathcal{Z}, \mathcal{W}_I)$. Conversely, for $l \geq 2$, $\mathfrak{B}(\mathcal{J})$ is a centreless division (B_l, G)-graded Lie algebra for any $\mathcal{J} = \mathcal{J}(S, L, K, \mathcal{Z}, \mathcal{W}_I)$.* ■

7. Lie G-tori of type B

We now specialize Theorem 6.7 to the case of Lie G-tori. Also, we show that there is a one-to-one correspondence between centreless Lie G-tori of type B_l and root systems extended by G of type B_l for $l \geq 3$ if the base field is algebraically closed and G is free.

A division G-graded Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W} = \oplus_{g \in G} (\mathcal{Z}_g \oplus \mathcal{W}_g)$ of a symmetric bilinear form is called a *Clifford G-torus* over F if $\dim_F(\mathcal{Z}_g \oplus \mathcal{W}_g) \leq 1$ for all $g \in G$ (1-dimensionality).

For any root system $R(S, L)_{\mathrm{B}_l}$ extended by G for $l \geq 3$, we construct such a Clifford G-torus $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$ with $\operatorname{supp} \mathcal{J} = S$ and $\operatorname{supp} \mathcal{Z} = L$. Although the construction is just a special case of division G-graded Jordan algebras in §6, they can be described in a simpler way. As in §6, we choose a collection of coset representatives $\{u_t\}_{t \in T}$ in G/L so that

$$G = \bigsqcup_{t \in T} (u_t + L),$$

$$S = \bigsqcup_{t \in I \sqcup \{0\} \subset T} (u_t + L) \quad \text{with } u_0 = 0$$

and

$$l_i := 2u_i \in L \quad \text{for all } i \in I \sqcup \{0\}.$$

Let

$$\mathcal{Z} = \bigoplus_{l \in L} \mathcal{Z}_l$$

be a commutative associative L-torus over F, i.e., a division L-graded commutative associative algebra with $\dim_F \mathcal{Z}_l \leq 1$ for all $l \in L$. So $\operatorname{supp} \mathcal{Z} = L$ and $\dim_F \mathcal{Z}_l = 1$ for all $l \in L$. Let $\mathcal{W}$ be a free $\mathcal{Z}$-module with basis $\{w_i\}_{i \in I}$ and choose $0 \neq z_i \in \mathcal{Z}_{l_i}$. Define a symmetric bilinear form f on $\mathcal{W}$ over $\mathcal{Z}$ as

$$f(w_i, w_j) = \delta_{i,j} z_i.$$

We claim that the Jordan algebra $\mathcal{J} := \mathcal{Z} \oplus \mathcal{W}$ over $\mathcal{Z}$ determined by the symmetric bilinear form f is a Clifford G-torus. In fact we put

$$w_0 := 1 \quad \text{and} \quad w_t := 0 \quad \text{for } t \in T \setminus (I \sqcup \{0\}).$$

For $g \in G$, there exist unique $u_t \in G$ and $l \in L$ such that $g = u_t + l$, and so we set

$$\mathcal{J}_g = \mathcal{J}_{u_t + l} := \mathcal{Z}_l w_t.$$

Then

$$\mathcal{J} = \mathcal{Z} \oplus \mathcal{W} = \bigoplus_{(t,l) \in T \times L} \mathcal{Z}_l w_t = \bigoplus_{g \in G} \mathcal{J}_g$$

as F-vector spaces. Let $g = l + u_t$, $g' = l' + u_{t'} \in G$ $(l, l' \in L)$, $z \in \mathcal{Z}_l$ and $z' \in \mathcal{Z}_{l'}$. Then

$$(z w_t)(z' w_{t'}) = \begin{cases} \delta_{t,t'} z z' z_t & \text{if } t, t' \in I \\ z z' w_t w_{t'} & \text{otherwise.} \end{cases}$$

Since $z_t \in \mathcal{Z}_{l_t} = \mathcal{Z}_{2t}$, we obtain $\mathcal{J}_g \mathcal{J}_{g'} \subset \mathcal{J}_{g+g'}$ for all $g, g' \in G$. Since $\operatorname{supp} \mathcal{J} = S$ which generates G, $\mathcal{J}$ is a G-graded algebra. For any $0 \neq z w_t \in \mathcal{J}_{l + u_t}$, we have $(z w_t)^2 = z^2 z_t$, which is invertible. Hence $z w_t$ is invertible. Thus we have shown that

$$\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i \in I})$$

is a Clifford G-torus.

Conversely, let $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W} = \oplus_{g \in G} (\mathcal{Z}_g \oplus \mathcal{W}_g)$ be a Clifford G-torus. Let $L = \operatorname{supp} \mathcal{Z}$, $S = \operatorname{supp} \mathcal{J}$ and $S' = \operatorname{supp} \mathcal{W}$. Then, by the same reason as in §6, the pair of S and L satisfies the conditions of a root system extended by G of type B_l for $l \geq 3$ (so $l \geq 2$), and we have

$$S = L \sqcup S' = L \sqcup \left(\bigsqcup_{i \in I} (u_i + L) \right)$$

for some $0 \neq u_i \in S'$ indexed by some set I. Now $\mathcal{Z} = \oplus_{l \in L} \mathcal{Z}_l$, which is a commutative associative L-torus. Let $0 \neq w_i \in \mathcal{W}_{u_i}$ for all $i \in I$. Then $\{w_i\}_{i \in I}$ is a $\mathcal{Z}$-basis of $\mathcal{W}$ so that

$$\mathcal{W} = \bigoplus_{s \in S'} \mathcal{W}_s = \bigoplus_{i \in I} \mathcal{Z} \mathcal{W}_{u_i} = \bigoplus_{i \in I} \mathcal{Z} w_i.$$

Let $l_i := 2u_i \in L$ and $z_i := w_i^2 \in \mathcal{Z}_{l_i}$ for all $i \in I$. Then by the same argument as in §6, we get $f(w_i, w_j) = \delta_{i,j} z_i$ for all $i, j \in I$, and so $\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i \in I})$ constructed above. Thus:

Theorem 7.1. *For any root system $R(S,L)_{\mathrm{B}_l}$ extended by G for $l \geq 3$, there exists a Clifford G-torus $\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i\in I})$.*

Conversely, for any Clifford G-torus $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$, the pair of $S := \operatorname{supp} \mathcal{J}$ and $L := \operatorname{supp} \mathcal{Z}$ satisfies the conditions of a root system $R(S,L)_{\mathrm{B}_l}$ extended by G for $l \geq 2$, and $\mathcal{J}$ is graded isomorphic to some $\mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i\in I})$. ∎

Since 1-dimensionality of (B_l, G)-graded Lie algebras reflects 1-dimensionality of G-graded Jordan algebras, we obtain the following as a corollary of Theorem 6.6:

Theorem 7.2. *Let $\mathcal{L}$ be a centreless Lie G-torus of type B_l for $l \geq 3$ with root system $R(S,L)_{\mathrm{B}_l}$. Then $\mathcal{L} = \mathfrak{B}(\mathcal{J})$ for some $\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i\in I})$. Conversely, for $l \geq 2$, $\mathfrak{B}(\mathcal{J})$ is a centreless Lie G-torus for any $\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i\in I})$.* ∎

Example 7.3. (1) Let $F = \mathbb{Q}$, and let G, S and L be in Example 3.5(1). Let $\mathcal{Z} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, and $\mathcal{W}$ a vector space over $\mathcal{Z}$ with basis $\{w_1, w_2\}$. Define a $\mathcal{Z}$-bilinear form f on $\mathcal{W}$ by

$$f(w_i, w_j) = \begin{cases} \sqrt{2} & \text{if } i = j = 1 \\ \sqrt{3} & \text{if } i = j = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the Jordan algebra $\mathcal{J} = \mathcal{Z} \oplus \mathcal{W}$ of f. Let $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$. Then

$$\mathcal{J}_{\varepsilon_1} := \mathbb{Q}w_1, \quad \mathcal{J}_{\varepsilon_2} := \mathbb{Q}w_2 \quad \text{and} \quad \mathcal{J}_{\varepsilon_1+\varepsilon_2} := 0$$

determine a unique G-grading of $\mathcal{J}$ so that $\mathcal{Z}$ is L-graded. Thus $\mathcal{J}$ is a Clifford G-torus with $S = \operatorname{supp} \mathcal{J}$ and $L = \operatorname{supp} \mathcal{Z}$. The Lie G-torus $\mathfrak{B}(\mathcal{J})$ is a finite-dimensional simple Lie algebra over $\mathbb{Q}$ since $\mathcal{J}$ is a finite-dimensional Jordan division algebra.

(2) Any centreless Lie $\mathbb{Q}$-torus $\mathcal{L}$ over F of type B_l for $l \geq 3$ is untwisted, i.e., $\mathcal{L} = \mathfrak{B}(\mathcal{Z}) = \mathfrak{g} \otimes_F \mathcal{Z}$, where $\mathcal{Z}$ is a commutative associative $\mathbb{Q}$-torus over F, by the reason in Example 3.5(2).

(3) If $G = \mathbb{Z}$, then for $l \geq 2$, we have $R = R(\mathbb{Z}, \mathbb{Z})_{\mathrm{B}_l}$ or $R(\mathbb{Z}, 2\mathbb{Z})_{\mathrm{B}_l}$ (see Example 3.5(3)). For the first case, the Clifford $\mathbb{Z}$-torus $\mathcal{J} = \mathcal{Z}$ can be identified with the group algebra $F[\mathbb{Z}]$ or equivalently the algebra of Laurent polynomials $F[t, t^{-1}]$, and the centreless Lie 1-torus $\mathfrak{B}(\mathcal{Z})$ is an untwisted loop algebra $\mathfrak{g} \otimes_F F[t, t^{-1}]$ of type $\mathrm{B}_l^{(1)}$.

For the second case, the Clifford $\mathbb{Z}$-torus $\mathcal{J} = \mathcal{J}(\mathbb{Z}, 2\mathbb{Z}, \mathcal{Z}, \{z\})$ is written as $\mathcal{J} = \mathcal{Z} \oplus \mathcal{Z}w$, where $0 \neq w \in \mathcal{J}_1$ and $w^2 = z \in \mathcal{Z}_2$. Also, $\mathcal{J}$ is associative since $\mathcal{J}$ has rank 2 over $\mathcal{Z}$. We identify $\mathcal{Z}$ with the group algebra $F[2\mathbb{Z}]$, say $\mathcal{Z} = F[2\mathbb{Z}] = \oplus_{m\in\mathbb{Z}} F\overline{2m}$. Since $2\mathbb{Z}$ is a free group generated by 2, we can put $z = \overline{2}$. Hence $\mathcal{J}$ is uniquely determined by the root system $R(\mathbb{Z}, 2\mathbb{Z})_{\mathrm{B}_l}$, and $\mathcal{J} = F[\mathbb{Z}] = \oplus_{m\in\mathbb{Z}} F\overline{m}$ is also a group algebra, where $\overline{1} = w$. Thus $\mathcal{J}$ can be again identified with $F[t, t^{-1}]$, and the centreless Lie 1-torus $\mathfrak{B}(\mathcal{J})$ is a twisted loop algebra $(\mathfrak{g} \otimes_F F[t^{\pm 2}]) \oplus (V \otimes_F tF[t^{\pm 2}])$ of type $\mathrm{D}_{l+1}^{(2)}$.

In the construction of a Clifford torus $\mathcal{J} = \mathcal{J}(S, L, \mathcal{Z}, \{z_i\}_{i\in I})$ above, the commutative associative L-torus $\mathcal{Z}$ over F is a group algebra

$$\mathcal{Z} = F[L] = \bigoplus_{l\in L} F\bar{l}$$

if L is free. Also, if any element of F has a square root in F, then one can make $z_i = \bar{l}_i$ for all $i \in I$ by switching w_i to

$$\frac{1}{\sqrt{a_i}} w_i \quad \text{where } a_i = z_i \bar{l}_i^{-1} \in F.$$

Thus $\mathcal{J}$ is uniquely determined by S and L, say $\mathcal{J} = \mathcal{J}(S, L)$. If G is free, so is L. Hence we have:

Theorem 7.4. *If G is free and if any element of F has a square root in F, e.g. F is an algebraically closed field, then there is a one-to-one correspondence between the centreless Lie G-tori of type B_l and the root systems extended by G of type B_l for $l \geq 3$.* ■

Remark 7.5. Division (Δ, G)-graded Lie algebras for the other types, $\Delta = \mathrm{A}_1$, C_l, F_4, G_2 or BC_l (see Example 4.3(3)) are not classified yet, even in the case $G = \mathbb{Z}^n$.

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Yoji Yoshii
Department of Mathematics and Statistics
University of Saskatchewan
106 Wiggins Rd, Saskatoon,
SK., S7N 5E6 Canada
yoshii@math.usask.ca

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