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Extended Affine Root Systems of Type BC

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Abstract. We classify the BC-type extended affine root systems for nullity ≤ 3, in its most general sense. We show that these abstractly defined root systems are the root systems of a class of Lie algebras which are axiomatically defined and are closely related to the class of extended affine Lie algebras.

0 Introduction

The term *extended affine root system* was axiomatically introduced in 1985 by K. Saito [11]. This term was also introduced in a different set of axioms in 1997 by B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola. The relation between these two sets of axioms is clarified in [5], in particular, it is shown that there is a one to one correspondence between these two sets of axioms. More precisely, the set of nonisotropic roots of an extended affine root system is a Saito extended affine root system, and a Saito extended affine root system can be enlarged (by certain isotropic elements) in some prescribed way to get an extended affine root system. If one rearranges the axioms of [1] in some particular way, a better relation can be seen between these two classes of root systems. This allows to use the term extended affine root system (EARS for short) for both classes.

Extended affine root systems are natural generalization of finite and affine root systems. According to the original definition of Saito, an EARS (when added with certain isotropic roots) in a real vector space $\mathcal{V}$ equipped with a positive semidefinite symmetric bilinear form, is a discrete subset R of $\mathcal{V}$ such that R spans $\mathcal{V}$, $R=-R$ and that the root string property holds for the set of roots (see Definition 1.9). The dimension ν of the radical $\mathcal{V}^0$ of the form is called the *nullity* of R. Finite and affine root systems are EARS's of nullity 0 and 1, respectively. It follows that the image of R in the quotient space $\mathcal{V}/\mathcal{V}^0$ is a finite root system (possibly non-reduced) whose rank and type are called the *rank* and the *type* of R, respectively. A root belonging to the radical of the form is called an *isotropic* root. R is called *reduced* if two times of a nonisotropic root is not a root.

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Finally, R is called *irreducible* if the set of nonisotropic roots is indecomposable and isotropic roots are nonisolated (see Definition 1.9). Our goal is to classify irreducible BC-type EARSs of nullity ≤ 3.

The classification of EARS started in 1985 with the work of K. Saito [11], where he achieved a complete classification of reduced EARS of nullity 2, up to a notion of marking, meaning that the root system modulo a certain one-dimensional space, called a *marking*, is reduced. In 1997 a systematic study of EARS was carried out in [1] and a complete description of (irreducible reduced) EARS's was achieved, using a concept of semilattice (or translated semilattice). The methods used in [1] allow the authors to give a complete classification of such root systems for certain types (of arbitrary nullity) and to give a complete classification list for other types, up to some specific nullity. In particular, they list the classification table of (irreducible reduced) BC–type EARS of nullity less than or equal 2. Because of the notion of marking, the Saito's classification list contains less root systems than [1]. Using the methods in [1] and a notion of duality, the classification for irreducible BC-type EARSs of nullity 3 is obtained in [6]. In [5, Remarks 1.2, 1.3] it is shown that by applying the notion of marking one reduces significantly the number of root systems.

The objective of this work is to classify all irreducible EARS's of nullity ≤ 3 (which are not necessarily reduced or marked). Such root systems are appearing as the root systems of a class of Lie algebras over a filed of characteristic zero which we define axiomatically, and we call them *toral type extended affine Lie algebras*, see Definition 1.2. The axioms of toral type extended affine Lie algebras can be considered as a generalization of the axioms of extended affine Lie algebras. In particular, when the base field is the field of complex numbers, they contain extended affine Lie algebras. The root systems under consideration also arise as the root systems of the so called *division* $(\Delta,\mathbb{Z}^\nu)$- *graded Lie algebras* (see [14]). For the study of extended affine Lie algebras and their close counterparts we refer reader to [10, 8, 7, 1, 3, 2, 12].

The paper is arranged as follows. In Section 1, we introduce the axioms for toral type extended affine Lie algebras. Starting from one of such Lie algebras, we extract from the axioms the properties of the corresponding root system which turns out to be the same as an EARS. Using the *quantum torus* as the coordinate algebra, we construct a typical example of a toral type extended affine Lie algebra, and describe its root system.

In Section 2, we describe the structure of an EARS in terms of the so called *semilattices* and *translated semilattices*. This allows us to reduce the notion of *isomorphism* (between root systems) to the notion of *similarity* (between triples of semilattices and translated semilattices). In the remaining sections we restrict our attention to the case $\nu = 3$. In Section 3, we show that many facts about (translated) semilattices can be read from some rather simple combinatorics inside the so called (*shifted*) *large* sets. In Sections 4 and 5, we classify BC-triples, the triples which are in a one to one correspondence with EARS. The paper ends with Section 6 which contains the classification tables, where the tables for the cases $\nu = 1, 2$ are given without any details.

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1 Toral type extended affine Lie algebras

In this section we introduce a set of axioms for a new class of Lie algebras over a field $\mathbb{F}$ of characteristic zero. These Lie algebras are closely related to the class of extended affine Lie algebras, however instead of fixing Cartan subalgebras as in [1] or [12], we fix the so called toral subalgebras. We call such a Lie algebra a *toral type extended affine Lie algebra.* With this terminology, the usual extended affine Lie algebras which are examples of our Lie algebras (when $\mathbb{F}=\mathbb{C}$) should be considered as Cartan type. We extract from the axioms the properties of the corresponding root systems, which turn out to be the same as those for an EARS. We construct a typical example of a toral type extended affine Lie algebra and describe its root system in terms of the involved semilattices.

Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic zero. Consider the following axioms for $\mathcal{L}$:

A1) $\mathcal{L}$ is equipped with a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$: $\mathcal{L}\times\mathcal{L}\to\mathbb{F}$.

A2) $\mathcal{L}$ has a non-trivial finite dimensional abelian subalgebra $\mathcal{H}$ such that $\mathrm{ad}_{\mathcal{L}}h$ is diagonalizable for all $h\in\mathcal{H}$ and that $(\cdot,\cdot)_{|\mathcal{H}\times\mathcal{H}}$ is nondegenerate.

Axiom A2 allows us to write

$$\mathcal{L}=\bigoplus_{\alpha\in\mathcal{H}^\star}\mathcal{L}_\alpha,\ \text{where } \mathcal{L}_\alpha=\{x\in\mathcal{L}\mid [h,x]=\alpha(h)x \text{ for all } h\in\mathcal{H}\}.$$

Consider the root system $R:=\{\alpha\in\mathcal{H}^\star\mid \mathcal{L}_\alpha\neq\{0\}\}$ of $\mathcal{L}$ and set

$$R^\times=\{\alpha\in R\mid (\alpha,\alpha)\neq 0\}\quad\text{and}\quad R^0=\{\alpha\in R\mid(\alpha,\alpha)=0\}.$$

Elements of $R^\times$ (R^0) are called *non-isotropic (isotropic)* roots of R. For $\alpha\in\mathcal{H}^\star$, let t_α be the unique element in $\mathcal{H}$ which represents α through the non-degenerate bilinear form on $\mathcal{H}$. For each $\alpha\in R^\times$, set $h_\alpha=2t_\alpha/(t_\alpha,t_\alpha)$.

Our following axioms guarantee the existence of $\mathfrak{sl}_2$-cells, and ad-nilpotency of non-isotropic root vectors, namely:

A3) For any $\alpha\in R$, there exist $x\in\mathcal{L}_\alpha$, $y\in\mathcal{L}_{-\alpha}$ such that $[x,y]=t_\alpha$.

We note that this axiom for $\alpha\in R^\times$ is equivalent to saying that there exist $e_{\pm\alpha}\in\mathcal{L}_{\pm\alpha}$ such that $(e_\alpha,h_\alpha,e_{-\alpha})$ is an $\mathfrak{sl}_2$-triple.

A4) If $\alpha\in R^\times$ and $x_\alpha\in\mathcal{L}_\alpha$, then $\mathrm{ad}_{\mathcal{L}}x_\alpha$ acts locally nilpotently on $\mathcal{L}$.

The next three axioms are related to the root system.

A5) R is *irreducible*, in the sense that it satisfies the following two conditions:

(a) $R^\times$ cannot be written as a disjoint union of two non-empty subsets which are orthogonal with respect to the form.

(b) For $\delta\in R^0$ there exists $\alpha\in R^\times$ such that $\alpha+\delta\in R$.

A6) $\mathcal{V}_{\mathbb{Q}} := \text{span}_{\mathbb{Q}} R$ is finite dimensional over $\mathbb{Q}$.

Let $\mathcal{V}$ be the real vector space obtained by extending the base field from $\mathbb{Q}$ to $\mathbb{R}$, namely

$$\mathcal{V} := \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{V}_{\mathbb{Q}}. \tag{1.1}$$

Considering R as a subset of $\mathcal{V}$, our last axiom says that

A7) R is a discrete subset of $\mathcal{V}$.

Definition 1.2. We call a triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ satifying axioms A1-A7 a *toral type extended affine Lie algebra* (toral type EALA for short). When there is no confusion we simply write $\mathcal{L}$ instead of $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$.

Remark 1.3. We ask the reader to compare the axioms of a toral type EALA with those of an EALA. In particular, one should note that in the definition of an EALA as in [1], it is assumed that $\mathcal{H}$ is a Cartan subalgebra while in our definition $\mathcal{H}$ is just a toral subalgebra, meaning that it is abelian and its elements are ad-diagonalizable. This is why we have used the term *toral type* for these class of Lie algebras. Note that if $\mathcal{H}$ is Cartan, then axiom A1 and ad-diagonalizability of $\mathcal{H}$, implies the existence of $\mathfrak{sl}_2$-triples (axiom A3), and that $\mathcal{L}_0 = \mathcal{H}$. From A2 we have $\mathcal{H} \subseteq \mathcal{L}_0$ and so $0 \in R$. This together with A5(b) gives $R^\times \neq \emptyset$.

In this work we are primarily interested in the structure of root systems arising from toral type EALA. The structure of Lie algebras satisfying the above axioms (or a part of that) will be addressed in another work.

Let us start with a toral type EALA $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$. We may transfer the form from $\mathcal{H}$ to $\mathcal{H}^\star$ by setting $(\alpha, \beta) := (t_\alpha, t_\beta)$ for any $\alpha, \beta \in \mathcal{H}^\star$. This allows us to define for $\alpha \in R^\times$ the reflection $w_\alpha \in GL(\mathcal{H}^\star)$ by

$$w_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

An argument similar to that of [1, Theorem 1.29] gives the following:

Proposition 1.4. *Let $\mathcal{L}$ satisfy A1-A4 and let $\alpha \in R^\times$. Then*
a) For $\beta \in R$, we have $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$.
b) For any $\beta \in R$, we have $w_\alpha(\beta) \in R$.

Proposition 1.5. *Let $\mathcal{L}$ satisfy A1-A4 and let $\alpha \in R^\times$. Then*
a) $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = \mathbb{F}h_\alpha \Rightarrow \mathbb{F}\alpha \cap R = \{0, \pm\alpha\}$.
b) $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = \mathbb{F}h_\alpha \Leftrightarrow \dim \mathcal{L}_\alpha = \dim \mathcal{L}_{-\alpha} = 1$.

Proof. a) Let $k \in \mathbb{F}$ and $k\alpha \in R$. By Proposition 1.4(a) we have $\frac{2(k\alpha,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, and so $k \in \mathbb{Q}$. Now an argument identical to that of [1, Theorem 1.29(c)] gives (a).

b) Let $[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = \mathbb{F}h_\alpha$. By (a), we have $\mathbb{F}\alpha \cap R = \{0, \pm\alpha\}$. Again an argument identical to that of [1, Theorem 1.29(d)] gives $\dim \mathcal{L}_\alpha = \dim \mathcal{L}_{-\alpha} = 1$. The converse implication follows immediately from A3. ☐

Assume that $\mathcal{L}$ satisfies A1-A5. For a fixed $\alpha \in R^\times$ consider $k \in \mathbb{F}$ such that $k(\alpha,\alpha) = 1$. Then it follows from Proposition 1.4(a) and A5(a) that

$$k(\gamma,\beta) \in \mathbb{Q} \quad \text{for all} \quad \beta, \gamma \text{ in the } \mathbb{Q}\text{-span of } R. \tag{1.6}$$

This allows us to assume that *the form $(\cdot,\cdot)$ is $\mathbb{Q}$-valued on $\mathcal{V}_{\mathbb{Q}}$.*

Proposition 1.7. *Let $\mathcal{L}$ satisfy A1-A5. Then*

(a) For $\alpha \in R^\times$ and $\beta \in R$ there exist two non-negative integers u, d such that for any $n \in \mathbb{Z}$ we have $\beta + n\alpha \in R$ if and only if $-d \le n \le u$. Moreover, $d - u = 2(\beta,\alpha)/(\alpha,\alpha)$.

(b) $(R, R^0) = \{0\}$.

Proof. To see (a) just follow the argument in the proof of [1, Theorem 1.29(e)], keeping in mind that the form on R is $\mathbb{Q}$-valued.

(b) First we prove $(R^\times, R^0) = \{0\}$. Suppose to the contrary that $\alpha \in R^\times$, $\delta \in R^0$ and $(\alpha,\delta) \neq 0$. Since A3 holds, one can use the same proof as [1, Lemma I.1.30] to see that $\alpha + n\delta \in R$ for infinitely many $n \in \mathbb{Z}$. But at most for one n, $\alpha + n\delta \in R^0$. So by (a), $(\delta, (\alpha+n\delta)^\vee) \in \mathbb{Z}$ for infinitely many $n \in \mathbb{Z}$ which is a contradiction. Now using A5(b) it follows easily that $(R^0, R^0) = \{0\}$. (For a proof without using A5(b) see [4]). □

¿From now on we assume that $\mathcal{L}$ satisfies A1-A6. Let R be the root system of $\mathcal{L}$ with respect to $\mathcal{H}$ and let $\mathcal{V}$ be as in (1.1). The form on $\mathcal{V}_\mathbb{Q}$ extends canonically to $\mathcal{V}$ and is real valued by (1.6). Moreover, we can assume that $(\alpha,\alpha) > 0$ for some $\alpha \in R^\times$. In [1], the authors prove a conjecture of V. Kac which was reported in [10], namely the form restricted to the real span of roots is positive semidefinite (when $\mathbb{F} = \mathbb{C}$). We reproduce this proof for general $\mathbb{F}$.

Proposition 1.8. *(The Kac conjecture for fields of characteristic zero) The form on $\mathcal{V}$ is positive semidefinite.*

Proof. We first claim that for each $\beta \in R^\times$, $(\beta,\beta) > 0$. If not, then from A5(a) we have that there exist $\alpha, \beta \in R^\times$ such that $(\alpha,\alpha) > 0$, $(\beta,\beta) < 0$ and $(\alpha,\beta) \neq 0$. Using Proposition 1.7 and replacing α or β with 2α or 2β if necessary, we may assume that $2\alpha, 2\beta \notin R$. If $\alpha+\beta$ or $\alpha-\beta \notin R$, then we get a contradiction as in [1, Lemma I.2.3]. So assume $\alpha \pm \beta \in R$. By A3, we can choose elements $x_{\pm(\alpha\pm\beta)} \in \mathcal{L}_{\pm(\alpha\pm\beta)}$ such that

$$[x_{\alpha+\beta}, x_{-(\alpha+\beta)}] = t_{\alpha+\beta} \quad \text{and} \quad [x_{\alpha-\beta}, x_{-(\alpha-\beta)}] = t_{\alpha-\beta}.$$

Let $S_0 = \mathbb{F}t_\alpha \oplus \mathbb{F}t_\beta$ and let S be the $\mathbb{F}$-span of $\{t_{\alpha\pm\beta}, x_{\pm(\alpha\pm\beta)}\}$. As in [1, Lemma I.2.3], it follows that S is a 6-dimensional simple subalgebra of $\mathcal{L}$. We note that S_0 is a split Cartan subalgebra in the sense of [10, Chapter IV], so S is a split simple Lie algebra with $\dim(S) = 6$ which is a contradiction (see [10, Chapter IV]). This contradiction proves our claim. To show that the form $(\cdot,\cdot)$ restricted to $\mathcal{V}$ is positive semidefinite just follow Lemmas 2.6, 2.10, 2.11 and Theorem 2.14 of [1, Chapter I]). □

Now suppose that $\mathcal{L}$ is toral type EALA. It follows from Propositions 1.8, 1.4 and 1.7 that R is an irreducible extended affine root system in the sense of the following definition.

Definition 1.9. Let $\mathcal{V}$ be a non-trivial finite dimensional real vector space with a positive semidefinite symmetric bilinear form $(.,.)$ and let R be a subset of $\mathcal{V}$. Let

$$R^\times = \{\alpha \in R : (\alpha,\alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R : (\alpha,\alpha) = 0\}.$$

Then $R = R^\times \uplus R^0$ where $\uplus$ means disjoint union. We say that R is an *extended affine root system* (EARS) in $\mathcal{V}$ if R satisfies the following 4 axioms:

(R1) $R=-R$,

(R2) R spans $\mathcal{V}$,

(R3) R is discrete in $\mathcal{V}$,

(R4) if $\alpha\in R^\times$ and $\beta\in R$, then there exist $d,u\in\mathbb{Z}_{\geq 0}$ such that

$$\{\beta+n\alpha : n\in\mathbb{Z}\}\cap R=\{\beta-d\alpha,\ldots,\beta+u\alpha\} \text{ and } d-u=2\frac{(\alpha,\beta)}{(\alpha,\alpha)}.$$

The EARS R is called *irreducible* if it satisfies the following two conditions,

(R5) (a) $R^\times$ cannot be written as a disjoint union of two nonempty subsets which are orthogonal with respect to the form.

(b) Isotropic roots are nonisolated in the sense that for any $\delta\in R^0$, there exists $\alpha\in R^\times$ such that $\alpha+\delta\in R$.

Finally R is called *reduced* if it satisfies:

(R6) $\alpha\in R^\times\Rightarrow 2\alpha\notin R$.

Suppose that R is an irreducible EARS. From (R2), we have $R\neq\emptyset$. Then (R5)(b) implies $R^\times\neq\emptyset$. This in turn implies that $0\in R$. Note that an EARS as defined here could have both isolated and nonisolated roots. However, if axiom (R5) is satisfied, the isotropic roots are nonisolated (see [7] for examples of Lie algebras which do not satisfy this condition). By [5], there is a one to one correspondence between irreducible (reduced) EARS and indecomposable (reduced) extended affine root systems defined by K. Saito [13].

We close this section with a typical example of a toral type EALA of type BC (for the definition of type see Section 2). We show that the root system R of $\mathcal{L}$ does not satisfy axiom (R6). In other words, R is not reduced. Our setting will be similar to [1, III.3] and our coordinate algebra will be the quantum torus. However, we consider the real numbers as the base field and we work with a semi-linear involution.

Example 1.10. Consider the quantum torus $\mathcal{A}=\mathbb{C}_{-1}[t_1^{\pm1},\cdots,t_\nu^{\pm1}]$. That is, $\mathcal{A}$ is the associative algebra with generators $t_i^{\pm1}$, $1\leq i\leq\nu$ subject to the relations $t_it_j=-t_jt_i$ for $i\neq j$. Then

$$\mathcal{A}=\bigoplus_{\sigma\in\mathbb{Z}^\nu}\mathbb{C}t^\sigma=\bigoplus_{\sigma\in\mathbb{Z}^\nu}(\mathbb{R}t^\sigma\oplus\mathbb{R}it^\sigma) \text{ where}$$
$$t^\sigma=t_1^{n_1}\cdots t_\nu^{n_\nu} \text{ for } \sigma=(n_1,\ldots,n_\nu)\in\mathbb{Z}^\nu.$$

Consider the opposite algebra $\mathcal{A}^{\text{op}}$ with the scalar product defined by

$$a\cdot x=\bar{a}x,\quad x\in\mathcal{A}, a\in\mathbb{C}.$$

Then $\mathcal{A}^{\text{op}}$ is an associative algebra satisfying $t_it_j=-t_jt_i$, for $i\neq j$. Hence there exists a linear map $\bar{\ }:\mathcal{A}\to\mathcal{A}^{\text{op}}$ such that $\bar{t_i}\mapsto t_i$. In fact $\bar{\ }$ is the semi-linear involution on $\mathcal{A}$ defined by

$$\bar{t_i}=t_i,\quad 1\leq i\leq\nu \quad\text{and}\quad \overline{xt^\sigma}=\bar{x}\bar{t^\sigma},\quad x\in\mathbb{C},\ \sigma\in\mathbb{Z}^\nu.$$

Let

$$\Lambda = \mathbb{Z}^\nu \quad \text{and} \quad \tilde{\Lambda} = \Lambda/2\Lambda = \mathbb{F}_2^\nu.$$

For $\sigma = (n_1, \ldots, n_\nu) \in \Lambda$ set $\kappa_\sigma := \sum_{i<j} n_i n_j$ and define a quadratic form $Q : \tilde{\Lambda} \to \mathbb{F}_2$ by

$$Q(\tilde{\sigma}) = \kappa_\sigma \ (\text{mod}\ \mathbb{F}_2).$$

Set $Z = \{\sigma \in \mathbb{Z}^\nu \mid Q(\tilde{\sigma}) = 0\}$. Let us call $\sigma \in \mathbb{Z}^\nu$ *even* if κ_σ is even and call it *odd* if κ_σ is odd. So $Q(\tilde{\sigma}) = 0$ if and only if σ is even. Note that $\overline{t^\sigma} = (-1)^{\kappa_\sigma} t^\sigma$.

Now suppose that $m \geq 1$ and $\tau_1, \ldots, \tau_m$ are elements of $\mathbb{Z}^\nu$ such that

$\tau_1 = 0,$
$\tau_1, \cdots, \tau_m$ represent distinct cosets of $2\mathbb{Z}^\nu$ in $\mathbb{Z}^\nu$,
$\tau_i \in Z$ for $i = 1, \ldots, m$.

Let $\ell \geq 1$ and set

$$J = \begin{pmatrix} 0 & I_\ell & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \text{ where } F = \begin{pmatrix} t^{\tau_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\tau_m} \end{pmatrix}.$$

Set $n = 2\ell + m$ and consider the involution * on the associative algebra $M_n(\mathcal{A})$ defined by

$$X^* = J^{-1} \bar{X}^t J.$$

Set $\mathcal{G} = \{X \in M_n(\mathcal{A}) \mid X^* = -X\}$. Then $\mathcal{G}$ is a subalgebra of $\mathfrak{gl}_n(\mathcal{A})$, with involution *. It follows that $X \in \mathcal{G}$ if and only if

$$X = \begin{pmatrix} A & S & -\bar{N}^t F \\ T & -\bar{A}^t & -\bar{M}^t F \\ M & N & B \end{pmatrix} \text{ with } \bar{B}^t F = -FB,\ \bar{S}^t = -S,\ \bar{T}^t = -T,$$

where $A, S, T \in M_\ell(\mathcal{A})$, $M, N \in M_{\ell,m}(\mathcal{A})$ and $B \in M_m(\mathcal{A})$. Next set $\dot{\mathcal{H}} = \sum_{i=1}^{\ell} \mathbb{R}\dot{h}_i$ where $\dot{h}_i = e_{i,i} - e_{\ell+i,\ell+i}$ for $1 \leq i \leq \nu$. Then

$$[\dot{h}_t, e_{2\ell+i,2\ell+j}] = 0 \ \text{ for } \ 1 \leq t \leq \ell, \ \ 1 \leq i, j \leq m.$$

For $1 \leq i \leq \ell$, define $\epsilon_i \in \dot{\mathcal{H}}^\star$ by $\epsilon_i(\dot{h}_j) = \delta_{ij}$. Now $\mathcal{G} = \sum_{\dot{\alpha} \in \dot{\mathcal{H}}^\star} \mathcal{G}_{\dot{\alpha}}$ where $\mathcal{G}_{\dot{\alpha}} = \{x \in \mathcal{G} \mid [h, x] = \dot{\alpha}(h)x, \text{ for all } h \in \dot{\mathcal{H}}\}$. Set $\dot{R} = \{\dot{\alpha} \in \dot{\mathcal{H}}^\star \mid \mathcal{G}_{\dot{\alpha}} \neq \{0\}\}$. Then $\dot{R} \setminus \{0\} = \dot{R}_{sh} \cup \dot{R}_{lg} \cup \dot{R}_{ex}$, where

$$\dot{R}_{sh} = \{\pm\epsilon_i \mid 1 \leq i \leq \ell\}, \quad \dot{R}_{lg} = \{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i \neq j \leq \ell\} \text{ and}$$
$$\dot{R}_{ex} = \{\pm 2\epsilon_i \mid 1 \leq i \leq \ell\}.$$

In fact

$$\begin{aligned}
\mathcal{G}_{\epsilon_i - \epsilon_j} &= \{a e_{ij} - \bar{a} e_{\ell+j,\ell+i} \mid a \in \mathcal{A}\}, \\
\mathcal{G}_{\epsilon_i + \epsilon_j} &= \{a e_{i,\ell+j} - \bar{a} e_{j,\ell+i} \mid a \in \mathcal{A}\}, \\
\mathcal{G}_{-\epsilon_i - \epsilon_j} &= \{a e_{\ell+i,j} - \bar{a} e_{\ell+j,i} \mid a \in \mathcal{A}\}, \\
\mathcal{G}_{\epsilon_i} &= \{\textstyle\sum_{j=1}^{m} (a_j e_{2\ell+j,\ell+i} - \bar{a}_j t^{\tau_j} e_{i,2\ell+j}) \mid a_1, \ldots, a_m \in \mathcal{A}\}, \\
\mathcal{G}_{-\epsilon_i} &= \{\textstyle\sum_{j=1}^{m} (a_j e_{2\ell+j,i} - \bar{a}_j t^{\tau_j} e_{\ell+i,2\ell+j}) \mid a_1, \ldots, a_m \in \mathcal{A}\}, \\
\mathcal{G}_{2\epsilon_i} &= \{a e_{i,\ell+i} \mid a \in \mathcal{A}, \bar{a} = -a\}, \\
\mathcal{G}_{-2\epsilon_i} &= \{a e_{\ell+i,i} \mid a \in \mathcal{A}, \bar{a} = -a\},
\end{aligned} \tag{1.11}$$

and

$$\mathcal{G}_0=\left\{\begin{pmatrix} A & 0 & 0\\ 0 & -\bar{A}^t & 0\\ 0 & 0 & B\end{pmatrix} \mid A \text{ is diagonal, } F^{-1}\bar{B}^tF=-B\right\}. \tag{1.12}$$

More precisely,

$$\begin{aligned}\mathcal{G}_0=\text{span}_{\mathbb{R}}\{\bar{a}t^{\tau_j}e_{2\ell+i,2\ell+j}-at^{\tau_i}e_{2\ell+j,2\ell+i}\mid a\in\mathcal{A},\ 1\le i\ne j\le m\}\\ \oplus\text{span}_{\mathbb{R}}\{bt^{\tau_j}e_{2\ell+j,2\ell+j}\mid b\in\mathcal{A},\ \bar{b}=-b,\ 1\le j\le m\}\\ \oplus\text{span}_{\mathbb{R}}\{ae_{kk}-\bar{a}e_{k+\ell,k+\ell}\mid a\in\mathcal{A},\ 1\le k\le\ell\}.\end{aligned} \tag{1.13}$$

Next, we would like to consider $\mathcal{G}$ as a $\mathbb{Z}^\nu$-graded Lie algebra. We start with a gradation on $M_n(\mathcal{A})$, as a vector space. For $1\le p,q\le n$, set

$$\begin{gathered}\deg(it^\sigma e_{pq})=\deg(t^\sigma e_{pq})=2\sigma+\lambda_p-\lambda_q \text{ where}\\ \lambda_1=\cdots=\lambda_{2\ell}=0 \text{ and } \lambda_{2\ell+1}=\tau_1,\ldots,\lambda_n=\tau_m.\end{gathered}$$

This defines a $\mathbb{Z}^\nu$-grading on $M_n(\mathcal{A})$ and in turn on $\mathfrak{gl}_n(\mathcal{A})$. Moreover, the involution * preserves the grading on $M_n(\mathcal{A})$, thus $\mathcal{G}$ is also a $\mathbb{Z}^\nu$-graded subalgebra of $\mathfrak{gl}_n(\mathcal{A})$, $\mathcal{G}=\sum_{\sigma\in\mathbb{Z}^\nu}\mathcal{G}^\sigma$. It then can be read easily from (1.11) and (1.13) that for any $\dot\alpha\in\dot R$

$$\mathcal{G}_{\dot\alpha}=\bigoplus_{\sigma\in\mathbb{Z}^\nu}(\mathcal{G}_{\dot\alpha}\cap\mathcal{G}^\sigma).$$

Now we want to define a form $(\cdot,\cdot)$ on $\mathcal{G}$. For this, we first define $\epsilon:\mathcal{A}\to\mathbb{R}$ by linear extension of

$$\epsilon(t^\sigma)=\begin{cases}1 & \text{if } \sigma=0\\ 0 & \text{if } \sigma\ne 0\end{cases} \quad\text{and}\quad \epsilon(it^\sigma)=0.$$

Then $(a,b)\mapsto\epsilon(ab)$ is a non-degenerate symmetric bilinear form on $\mathcal{A}$ preserved by $^-$. Therefore, the form on $M_n(\mathcal{A})$ defined by

$$(A,B)=\epsilon(tr(AB))$$

is an invariant symmetric non-degenerate bilinear form. Since $\epsilon(tr(A))=\epsilon(tr(\bar{A}))$ we get $(A^*,B^*)=(A,B)$. It then follows that $(\cdot,\cdot)$ is a non-degenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{gl}_n(\mathcal{A})$ whose restriction to $\mathcal{G}$ is also non-degenerate, see [1, Lemma III.3.21].

We now want to extend the Lie algebra $\mathcal{G}$ to a bigger Lie algebra satisfying axioms A1-A7, as follows. For $1\le i\le\nu$ define $d_i\in\text{Der}(\mathcal{G})$ by

$$d_ix=n_ix$$

for $x\in\mathcal{G}^{(n_1,\cdots,n_\nu)}$. It follows that d_i's are linearly independent. Set $\mathcal{D}=\bigoplus_{i=1}^\nu\mathbb{R}d_i\subseteq\text{Der}(\mathcal{G})$. Consider a ν-dimensional real vector space $\mathcal{C}=\bigoplus_{i=1}^\nu\mathbb{R}c_i$ and set

$$\mathcal{L}=\mathcal{G}\oplus\mathcal{C}\oplus\mathcal{D},\quad\text{and}\quad \mathcal{H}=\dot{\mathcal{H}}\oplus\mathcal{C}\oplus\mathcal{D}. \tag{1.14}$$

Define an anti-commutative product $[.,.]'$ on $\mathcal{L}$ as follows:

$$\begin{gathered}[\mathcal{L},\mathcal{C}]'=\{0\},\ \ [\mathcal{D},\mathcal{D}]'=\{0\},\\ [d_i,x]'=d_ix \text{ for all } x\in\mathcal{G} \text{ and}\\ [x,y]'=[x,y]+\textstyle\sum_{i=1}^\nu(d_ix,y)c_i \text{ for all } x,y\in\mathcal{G}.\end{gathered}$$

Then $\mathcal{L}$ is a Lie algebra over $\mathbb{R}$ and $\mathcal{H}$ is an abelian subalgebra of $\mathcal{L}$.

Next, we extend the form $(.,.)$ on $\mathcal{G}$ to $\mathcal{L}$ by requiring

$$\begin{gathered}(c_i, d_j) = \delta_{ij}, 1 \leq i, j \leq \nu, \text{ and} \\ (\mathcal{C},\mathcal{C}) = (\mathcal{D},\mathcal{D}) = \{0\} = (\mathcal{C},\mathcal{G}) = (\mathcal{D},\mathcal{G}).\end{gathered} \tag{1.15}$$

This defines a non-degenerate symmetric invariant form on $\mathcal{L}$ whose restriction to $\mathcal{H}$ is also non-degenerate.

We can identify

$$\mathcal{H}^\star = \dot{\mathcal{H}}^\star \oplus \mathcal{C}^\star \oplus \mathcal{D}^\star.$$

Let $\{\delta_1, \cdots, \delta_\nu\}$ be the basis of $\mathcal{D}^\star$ dual to $\{d_1, \cdots, d_\nu\}$ and $\{\gamma_1, \cdots, \gamma_\nu\}$ be the basis of $\mathcal{C}^\star$ dual to $\{c_1, \cdots, c_\nu\}$. Identify $\mathbb{Z}^\nu \subset \mathcal{D}^\star$ by considering an element $(n_1, \cdots, n_\nu) \in \mathbb{Z}^\nu$ as the element $\sum_{i=1}^{\nu} n_i\delta_i \in \mathcal{D}^\star$. Then

$$[d, x]' = \sigma(d)x \text{ for } d \in \mathcal{D}, x \in \mathcal{G}^\sigma, \sigma \in \mathbb{Z}^\nu. \tag{1.16}$$

One can check easily that c_i and d_i, represent δ_i and γ_i for $1 \leq i \leq \nu$, respectively and $\dot{h}_j/2 \in \mathcal{H}$ represents ϵ_j, for $1 \leq j \leq \ell$.

For $\alpha \in \mathcal{H}^\star$ set $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$ and

$$R = \{\alpha \in \mathcal{H}^\star \mid \mathcal{L}_\alpha \neq \{0\}\}.$$

Then it follows from our gradation on $\mathcal{G}$ and the above setting that

$$\mathcal{L} = \sum_{\alpha \in \mathcal{H}^\star} \mathcal{L}_\alpha = \sum_{\alpha \in R} \mathcal{L}_\alpha = \sum_{\sigma \in \mathbb{Z}^\nu} \sum_{\dot{\alpha} \in \dot{R}} \mathcal{L}_{\dot{\alpha}+\sigma}, \tag{1.17}$$

where

$$\mathcal{L}_0 = (\mathcal{G}_0 \cap \mathcal{G}^0) \oplus \mathcal{C} \oplus \mathcal{D} \quad \text{and} \quad \mathcal{L}_{\dot{\alpha}+\sigma} = \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^\sigma \tag{1.18}$$

for $\dot{\alpha} \in \dot{R}$, $\sigma \in \mathbb{Z}^\nu$ with $\dot{\alpha} + \sigma \neq 0$. This completes our construction of a triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$.

In the next proposition we summarize the results obtained about the Lie algebra $\mathcal{L}$ constructed in Example 1.10.

Proposition 1.19. *Let $\mathcal{L}$ and $\mathcal{H}$ be defined by (1.14), and let $(\cdot,\cdot)$ be defined by (1.15). Then $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ is a toral type EALA with corresponding root system*

$$\begin{gathered}R = (S + S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + 2\mathbb{Z}^\nu) \cup (\dot{R}_{ex} + 2\mathbb{Z}^\nu) \text{ where} \\ S = \bigcup_{i=1}^{m} (2\mathbb{Z}^\nu + \tau_i).\end{gathered}$$

Moreover,

$$\mathcal{L}_0 = \dot{\mathcal{H}} \oplus \big(\bigoplus_{k=1}^{\ell} i\mathbb{R}(e_{kk} + e_{\ell+k,\ell+k})\big) \oplus \sum_{j=1}^{m} i\mathbb{R}e_{2\ell+j,2\ell+j} \oplus \mathcal{C} \oplus \mathcal{D}.$$

In particular, R is a non-reduced irreducible EARS and $\mathcal{H} \subsetneq \mathcal{L}_0$.

Proof. From (1.15) and (1.17) we have that axioms A1, A2 hold. It also follows from (1.18), (1.13) and (1.11) that $\mathcal{L}_0$ and R are of the forms as in the statement. In particular it is clear from the structure of R that axioms A4-A7 hold and that $2S\cap 2\mathbb{Z}^\nu\neq\emptyset$ and so R is not reduced. Therefore, it only remains to check axiom A3. For this, let us first introduce some symbols. For $1\leq i\neq j\leq \ell$, $1\leq k\leq m$ and $\sigma\in\mathbb{Z}^\nu$ set

$$\begin{gathered}
A_{i,j,\sigma}=t^\sigma(e_{i,j}+(-1)^{\kappa_\sigma+1}e_{\ell+j,\ell+i}),\quad B_{i,j,\sigma}=t^\sigma(e_{i,j}+(-1)^{\kappa_\sigma}e_{\ell+j,\ell+i}),\\
C_{i,j,\sigma}=t^\sigma(e_{i,\ell+j}+(-1)^{\kappa_\sigma+1}e_{j,\ell+i}),\quad D_{i,j,\sigma}=t^\sigma(e_{i,\ell+j}+(-1)^{\kappa_\sigma}e_{j,\ell+i}),\\
M_{i,k,\sigma}=t^\sigma(e_{2\ell+k,\ell+i}+(-1)^{\kappa_\sigma+1}t^{\tau_k}e_{i,2\ell+k}),\\
N_{i,k,\sigma}=t^\sigma(e_{2\ell+k,\ell+i}+(-1)^{\kappa_\sigma}t^{\tau_k}e_{i,2\ell+k}),\\
A'_{i,j,\sigma}=A_{j,i,-\sigma},\qquad B'_{i,j,\sigma}=B_{j,i,-\sigma},\\
C'_{i,j,\sigma}=t^\sigma(e_{i+\ell,j}+(-1)^{\kappa_\sigma+1}e_{j+\ell,i}),\quad D'_{i,j,\sigma}=t^\sigma(e_{i+\ell,j}+(-1)^{\kappa_\sigma}e_{j+\ell,i}),\\
M'_{i,k,\sigma}=t^\sigma(e_{2\ell+k,i}+(-1)^{\kappa_\sigma+1}t^{\tau_k}e_{\ell+i,2\ell+k}),\\
N'_{i,k,\sigma}=t^\sigma(e_{2\ell+k,i}+(-1)^{\kappa_\sigma}t^{\tau_k}e_{\ell+i,2\ell+k}).
\end{gathered}$$

It can be read from (1.11) and (1.18) that

$$\begin{aligned}
\mathcal{L}_{\epsilon_i-\epsilon_j+2\sigma} &= \mathbb{R}A_{i,j,\sigma}+i\mathbb{R}B_{i,j,\sigma},\\
\mathcal{L}_{\epsilon_i+\epsilon_j+2\sigma} &= \mathbb{R}C_{i,j,\sigma}+i\mathbb{R}D_{i,j,\sigma},\\
\mathcal{L}_{-\epsilon_i-\epsilon_j+2\sigma} &= \mathbb{R}C'_{i,j,\sigma}+i\mathbb{R}D'_{i,j,\sigma},\\
\mathcal{L}_{\epsilon_i+2\sigma+\tau_k} &= \mathbb{R}M_{i,k,\sigma}+i\mathbb{R}N_{i,k,\sigma},\\
\mathcal{L}_{-\epsilon_i+2\sigma+\tau_k} &= \mathbb{R}M'_{i,k,\sigma}+i\mathbb{R}N'_{i,k,\sigma},
\end{aligned}$$

$$\mathcal{L}_{2\epsilon_i+2\sigma}=\begin{cases}\mathbb{R}t^\sigma e_{i,\ell+i} & \text{if } \sigma \text{ is odd}\\ i\mathbb{R}t^\sigma e_{i,\ell+i} & \text{if } \sigma \text{ is even}\end{cases}$$

$$\mathcal{L}_{-2\epsilon_i+2\sigma}=\begin{cases}\mathbb{R}t^\sigma e_{\ell+i,i} & \text{if } \sigma \text{ is odd}\\ i\mathbb{R}t^\sigma e_{\ell+i,i} & \text{if } \sigma \text{ is even}\end{cases}$$

Now for each $\alpha\in R^\times$ we introduce $e_\alpha\in\mathcal{L}_\alpha$, $e_{-\alpha}\in\mathcal{L}_{-\alpha}$ as follows:

$$\begin{array}{lll}
\alpha=\epsilon_i-\epsilon_j+2\sigma: & e_\alpha:=(-1)^{\kappa_\sigma}A_{i,j,\sigma}, & e_{-\alpha}=A'_{i,j,\sigma},\\
\alpha=\epsilon_i+\epsilon_j+2\sigma: & e_\alpha:=C_{i,j,\sigma}, & e_{-\alpha}=-C'_{i,j,-\sigma},\\
\alpha=2\epsilon_i+2\sigma,\ (\sigma\text{ odd }): & e_\alpha:=t^\sigma e_{i,\ell+i}, & e_{-\alpha}=-t^{-\sigma}e_{\ell+i,i},\\
\alpha=2\epsilon_i+2\sigma,\ (\sigma\text{ even }): & e_\alpha:=it^\sigma e_{i,\ell+i}, & e_{-\alpha}=-it^{-\sigma}e_{\ell+i,i},\\
\alpha=\epsilon_i+2\sigma+\tau_k,: & e_\alpha: mM_{i,k,\sigma}, & e_{-\alpha}=m'M'_{i,k,\sigma'},
\end{array}$$

in which σ', m' are given by

$$\sigma'=-\sigma-\tau_k,\quad mm'g(\sigma,\tau_k)(-1)^{\kappa_\sigma+\kappa_{\sigma'}+1}=2,$$

where $g:\mathbb{Z}^\nu\times\mathbb{Z}^\nu\to\{\pm1\}$ is defined by

$$g(a,b)=\prod_{1\leq i<j\leq\nu}(-1)^{a_jb_i}.$$

The properties of the function g, defined in a more general setting, are studied in [7, §2].

Also, for each $\delta\in R^0$, consider $x_\delta\in\mathcal{L}_\delta$, $x_{-\delta}\in\mathcal{L}_{-\delta}$ as follows: If $i\neq j,\sigma\in\mathbb{Z}^\nu$ and $\delta=2\sigma+\tau_i+\tau_j$, set

$$x_\delta=rt^\sigma t^{\tau_i}e_{2l+j,2l+i}-r(-1)^{\kappa_\sigma}t^\sigma t^{\tau_j}e_{2l+i,2l+j},$$

$$x_{-\delta} = st^{\sigma'}t^{\tau_i}e_{2l+j,2l+i} - s(-1)^{\kappa_{\sigma'}}t^{\sigma'}t^{\tau_j}e_{2l+i,2l+j}$$

in which $\sigma' = -(\sigma + \tau_i + \tau_j)$ and $r, s \in \mathbb{R}$ satisfy $-2rsg(\sigma, \tau_i)g(\tau_j, \sigma)g(\tau_j, \tau_i) = 1$. Also for $\delta = 2\sigma \in 2\mathbb{Z}^\nu$ set

$$x_\delta = at^\sigma e_{kk} - a(-1)^{\kappa_\sigma}t^\sigma e_{k+l,k+l}, \quad x_{-\delta} = ct^{-\sigma}e_{kk} - c(-1)^{\kappa_\sigma}t^{-\sigma}e_{k+l,k+l}$$

where $a, c \in \mathbb{R}$ and $2ac(-1)^{\kappa_\sigma} = 1$.

Having elements $e_{\pm\alpha}$ and $x_{\pm\delta}$ defined above, it is now straightforward to check that axiom A3 holds for $\mathcal{L}$, that is, for each $\alpha \in R^\times$

$$(e_\alpha, h_\alpha, e_{-\alpha}) \text{ is an } \mathfrak{sl}_2\text{-triple},$$

and for each $\delta \in R^0$, $t_\delta = [x_\delta, x_{-\delta}]$. This completes the proof. □

2 From isomorphism to similarity

Our goal is to classify all irreducible EARS of nullity ≤ 3. This classification is already achieved for reduced irreducible EARS of nullity ≤ 2 by [13] and [1] and for nullity 3 by [6].

Assume that R is an irreducible EARS in $\mathcal{V}$. In particular, R can be the root system of a toral type EALA $\mathcal{L}$. In [1, Chapter II] the structure of an irreducible reduced EARS is described. In our case the root system R is not necessarily reduced, however by mimicking the arguments in [1] we can get an analogue of the structure obtained there. In what follows we just state the results and leave details to the reader. Let $\mathcal{V}^0$ be the radical of the form. Set $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and let $^- : \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{V}^0$ be the canonical map. One can show that the image $\bar{R}$ of R under the map $^-$ is an irreducible finite root system in $\bar{\mathcal{V}}$. Take a preimage $\dot{R}$ of $\bar{R}$ so that if $\dot{\mathcal{V}}$ is the real span of $\dot{R}$, then $\dot{R}$ is an irreducible finite root system in $\dot{\mathcal{V}}$ isometrically isomorphic to $\bar{R}$. The *rank* and the *type* of R is defined to be the rank and the type of $\bar{R}$, and the nullity of R is defined to be the dimension ν of $\mathcal{V}^0$. If $\mathcal{L}$ is a toral type EALA, the *type* of $\mathcal{L}$ is defined to be the type of its root system R. Note that if $\bar{R}$ has one of the types A, B, C, D, E, F_4 or G_2, the axiom (R6) of Definition 1.9 is automatically satisfied, and so our work is identical to [1] and [6]. Thus from now on we assume that R *is an irreducible EARS of type* BC.

Let $\dot{R}_{sh}$, $\dot{R}_{lg}$ and $\dot{R}_{ex}$ be the set of short, long and extra long roots of $\dot{R}$, respectively. Set

$$S = \{\delta \in \mathcal{V}^0 \mid \delta + \dot{R}_{sh} \subseteq R\}, \quad L = \{\delta \in \mathcal{V}^0 \mid \delta + \dot{R}_{lg} \subseteq R\}, \text{ if } \dot{R}_{lg} \neq \emptyset, \text{ and}$$
$$E = \{\delta \in \mathcal{V}^0 \mid \delta + \dot{R}_{ex} \subseteq R\}.$$

It follows that S, L are semilattices and E is a translated semilattice (see [1, Chapter II] for terminology). Moreover,

$$R = \begin{cases} (S+S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{ex} + E) & \text{if } \dot{R}_{lg} = \emptyset \\ (S+S) \cup (\dot{R}_{sh} + S) \cup (\dot{R}_{lg} + L) \cup (\dot{R}_{ex} + E) & \text{if } \dot{R}_{lg} \neq \emptyset \end{cases} \tag{2.1}$$

where

$$\begin{cases} S + E \subseteq S, \ \ 4S + E \subseteq E & \text{if } \dot{R}_{lg} = \emptyset \\ S + L \subseteq S, \ \ 2S + L \subseteq L, \ \ L + E \subseteq L, \ \ 2L + E \subseteq E, & \text{if } \dot{R}_{lg} \neq \emptyset \end{cases} \tag{2.2}$$

A pair (S, E) (triple (S, L, E)) in $\mathcal{V}^0$ satisfying (2.2) is called *a BC-pair* (*a BC-triple*) in $\mathcal{V}^0$. This description of R leads us to introduce the following construction. Let $\dot{R}$ be an irreducible finite root system of type BC and set

$$R(S,E) = (S+S) \cup (\dot{R}_{sh}+S) \cup (\dot{R}_{ex}+E) \text{ where } (S,E) \text{ is a } BC\text{-pair}, \tag{2.3}$$

$$R(S,L,E) = (S+S) \cup (\dot{R}_{sh}+S) \cup (\dot{R}_{lg}+L) \cup (\dot{R}_{ex}+E) \text{ where } (S,L,E) \text{ is a } BC\text{-triple}. \tag{2.4}$$

It follows that $R(S,E)$ is an EARS of type BC_1 and $R(S,L,E)$ is an EARS of type BC_ℓ $(\ell \geq 2)$, and as we have already seen any EARS of type BC is of the form $R(S,E)$ or $R(S,L,E)$. Similar to [1] one can show that the classification of irreducible EARS of type BC reduces to the classification of BC-triples in $\mathcal{V}^0$, up to similarity. Let us state what we mean by two triples to be isomorphic or similar.

Two BC-triples (S,L,E) and (S',L',E') in $\mathcal{V}^0$ are said to be *isomorphic*, written $(S,L,E) \cong (S',L',E')$, if there exists a linear isomorphism φ of $\mathcal{V}^0$ such that $\varphi(S) = S'$, $\varphi(L) = L'$ and $\varphi(E) = E'$, and are said to be *similar*, written $(S,L,E) \sim (S',L',E')$, if there exists a linear isomorphism φ of $\mathcal{V}^0$ such that $\varphi(S) = S'+\sigma'$, $\varphi(L) = L'+\lambda'$ and $\varphi(E) = E'+2\sigma'$ for some $\sigma' \in S'$ and $\lambda' \in L'$. The relations $\cong$ and $\sim$ are both equivalence relations. Denote by BC the set of BC-triples in $\mathcal{V}^0$. Set

$$\mathcal{T}_1 = \{(S,L,E) \in BC \mid E \text{ is a semilattice}\},$$
$$\mathcal{T}_2 = \{(S,L,E) \in BC \mid 2S \cap E = \emptyset\},$$
$$\mathcal{T}_3 = \{(S,L,E) \in BC \mid 2S \cap E \neq \emptyset \text{ and } E \text{ is not a semilattice}\}.$$

Then $BC = \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3$. Let us denote by $[BC]$ and $[\mathcal{T}_i]$, $1 \leq i \leq 3$, the similarity classes of triples in BC and $\mathcal{T}_i$, $1 \leq i \leq 3$, respectively. Note that by [1, Chapter II], the similarity classes of triples in $\mathcal{T}_2$ are in a 1-1 correspondence with the isomorphism classes of reduced irreducible EARS (in [1] the set $\mathcal{T}_1$ is denoted by $\mathcal{T}$).

Remark 2.5. Since $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are mutually disjoint, it is conceivable that their similarity classes are also mutually disjoint. However, this is not the case. To see this let $S = 2\Lambda \cup (\sigma_1+2\Lambda) \cup (\sigma_2+2\Lambda) \cup (\sigma_3+2\Lambda)$ where $\Lambda = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus \mathbb{Z}\sigma_3$. Then $(S, 2\Lambda, 2(S+\sigma_3)) \in \mathcal{T}_1$, $(S+\sigma_2, 2\Lambda, 2(S+\sigma_2+\sigma_3)) \in \mathcal{T}_3$ and

$$(S+\sigma_2, 2\Lambda, 2(S+\sigma_2+\sigma_3)) \sim (S, 2\Lambda, 2(S+\sigma_3)).$$

This is one of the reasons which makes the classification of BC-type EARS to a complicated problem. One can easily see that $([\mathcal{T}_1] \cup [\mathcal{T}_3]) \cap [\mathcal{T}_2] = \emptyset$.

We classify the similarity classes of triples in BC for $\nu \leq 3$ through the classification of a set of triples closely related to $\mathcal{T}_1$ with respect to a different notion of similarity. We say two triples (S,L,F) and (S',L',F') in $\mathcal{T}_1$ are *(weakly) similar*, denoted $(S,L,F) \approx (S',L',F')$, if there exists $\psi \in GL(\mathcal{V}^0)$ such that $\psi(S) = S'+\sigma'$, $\psi(L) = L'+\lambda'$ and $\psi(F) = F'+\gamma'$ for some $\sigma' \in S'$, $\lambda' \in L'$ and $\gamma' \in F'$. Define a notion of *twist-triple* for a BC-triple exactly as in [1]. Twist-triple is a similarity invariant of BC-triples (triples in $\mathcal{T}_1$) with respect to

$\sim$ ($\approx$). Denote by $BC(t_1,t,t_2)$ the set of BC-triples with twist-triple (t_1,t,t_2). Also denote by $[BC(t_1,t,t_2)]$ ($[[\mathcal{T}_1(t_1,t,t_2)]]$) the set of similarity classes of triples in $BC(t_1,t,t_2)$ ($\mathcal{T}_1(t_1,t,t_2)$) with respect to $\sim$ ($\approx$). Use a similar notation for triples in $\mathcal{T}_2$ and $\mathcal{T}_3$.

The following proposition shows how we can construct all BC-triples out of the similarity classes in $[[\mathcal{T}_1]]$.

Proposition 2.6. *Let $\mathcal{R}_1(t_1,t,t_2)$ be a set of representatives for the similarity classes in $[[\mathcal{T}_1(t_1,t,t_2)]]$. Then*

$$\mathcal{R}(t_1,t,t_2):=\{(S,L,F+\eta)\mid (S,L,F)\in\mathcal{R}_1(t_1,t,t_2),\ \ \eta\in L,\ \ L+\eta\subseteq L\}$$

is a set of representatives for the similarity classes in $BC(t_1,t,t_2)$.

Proof. It is easy to see that for any $(S,L,F)\in\mathcal{T}_1(t_1,t,t_2)$ and any $\eta\in L$ with $L+\eta\subseteq L$, we have $(S,L,F+\eta)\in BC(t_1,t,t_2)$. Conversely, let $(S,L,E)\in BC(t_1,t,t_2)$. Take any $\eta\in E$ and set $F=E-\eta$. Then $\eta+L\subseteq L$ and $(S,L,F)\in\mathcal{T}_1(t_1,t,t_2)$. So there exists $(S',L',F')\in\mathcal{R}_1(t_1,t,t_2)$ and $\psi\in GL(\mathcal{V}^0)$ such that $\psi(S)=S'+\sigma'$, $\psi(L)=L'+\lambda'$, $\psi(F)=F'+\gamma'$ for some $\sigma'\in S'$, $\lambda'\in L'$ and $\gamma'\in F'$. Set

$$\eta':=\psi(\eta)-2\sigma'+\gamma'.$$

Then $(S',L',F'+\eta')$ is in the right hand side of the equality in the statement and $\psi(E)=\psi(F+\eta)=F'+\gamma'+\psi(\eta)=F'+\eta'+2\sigma'$. This shows that $(S,L,E)\sim(S',L',F'+\eta')$. □

There is a very effective tool in the classification of EARS, called *duality*, which reduces sharply the number of twist-triples which we must consider. The notion of duality for extended affine root systems induces a notion of duality for both BC-triples and twist-triples (see [6] for details). It follows that the classification table for similarity classes of BC-triples for a particular twist-triple (t_1,t,t_2) can be obtained routinely from the classification table for its dual twist-triple $(t_1,t,t_2)^\vee$. Accordingly, the triples which we must consider are $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,0,3)$, $(1,1,1)$ $(1,1,2)$, $(0,1,3)$, $(0,1,2)$, $(0,2,3)$, $(1,2,2)$, $(0,1,1)$, $(0,2,2)$, $(0,3,3)$.

3 Large and shifted-large sets

The main tool in our work is the so called *large (shifted-large) sets.* They are certain subsets of a ν-dimensional vector space over the field of two elements. The point is that many interesting facts about semilattices and translated semilattices can be obtained through some rather simple calculations in the corresponding large and shifted-large sets.

Let R be an irreducible EARS and let Λ be the $\mathbb{Z}$-span of isotropic roots in R. It follows that $\Lambda=\mathbb{Z}\sigma_1\oplus\cdots\oplus\mathbb{Z}\sigma_\nu$, for some isotropic roots $\sigma_1,\ldots,\sigma_\nu$. We fix this $\mathbb{Z}$-basis of Λ throughout our work. Set $\tilde\Lambda=\Lambda/2\Lambda$, and let $\tilde{}:\Lambda\to\tilde\Lambda$ be the canonical map. For $\psi\in\mathrm{Aut}(\Lambda)$ consider $\tilde\psi\in GL(\tilde\Lambda)$ by $\tilde\psi(\tilde\sigma)=\widetilde{(\psi(\sigma))}$. The map $\psi\to\tilde\psi$ is a surjective group homomorphism from $\mathrm{Aut}(\Lambda)$ onto $\mathrm{Aut}(\tilde\Lambda)$. A subset T of $\tilde\Lambda$ is called *large* if it spans $\tilde\Lambda$ and contains zero. Two large sets T and T' are called *similar* if there exists $\psi\in GL(\mathcal{V}^0)$ such that $\psi(T)=T'+\sigma'$ for some $\sigma'\in T'$. If we denote the preimage of a subset T of $\tilde\Lambda$ by $S(T)$, then it follows that the map $T \to S(T)$ induces a bijection from the set of similarity classes of large subsets of $\tilde{\Lambda}$ onto the set of similarity classes of semilattices in $\mathcal{V}^0$ with $\mathbb{Z}$-span Λ.

We would like to obtain a similar characterization for what we call a *shifted semilattice*, a translated semilattice which is not a semilattice. For this, call a subset T of $\tilde{\Lambda}$ *shifted-large* if it spans $\tilde{\Lambda}$ and does not contain zero. Two shifted-large sets T and T' are called *isomorphic* if there exists $\psi \in GL(\tilde{\Lambda})$ such that $\psi(T) = T'$. As before it follows that the map $T \to S(T)$ induces a bijection from the set of isomorphism classes of shifted-large subsets of $\tilde{\Lambda}$ onto the set of isomorphism classes of shifted semilattices in $\mathcal{V}^0$ with $\mathbb{Z}$-span Λ.

From now on we assume that $\dim \mathcal{V}^0 = \nu = 3$. Let E be a translated semilattice with $\langle E \rangle = \Lambda$. Then $E = S(T)$ for some large or shifted-large subset T of $\tilde{\Lambda}$. Define *index* of E by $\mathrm{ind}(E) = \#(T \cup \{0\}) - 1$. It is known that any semilattice in $\mathcal{V}^0$ with $\mathbb{Z}$-span Λ is of the form $S_\Lambda^{(i)} + \sigma$, $3 \leq i \leq 7$, for some $\sigma \in S_\Lambda^{(i)}$ (see [5, Remark 3.27] for notation and details) where $S_\Lambda^{(i)}$ is one of semilattices listed in the following tables:

$$\mathrm{ind}_\Lambda(3): \begin{cases} S_\Lambda^{(1)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3) \\ S_\Lambda^{(2)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3) \\ S_\Lambda^{(3)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(4)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(5)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(6)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(7)} = S(0,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \end{cases}$$

$$\mathrm{ind}_\Lambda(4): \begin{cases} S_\Lambda^{(8)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2) \\ S_\Lambda^{(9)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_3) \\ S_\Lambda^{(10)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(11)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(12)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(13)} = S(0,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3) \\ S_\Lambda^{(14)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_2+\tilde{\sigma}_3) \end{cases}$$

$$\mathrm{ind}_\Lambda(5): \begin{cases} S_\Lambda^{(15)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3) \\ S_\Lambda^{(16)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(17)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(18)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(19)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(20)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \\ S_\Lambda^{(21)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2+\tilde{\sigma}_3) \end{cases}$$

$$\mathrm{ind}_\Lambda(6): \left\{ S_\Lambda^{(22)} = S(0,\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_3,\tilde{\sigma}_1+\tilde{\sigma}_2,\tilde{\sigma}_1+\tilde{\sigma}_3,\tilde{\sigma}_2+\tilde{\sigma}_3) \right.$$

$$\mathrm{ind}_\Lambda(7): \left\{ S_\Lambda^{(23)} = \Lambda \right.$$

The next proposition classifies the semilattices (and related shifted semilattices) $S_\Lambda^{(i)}$ listed above. First, we need to state some lemmas. Let us define the length of an element v of a vector space with respect to a fixed basis X, denoted by $\ell_X(v)$, to be the number of nonzero coordinates of v with respect to X.

Lemma 3.1. *Let A and A' be two shifted-large subsets of $\tilde{\Lambda}$. Suppose that*

$$\#\{v \in A \mid \ell_X(v) = i\} = \#\{v \in A' \mid \ell_{X'}(v) = i\}, \quad 1 \leq i \leq 3, \tag{3.2}$$

for some bases $X \subseteq A$ and $X' \subseteq A'$. Then there exists an isomorphism $\varphi \in GL(\tilde{\Lambda})$ such that $\varphi(A) = A'$. Conversely, if $\varphi(A) = A'$ for some isomorphism $\varphi \in GL(\tilde{\Lambda})$, then there exist bases $X \subseteq A$ and $X' \subseteq A'$ such that 3.2 holds.

Proof. The proof is immediate as $\dim_{\mathbb{F}_2} \tilde{\Lambda} = 3$. □

Lemma 3.3. *i) Up to isomorphism there exist only two shifted-large subsets of $\tilde{\Lambda}$ of cardinal 4.*

ii) For $n = 3, 5, 6, 7$, up to isomorphism there exists only one shifted-large subset of $\tilde{\Lambda}$ of cardinal n.

Proof. Let A be a shifted-large subset of $\tilde{\Lambda}$ with $\#A = n$. The result for the cases $n = 3, 7$ is trivial. If $n = 4$, then for any basis $X \subseteq A$ exactly two cases can happen, either there is one element of length 3 or there is no element of this length. So the result follows from Lemma 3.1.

Now let $n = 5, 6$. It follows from Lemma 3.1 that we are done if we show that with respect to some basis $X \subseteq A$ no element of A has length 3. Let $A = \{a_i\}_{i=1}^{n}$. We may assume that $X' = \{a_i\}_{i=1}^{3}$ is a basis of $\tilde{\Lambda}$. If there exists no $v \in A$ such that $\ell_{X'}(v) = 3$ we are done. If not, we may assume that $\ell(a_4) = 3$. Then $a_5 = a_i + a_j$ for some distinct $i, j \in \{1, 2, 3\}$. Set $\{a_k\} = X' \setminus \{a_i, a_j\}$ and let $a_6 = a_i + a_k$ (if $n = 6$). Now $X = \{a_5, a_i, a_k\}$ is a basis of $\tilde{\Lambda}$ and with respect to X no element of A has length 3. □

Proposition 3.4. *Let S_1 and S_2 be two semilattices with $\langle S_1\rangle = \langle S_2\rangle = \Lambda$ and $\mathrm{ind}(S_1) = \mathrm{ind}(S_2) = n$.*

(i) For $i = 1, 2$, set $E_i = S_i + \eta_i$ where $\eta_i \in \Lambda$. If E_1, E_2 are shifted semilattices, then $E_1 \cong E_2$.

(ii) Up to isomorphism $S^{(8)}_\Lambda$ and $S^{(11)}_\Lambda$ are the only semilattices of index 4.

(iii) If $n \neq 4$, then $S_1 \cong S_2$.

(iv) Two semilattices in $\mathcal{V}^0$ with the same $\mathbb{Z}$-span are similar if and only if they have the same index.

Proof. (i) Let S be a semilattice with $\langle S\rangle = \Lambda$ and $\mathrm{ind}(S) = 3$. Let $\eta \in \Lambda$ be such that $E := S{+}\eta$ is a shifted semilattice. Then $E = S(T)$ where T is a shifted-large subset of $\tilde{\Lambda}$ and $\#T = 4$. So there exists $\tau \in T$ such that $S = S(T{+}\tau)$. Since S is a semilattice with $\langle S\rangle = \Lambda$ and $\mathrm{ind}(S) = 3$ we have $\tilde{S}\setminus\{0\}$ is a linearly independent subset of $\tilde{\Lambda}$ and so $\sum_{\tilde{\sigma}\in\tilde{S}} \tilde{\sigma} \neq 0$. This gives $\sum_{t\in T} t \neq 0$. Therefore, no element in T, with respect to any basis contained in T, has length 3. The result now follows from Lemma 3.3.

(ii) Recall that $T \mapsto S(T)$ induces a bijection from the set of isomorphism classes of large subsets of $\tilde{\Lambda}$ onto the set of isomorphism classes of semilattices in $\mathcal{V}^0$ with $\mathbb{Z}$-span Λ. Now if T is a large subset with $\#T = 5$ then $T \setminus \{0\}$ is a shifted-large subset with cardinal 4. But by Lemma 3.3, up to isomorphism there exist exactly two shifted-large subsets with cardinal 4 depending on whether their sum of elements is zero or not. So up to isomorphism there exist two large subsets with cardinal 5 depending on whether their sum of elements is zero or not. But the sum of elements in $(S^{(11)}_\Lambda)^\sim$ is zero while this is not the case for $(S^{(8)}_\Lambda)^\sim$. This completes the proof of (ii).

(iii) It follows immediately from Lemma 3.3(ii).

(iv) According to parts (ii) and (iii), we are done if we show that $S^{(8)}_\Lambda \sim S^{(11)}_\Lambda$. To see this take σ_1 to $\sigma_1 + \sigma_3$, σ_2 to $\sigma_2 + \sigma_3$ and σ_3 to σ_3. □

4 Classification of Λ-pairs and translated Λ-pairs

Let Λ be as in Section 3. In this section we classify the so called Λ-pairs and translated Λ-pairs. Our approach is as follows. We first find a set $\mathcal{R}$ of representatives for the class of Λ-pairs. We use this to classify the class of translated Λ-pairs. It then follows that the representatives in $\mathcal{R}$ are distinct, and so we get a classification of Λ-pairs. We later relate the classification of BC-triples (for some particular twist triples) to the classification of Λ-pairs. In this section and future sections we will face with a huge amount of computations regarding pairs or triples of semilattices and translated semilattices. To make our computations shorter, we agree to represent an expression such as

$$(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(5)}+\sigma_1)\overset{\varphi_{22}^{-1}}{\cong}(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(2)}) \text{ or } (S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_3)\overset{\varphi_1}{\cong}(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2)$$

by

$$(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(5)}+\sigma_1)\varphi_{22}^{-1}(, S_\Lambda^{(2)}) \text{ or } (S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2),$$

respectively. Some other simplifying notations that we will use would be clear from the context.

Let S and F be two semilattices in $\mathcal{V}^0$. The pair (S,F) is called a Λ*-pair* if $\langle S\rangle=\langle F\rangle=\Lambda$. Denote by $\mathcal{A}$ the set of Λ-pairs in $\mathcal{V}^0$, and set

$$\mathcal{A}'=\{(S,E)\mid E=F+\eta \text{ for some } (S,F)\in\mathcal{A},\ \eta\in\Lambda\}.$$

The elements of $\mathcal{A}'$ are called *translated* Λ*-pairs*. If F is a translated semilattice, then $\mathrm{ind}(F+\eta)=\mathrm{ind}(F+\gamma)$ for any $\eta,\gamma\in F$, so for a pair (Λ-pair or translated Λ-pair) (S,F) with $\mathrm{ind}(S)=i$, $\mathrm{ind}(F-\eta)=j$, $\eta\in F$, we call (i,j) the *index* of (S,F). Throughout this work we use 32 non-identity automorphisms, denoted by φ_i, $1\le i\le 32$, which are listed in Section 6.

Definition 4.1. (i) Let (S,F) and (S',F') be two Λ-pairs. We say that (S,F) and (S',F') are *isomorphic*, written $(S,F)\cong(S',F')$, if there exists $\varphi\in GL(\mathcal{V}^0)$ such that $\varphi(S)=S'$ and $\varphi(F)=F'$. Also, (S,F) and (S',F') are called *similar*, written $(S,F)\sim(S',F')$, if there exists $\varphi\in GL(\mathcal{V}^0)$ such that $\varphi(S)=S'+\sigma'$ and $\varphi(F)=F'+\gamma'$, for some $\sigma'\in S',\gamma'\in F'$. The relations $\cong$ and $\sim$ are equivalence relations.

(ii) Two translated Λ-pairs (S,E) and (S',E') are called *isomorphic*, written $(S,E)\cong(S',E')$, if there exists $\varphi\in GL(\mathcal{V}^0)$ such that $\varphi(S)=S'$ and $\varphi(E)=E'$. Also, (S,E) and (S',E') are called *directed similar*, written $(S,E)\approx(S',E')$, if there exists $\varphi\in GL(\mathcal{V}^0)$ such that $\varphi(S)=S'+\sigma'$ and $\varphi(E)=E'+\sigma'$, for some $\sigma'\in S'$. The relations $\cong$ and $\approx$ are equivalence relations.

Proposition 4.2. *Up to directed similarity, there exist 73 distinct translated* Λ*-pairs (for a complete list see Table 6.2).*

Proof. Step 1. (Reduction to similarity.) We first relate two notions, similarity and directed similarity. Let $\mathcal{R}$ be a set of representatives for the similarity classes in $\mathcal{A}$. Using an argument similar to Proposition 2.6, one can see that the set

$$\mathcal{R}'=\{(S,F+\eta)\mid (S,F)\in\mathcal{R},\eta\in\Lambda\} \tag{4.3}$$

is a set of representatives for the directed similarity classes in $\mathcal{A}'$.

Step 2. (Similar Λ-pairs.) In virtue of (4.3), we start by finding a set of representatives for similar Λ-pairs.

To find similar Λ-pairs, we must consider all pairs $(S+\sigma, F+\gamma)$ where $S, F \in \cup_{i=3}^{7}\mathrm{ind}_\Lambda(i)$ and $\sigma \in S$, $\gamma \in F$. Since index is a similarity invariant of semilattices we have $(S+\sigma, F+\gamma) \not\sim (S'+\sigma', F'+\gamma')$ if $\mathrm{ind}(S) \neq \mathrm{ind}(S')$. So for each i, $3 \leq i \leq 7$ it is enough to consider Λ-pairs of the form $(S+\sigma, F+\gamma)$ where $S \in \mathrm{ind}_\Lambda(i), F \in \mathrm{ind}_\Lambda(j)$, $3 \leq j \leq 7$, $\sigma \in S$, $\gamma \in F$. However, the answer is clear if F is of index 6 or 7, as for such indices there is only one semilattice (up to translation). So for each $3 \leq i \leq 7$ we restrict our computations to the cases $3 \leq j \leq 5$.

<u>$i = 3,\ 3 \leq j \leq 5:$</u>
We have

$$(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(5)}+\sigma_1)\varphi_{22}^{-1}(, S_\Lambda^{(2)})\varphi_5(, S_\Lambda^{(3)})\varphi_1(, S_\Lambda^{(4)}),\ \ (S_\Lambda^{(1)}, S_\Lambda^{(5)})\varphi_1(, S_\Lambda^{(6)})\varphi_5(, S_\Lambda^{(7)}).$$

So, $\{[(S_\Lambda^{(1)}+\sigma, F+\tau)] \mid F \in \mathrm{ind}_\Lambda(3), \sigma \in S_\Lambda^{(1)}, \tau \in F\} = \{[(S_\Lambda^{(1)}, S_\Lambda^{(1)})], [(, S_\Lambda^{(2)})]\}$. By Proposition 3.4, semilattices of the same index are similar, so for each $2 \leq i \leq 7$ and $F \in \mathrm{ind}_\Lambda(3)$ there exists $\varphi \in GL(\mathcal{V}^0)$ such that $\varphi(S_\Lambda^{(i)}) = S_\Lambda^{(1)}$. Now $\varphi(F)$ is a semilattice similar (in fact isomorphic) to F, so $\varphi(F) = E+\tau$ for some $E \in \mathrm{ind}_\Lambda(3)$, $\tau \in E$. So for $1 \leq i \leq 7$ and $F \in \mathrm{ind}_\Lambda(3)$, $[(S_\Lambda^{(i)}, F)] \in \{[(S_\Lambda^{(1)}, E)] \mid E \in \mathrm{ind}_\Lambda(3)\}$. Thus,

$$\{[(S+\sigma, F+\tau)] \mid S, F \in \mathrm{ind}_\Lambda(3), \sigma \in S, \tau \in F\} = \ \{[(S_\Lambda^{(1)}, S_\Lambda^{(1)})], [(, S_\Lambda^{(2)})]\}. \qquad (4.4)$$

Also $(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(18)}+\sigma_1)\varphi_8^{-1}(, S_\Lambda^{(16)})\varphi_5(, S_\Lambda^{(15)})\varphi_5(, S_\Lambda^{(16)})\varphi_1(, S_\Lambda^{(17)})$, and

$$(S_\Lambda^{(1)}+\sigma_1, S_\Lambda^{(11)}+\sigma_1)\varphi_8^{-1}(S_\Lambda^{(1)}, S_\Lambda^{(10)})\varphi_{24}(, S_\Lambda^{(8)})\varphi_1(, S_\Lambda^{(9)}),$$
$$(S_\Lambda^{(1)}, S_\Lambda^{(14)})\varphi_1(, S_\Lambda^{(12)})\varphi_{24}(, S_\Lambda^{(13)}),\ \ (S_\Lambda^{(1)}, S_\Lambda^{(18)})\varphi_1(, S_\Lambda^{(19)})\varphi_5(, S_\Lambda^{(20)}).$$

So as before for $\sigma \in S$ and $\tau \in F$,

$$\{[(S+\sigma, F+\tau)] \mid S \in \mathrm{ind}_\Lambda(3), F \in \mathrm{ind}_\Lambda(4)\} = \{[(S_\Lambda^{(1)}, S_\Lambda^{(8)})], [(, S_\Lambda^{(12)})]\} \qquad (4.5)$$

$$\{[(S+\sigma, F+\tau)] \mid S \in \mathrm{ind}_\Lambda(3), F \in \mathrm{ind}_\Lambda(5)\} = \{[(S_\Lambda^{(1)}, S_\Lambda^{(15)})], [(, S_\Lambda^{(21)})]\}. \qquad (4.6)$$

In a similar manner we can handle the remaining cases.

<u>$i = 4,\ 3 \leq j \leq 5:$</u> We have $(S_\Lambda^{(8)}+\sigma_2, S_\Lambda^{(3)})\varphi_5(\star+\sigma_1, S_\Lambda^{(2)})\varphi_2(, S_\Lambda^{(1)})\varphi_{29}(\star+\sigma_1+\sigma_2, S_\Lambda^{(5)})$, and $(S_\Lambda^{(8)}, S_\Lambda^{(10)})\varphi_5(, S_\Lambda^{(9)})\varphi_7(\star+\sigma_2, S_\Lambda^{(11)}+\sigma_2)$, $(S_\Lambda^{(8)}, S_\Lambda^{(7)})\varphi_5(, S_\Lambda^{(6)})\varphi_{10}(, S_\Lambda^{(4)})$.

Also we have, $(S_\Lambda^{(8)}, S_\Lambda^{(14)})\varphi_5(, S_\Lambda^{(13)})\varphi_9^{-1}(, S_\Lambda^{(11)})\varphi_2(\star+\sigma_1, S_\Lambda^{(12)})$,

and $(S_\Lambda^{(8)}+\sigma_1, S_\Lambda^{(21)})\varphi_2^{-1}(, S_\Lambda^{(20)})\varphi_5(, S_\Lambda^{(19)}), (S_\Lambda^{(8)}, S_\Lambda^{(18)})\varphi_7(\star+\sigma_2, S_\Lambda^{(15)}+\sigma_2)$, and $(S_\Lambda^{(8)}, S_\Lambda^{(15)})\varphi_5(, S_\Lambda^{(16)}), (S_\Lambda^{(8)}, S_\Lambda^{(17)})\varphi_2(\star+\sigma_1, S_\Lambda^{(19)})$. So, if $\mathrm{ind}(S) = 4$, $3 \leq \mathrm{ind}(F) \leq 5$ and $\sigma \in S$, $\tau \in F$, then

$$\{[(S+\sigma, F+\tau)]\} = \{[(S_\Lambda^{(8)}, S_\Lambda^{(i)})] \mid i = 1, 4, 8, 9, 15, 17\}. \qquad (4.7)$$

<u>$i = 5, 3 \leq j \leq 5:$</u> We have $(S_\Lambda^{(15)}+\sigma_2, S_\Lambda^{(5)})\varphi_{29}^{-1}(S_\Lambda^{(15)}, S^{(1)})\varphi_{26}(\star+\sigma_3, S_\Lambda^{(6)})$,

$(S_\Lambda^{(15)}, S_\Lambda^{(2)})\varphi_2(, S_\Lambda^{(1)})\varphi_3(\star+\sigma_2, S_\Lambda^{(3)}), (S_\Lambda^{(15)}, S_\Lambda^{(1)})\varphi_4(\star+\sigma_3, S_\Lambda^{(4)})$. Also, we have,

$$(S_\Lambda^{(15)}, S_\Lambda^{(9)})\varphi_1(, S_\Lambda^{(8)})\varphi_4(\star+\sigma_3, S_\Lambda^{(13)}+\sigma_3), (S_\Lambda^{(15)}, S_\Lambda^{(11)})\varphi_2(, S_\Lambda^{(12)}),$$

$(S_\Lambda^{(15)}, S_\Lambda^{(14)})\varphi_4(\star+\sigma_3, S_\Lambda^{(11)}), (S_\Lambda^{(15)}, S_\Lambda^{(10)})\varphi_{26}(\star+\sigma_3, S_\Lambda^{(12)}+\sigma_1)$. And

$$(S_\Lambda^{(15)}, S_\Lambda^{(16)})\varphi_1(, S_\Lambda^{(17)}),\ \ (S_\Lambda^{(15)}, S_\Lambda^{(18)})\varphi_1(, S_\Lambda^{(19)}), (S_\Lambda^{(15)}, S_\Lambda^{(17)})\varphi_3(\star+\sigma_2, S_\Lambda^{(20)}),$$
$$(S_\Lambda^{(15)}, S_\Lambda^{(19)})\varphi_3(\star+\sigma_2, S_\Lambda^{(21)}),\ \ (S_\Lambda^{(15)}, S_\Lambda^{(17)})\varphi_2(, S_\Lambda^{(19)})$$

Hence if $\mathrm{ind}(S)=5,\ 3\leq \mathrm{ind}(F)\leq 5$ and $\sigma\in S,\ \tau\in F$, then

$$\{[(S+\sigma, F+\tau)]\} = \{[(S_\Lambda^{(15)}, S_\Lambda^{(i)})] \mid i=1,7,8,10,15,16\}. \tag{4.8}$$

If $\mathrm{ind}(S)=\mathrm{ind}(F)$, then there exists $\varphi\in GL(\mathcal{V}^0)$ such that $\varphi(S)=F+\gamma$ for some $\gamma\in F$ and $\varphi(S_\Lambda^{(22)}) = S_\Lambda^{(22)}+\sigma$ for some $\sigma\in S_\Lambda^{(22)}$,so if $\mathrm{ind}(F)=3,4,5$ and $\tau\in F$, then

$$\{[(S_\Lambda^{(22)}, F+\tau)]\} = \{[(S_\Lambda^{(22)}, S_\Lambda^{(1)})], [(, S_\Lambda^{(8)})], [(, S_\Lambda^{(15)})]\}. \tag{4.9}$$

Step 3. (Directed similar translated Λ-pairs). Now we are ready to find directed similarity classes of translated Λ-pairs. As before, for each $3\leq i,j\leq 7$, it is enough to find directed similarity classes of the form $[(S+\sigma, F+\gamma)]$ where $S\in \mathrm{ind}_\Lambda(i), F\in\mathrm{ind}_\Lambda(j)$ and $\sigma\in S,\ \gamma\in\Lambda$.

In practice, one of the difficulties in dealing with pairs or triples of semilattices is to show that two pairs or triples are not similar (or isomorphic). For this, we record some very handy computational facts, which will be used very frequently in our computations. The proof of each fact is immediate as any $\mathbb{Z}$-module automorphism of Λ induces a vector space isomorphism of $\tilde\Lambda$.

Let $(S,E),(S',E')$ be two isomorphic translated Λ-pairs, then

$$\begin{aligned}
&(i)\ \#(\tilde S\cap\tilde E) = \#(\tilde S'\cap\tilde E').\\
&(ii)\ (\textstyle\sum_{\tilde\sigma\in\tilde S}\tilde\sigma+\sum_{\tilde\sigma\in\tilde E}\tilde\sigma)\in(S\cap E)^\sim \Rightarrow (\sum_{\tilde{\sigma'}\in\tilde S'}\tilde\sigma'+\sum_{\tilde{\sigma'}\in\tilde E'}\tilde\sigma')\in(S'\cap E')^\sim.\\
&(iii)\ \textstyle\sum_{\tilde\sigma\in\tilde S\setminus(S\cap E)^\sim}\tilde\sigma = \sum_{\tilde\sigma\in\tilde E\setminus(S\cap E)^\sim}\tilde\sigma \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'\setminus(S'\cap E')^\sim}\tilde\sigma' = \sum_{\tilde{\sigma'}\in\tilde E'\setminus(S'\cap E')^\sim}\tilde\sigma'.\\
&(iv)\ \textstyle\sum_{\tilde\sigma\in\tilde E\setminus(S\cap E)^\sim}\tilde\sigma\in(S\cap E)^\sim \Rightarrow \sum_{\tilde{\sigma'}\in\tilde E'\setminus(S'\cap E')^\sim}\tilde\sigma'\in(S'\cap E')^\sim.\\
&(v)\ \textstyle\sum_{\tilde\sigma\in\tilde S\setminus(S\cap E)^\sim}\tilde\sigma = 0 \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'\setminus(S'\cap E')^\sim}\tilde\sigma' = 0.\\
&(vi)\ \textstyle\sum_{\tilde\sigma\in\tilde S\cap\tilde E}\tilde\sigma = 0 \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'\cap\tilde E'}\tilde\sigma' = 0.\\
&(vii)\ \textstyle\sum_{\tilde\sigma\in\tilde S\cap\tilde E}\tilde\sigma\in\tilde E \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'\cap\tilde E'}\tilde\sigma'\in(E')^\sim,\\
&\qquad \textstyle\sum_{\tilde\sigma\in\tilde S\cap\tilde E}\tilde\sigma\in\tilde S \Rightarrow \sum_{\tilde{\sigma'}\in(S'\cap E')^\sim}\tilde\sigma'\in\tilde S'.\\
&(viii)\ \textstyle\sum_{\tilde\sigma\in\tilde S}\tilde\sigma+\sum_{\tilde\sigma\in\tilde E}\tilde\sigma = 0 \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'}\tilde\sigma'+\sum_{\tilde{\sigma'}\in\tilde E'}\tilde\sigma' = 0.\\
&(ix)\ \textstyle\sum_{\tilde\sigma\in\tilde S}\tilde\sigma = \sum_{\tilde\sigma\in\tilde E}\tilde\sigma \Rightarrow \sum_{\tilde{\sigma'}\in\tilde S'}\tilde\sigma' = \sum_{\tilde{\sigma'}\in\tilde E'}\tilde\sigma'.
\end{aligned}\tag{4.10}$$

According to (4.10), to prove that two pairs are not isomorphic, it is enough to show that at least one of the items (i)-(ix) does not hold. For this we only need to do some calculus inside the finite vector space $\tilde\Lambda$. Since for each $1\leq i,j\leq 23,\ \sigma\in S_\Lambda^{(i)}$ and $\tau\in\Lambda$ we have $(S_\Lambda^{(i)}, S_\Lambda^{(j)}+\tau)\overset{id}{\approx}(S_\Lambda^{(i)}+\sigma, S_\Lambda^{(j)}+\sigma+\tau)$, for each i,j we only need to find directed similarity classes of the form $[(S_\Lambda^{(i)}, S_\Lambda^{(j)}+\tau)]$ where $\tau\in\Lambda$.

<u>$i,j=1,1:$</u> We have

$$(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2)\varphi_{24}(,\star+\sigma_2+\sigma_3)\overset{\varphi_7}{\approx}(,\star+\sigma_3)\varphi_{24}(,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$$

and by (4.10)(i) $(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3)$.

$\underline{i,j=1,2:}$ We have

$$(S_\Lambda^{(1)}, S_\Lambda^{(2)})\varphi_1(,\star+\sigma_1),\ (S_\Lambda^{(1)}, S_\Lambda^{(2)}+\sigma_2+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2+\sigma_3)$$

and $(S_\Lambda^{(1)}, S_\Lambda^{(2)}+\sigma_1+\sigma_2) \overset{\varphi_8}{\approx} (,\star+\sigma_2) \overset{\varphi_{27}}{\approx} (,\star+\sigma_3) \overset{\varphi_8}{\approx} (,\star+\sigma_1+\sigma_3)$. From (4.10)$(i)$ we can see that $(S_\Lambda^{(1)}, S_\Lambda^{(2)}+\sigma_2+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_2+\sigma_3)$ and $(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma) \not\approx (\star, S_\Lambda^{(2)}+\tau)$, for $\sigma \in \{0,\sigma_2,\sigma_1+\sigma_2+\sigma_3\}$, $\tau \in \{0,\sigma_2,\sigma_2+\sigma_3\}$, except $\sigma=\sigma_2=\tau$, but by (4.10)(ix), $(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_2) \not\approx (,S_\Lambda^{(2)}+\sigma_2)$. So by (4.3) and (4.4), we get Table 4.2 for index (3, 3).

$\underline{i,j=1,8:}$ We have $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2) \overset{\varphi_{28}^{-1}}{\approx} (,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$. We also have $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_2+\sigma_3) \overset{\varphi_7^{-1}}{\approx} (,\star+\sigma_3) \overset{\varphi_8}{\approx} (,\star+\sigma_1+\sigma_3)$. From (4.10)$(i)$ we have

$$(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_1) \not\approx (,\star+\sigma_3) \not\approx (,\star)$$

and $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_1)$.

$\underline{i,j=1,12:}$ One can check that $(S_\Lambda^{(1)}, S_\Lambda^{(12)}+\sigma_3) \overset{\varphi_6}{\approx} (,\star) \overset{\varphi_{27}}{\approx} (,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$, and $(S_\Lambda^{(1)}, S_\Lambda^{(12)}+\sigma_1+\sigma_2+\sigma_3) \overset{\varphi_6}{\approx} (,\star+\sigma_1+\sigma_2) \overset{\varphi_{27}}{\approx} (,\star+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3)$. We have from (4.10)$(i)$ that $(S_\Lambda^{(1)}, S_\Lambda^{(12)}) \not\approx (,\star+\sigma_1+\sigma_3)$, $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,S_\Lambda^{(12)})$ and

$$\begin{gathered}(S_\Lambda^{(1)}, S_\Lambda^{(12)}+\sigma_1+\sigma_3) \not\approx (,S_\Lambda^{(8)}) \not\approx (,S_\Lambda^{(12)}) \not\approx (,S_\Lambda^{(8)}+\sigma_1+\sigma_2+\sigma_3),\\ (S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1) \not\approx (,S_\Lambda^{(12)}+\sigma_1+\sigma_3) \not\approx (,S_\Lambda^{(8)}+\sigma_1+\sigma_2+\sigma_3).\end{gathered}$$

By (4.10)(ii) $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1) \not\approx (,S_\Lambda^{(12)})$. Also we have from (4.10)(iv) that $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,S_\Lambda^{(12)}+\sigma_1+\sigma_3)$. Now by (4.3) and (4.5), we get Table 4.2 for index $(3,4)$.

$\underline{i,j=1,15:}$ We have $(S_\Lambda^{(1)}, S_\Lambda^{(15)}+\sigma_2)\varphi_1(,\star+\sigma_3)$, but from (4.10)$(i)$ we see that $(S_\Lambda^{(1)}, S_\Lambda^{(15)}) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_2+\sigma_3) \not\approx (,\star)$.

$\underline{i,j=1,21:}$ We have $(S_\Lambda^{(1)}, S_\Lambda^{(21)}+\sigma_2)\varphi_5(,\star+\sigma_1)\varphi_1(,\star) \overset{\varphi_{30}}{\approx} (,\star+\sigma_3)$. For $\sigma \in \{0,\sigma_2+\sigma_3\}$, we have from (4.10)$(i)$ that $(S_\Lambda^{(1)}, S_\Lambda^{(15)}+\sigma) \not\approx (,S_\Lambda^{(21)})$, and by (4.10)$(iv)$, $(S_\Lambda^{(1)}, S_\Lambda^{(15)}+\sigma_2) \not\approx (,S_\Lambda^{(21)})$. Therefore by (4.3) and (4.6), we obtain Table 4.2 for index $(3,5)$.

$\underline{i,j=1,22:}$ We have $(S_\Lambda^{(1)}, S_\Lambda^{(22)}) \overset{\varphi_8}{\approx} (,\star+\sigma_1)\ \varphi_5\ (,\star+\sigma_2)\,\varphi_1\,(,\star+\sigma_3)$. We also can see that $(S_\Lambda^{(1)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2+\sigma_3) \overset{\varphi_6}{\approx} (,\star+\sigma_1+\sigma_2)\,\varphi_1\,(,\star+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3)$, but by (4.10)$(i)$, $(S_\Lambda^{(1)}, S_\Lambda^{(22)}) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3)$. By (4.3), we have for $S \in \mathrm{ind}(3)$, $F \in \mathrm{ind}(6)$, $\sigma \in S$ and $\tau \in \Lambda$,

$$\{[(S+\sigma, F+\tau)]\} = \{[(S_\Lambda^{(1)}, S_\Lambda^{(22)}+\sigma)] \mid \sigma \in \{0,\sigma_1+\sigma_2+\sigma_3\}\}. \tag{4.11}$$

$\underline{i,j=8,1:}$ We have

$$(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2) \overset{\varphi_8}{\approx} (,\star+\sigma_2)\varphi_5(,\star+\sigma_1),\ (S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3) \overset{\varphi_7}{\approx} (,\star+\sigma_3),$$

but by (4.10)(i), $(S_\Lambda^{(8)}, S_\Lambda^{(1)}) \not\approx (,\star+\sigma_1) \not\approx (,\star+\sigma_3) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3)$, and similarly $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_1) \not\approx (,\star+\sigma_3)$.

$\underline{i,j=8,4:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(4)}+\sigma_3)\varphi_5(,\star) \overset{\varphi_3}{\approx} (,\star+\sigma_2)\varphi_5(,\star+\sigma_1+\sigma_3)$, and

$$(S_\Lambda^{(8)}, S_\Lambda^{(4)}+\sigma_2+\sigma_3)\varphi_5(,\star+\sigma_1) \overset{\varphi_{29}}{\approx} (,\star+\sigma_1+\sigma_2)\varphi_5(,\star+\sigma_1+\sigma_2+\sigma_3).$$

Now by (4.10) (i), $(S_\Lambda^{(8)}, S_\Lambda^{(4)}) \not\approx (,\star+\sigma_1)$, $(S_\Lambda^{(8)}, S_\Lambda^{(4)}+\sigma_1) \not\approx (,S_\Lambda^{(1)}) \not\approx (,S_\Lambda^{(4)})$, and $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_3) \not\approx (,S_\Lambda^{(4)}) \not\approx (,S_\Lambda^{(1)}+\sigma_1+\sigma_2+\sigma_3) \not\approx (,S_\Lambda^{(4)}+\sigma_1)$, and $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1) \not\approx (,S_\Lambda^{(4)}+\sigma_1)$. Also $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1) \not\approx (,S_\Lambda^{(4)})$, and $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_3) \not\approx (,S_\Lambda^{(4)}+\sigma_1)$, by (4.10) $(ii),(v)$. By (4.3) and (4.7), we obtain Table 4.2 for index $(4,3)$.

$\underline{i,j=8,8:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2)\varphi_9(,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$, and

$$(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_3)\overset{\varphi_3}{\approx}(,\star+\sigma_2+\sigma_3)\varphi_5(,\star+\sigma_1+\sigma_3)\varphi_9(,\star+\sigma_1+\sigma_2+\sigma_3).$$

By (4.10) (i), $(S_\Lambda^{(8)}, S_\Lambda^{(8)}) \not\approx (,\star+\sigma_1) \not\approx (,\star+\sigma_3) \not\approx (,\star)$.

$\underline{i,j=8,9:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_2)\overset{\varphi_8}{\approx}(,\star+\sigma_1+\sigma_2)$,

$$(S_\Lambda^{(8)}, S_\Lambda^{(9)})\varphi_{10}(,\star+\sigma_1),\text{ and }(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_3)\varphi_{10}(,\star+\sigma_1+\sigma_3).$$

By (4.10) (i), we have, $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_2+\sigma_3)$, and $(S_\Lambda^{(8)}, S_\Lambda^{(9)}) \not\approx (S_\Lambda^{(8)},\star+\sigma_2) \not\approx (,\star+\sigma_1+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_3) \not\approx (,\star)$. Also by (4.10) (vii), $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_3) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_3)$, and by (4.10) (i),

$$\begin{gathered}(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\tau) \not\approx (,S_\Lambda^{(8)}) \text{ for } \tau\in\{0,\sigma_2,\sigma_3,\sigma_2+\sigma_3,\sigma_1+\sigma_2+\sigma_3\} \text{ and}\\ (S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_1) \not\approx (,S_\Lambda^{(9)}+\tau) \text{ for } \tau\in\{\sigma_2,\sigma_3,\sigma_2+\sigma_3,\sigma_1+\sigma_2+\sigma_3\}.\end{gathered}$$

Also $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,S_\Lambda^{(9)}+\tau)$ where $\tau\in\{0,\sigma_2,\sigma_3,\sigma_2+\sigma_3\}$. By (4.10) (v), we have $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,S_\Lambda^{(9)}+\sigma_1+\sigma_2+\sigma_3)$ and by (4.10) (vi), $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_1) \not\approx (,S_\Lambda^{(9)})$. So by (4.3) and (4.7), we get Table 4.2 for index $(4,4)$.

$\underline{i,j=8,15:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(15)}+\sigma_3)\overset{\varphi_{31}}{\approx}(,\star+\sigma_2+\sigma_3)$. Also from (4.10) (i) we see that, $(S_\Lambda^{(8)}, S_\Lambda^{(15)}) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_3) \not\approx (,\star)$.

$\underline{i,j=8,17:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(17)}+\sigma_2)\varphi_9(,\star)\varphi_{10}(,\star+\sigma_1)$. Also from (4.10) (i) we see that, $(S_\Lambda^{(8)}, S_\Lambda^{(17)}) \not\approx (,\star+\sigma_1+\sigma_2)$. Again by (4.10) (i),

$$(S_\Lambda^{(8)}, S_\Lambda^{(15)}+\sigma_2) \not\approx (,S_\Lambda^{(17)}+\sigma_2+\sigma_1) \not\approx (,S_\Lambda^{(15)}) \not\approx (S_\Lambda^{(8)}, S_\Lambda^{(17)}) \not\approx (,S_\Lambda^{(15)}+\sigma_3).$$

Finally, from (4.10) $(viii)$ we have $(S_\Lambda^{(8)}, S_\Lambda^{(17)}) \not\approx (,S_\Lambda^{(15)}+\sigma_2)$, $(S_\Lambda^{(8)}, S_\Lambda^{(17)}+\sigma_1+\sigma_2) \not\approx (,S_\Lambda^{(15)}+\sigma_3)$. So by (4.3) and (4.7), we get Table 4.2 for index $(4,5)$.

$\underline{i,j=8,22:}$ We have $(S_\Lambda^{(8)}, S_\Lambda^{(22)})\varphi_{32}(,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$, and

$$(S_\Lambda^{(8)}, S_\Lambda^{(22)}+\sigma_3)\varphi_{10}(,\star+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3)\overset{\varphi_8}{\approx}(,\star+\sigma_1+\sigma_2+\sigma_3)$$

and by (4.10) (i) we have, $(S_\Lambda^{(8)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2) \not\approx (,\star) \not\approx (,\star+\sigma_3)$, and by (4.10) (vi), $(S_\Lambda^{(8)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2) \not\approx (,\star+\sigma_3)$. So by (4.3), we get Table 4.2 for index $(4,6)$.

$\underline{i,j=15,1:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(1)}+\sigma_2+\sigma_3)\varphi_8(,\star+\sigma_1+\sigma_2+\sigma_3)$, and

$$(S_\Lambda^{(15)}, S_\Lambda^{(1)})\varphi_8(,\star+\sigma_1),\ (S_\Lambda^{(15)}, S_\Lambda^{(1)}+\sigma_2)\varphi_1(,\star+\sigma_3)\overset{id}{\approx}(,\star+\sigma_1+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2).$$

But by (4.10) (i), $(S_\Lambda^{(15)}, S_\Lambda^{(1)}+\sigma_2) \not\approx (,\star) \not\approx (,\star+\sigma_2+\sigma_3) \not\approx (,\star+\sigma_2)$.

$\underline{i,j=15,7:}$ We see that $(S_\Lambda^{(15)}, S_\Lambda^{(7)}+\sigma_1+\sigma_2+\sigma_3)\varphi_2(,\star+\sigma_3)\varphi_1(,\star+\sigma_2)\varphi_2(,\star)$, and

$$(S_\Lambda^{(15)}, S_\Lambda^{(7)})\overset{id}{\approx}(,\star+\sigma_1)\varphi_{10}(,\star+\sigma_1+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2)\varphi_{10}(,\star+\sigma_2+\sigma_3).$$

By $(4.10)(i)(v)$,

$$(S_\Lambda^{(15)}, S_\Lambda^{(1)}) \not\approx (, S_\Lambda^{(7)}) \not\approx (, S_\Lambda^{(1)} + \sigma_2 + \sigma_3),\ (S_\Lambda^{(15)}, S_\Lambda^{(1)} + \sigma_2) \not\approx (, S_\Lambda^{(7)}).$$

So by (4.3) and (4.8), get Table 4.2 for index $(5,3)$.

$\underline{i,j = 15,8:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(8)})\varphi_8(,\star+\sigma_1)$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_2)\varphi_2(,\star+\sigma_1+\sigma_2)$ and $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_3)\overset{id}{\approx}(,\star+\sigma_1+\sigma_3)\overset{\varphi_{31}}{\approx}(,\star+\sigma_1+\sigma_2+\sigma_3)\overset{id}{\approx}(,\star+\sigma_2+\sigma_3)$. Now by $(4.10)(i)$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (,\star) \not\approx (,\star+\sigma_2) \not\approx (,\star+\sigma_3)$.

$\underline{i,j = 15,10:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(10)}+\sigma_2+\sigma_3)\overset{id}{\approx}(,\star+\sigma_1+\sigma_2+\sigma_3)$. Also

$$(S_\Lambda^{(15)}, S_\Lambda^{(10)}+\sigma_1)\overset{id}{\approx}(,\star)\overset{\varphi_3}{\approx}(,\star+\sigma_2)\varphi_1(,\star+\sigma_3)\overset{id}{\approx}(,\star+\sigma_1+\sigma_3)\overset{\varphi_1}{\approx}(,\star+\sigma_1+\sigma_2).$$

By $(4.10)(i)$, $(S_\Lambda^{(15)}, S_\Lambda^{(10)}) \not\approx (,\star+\sigma_2+\sigma_3)$, $(S_\Lambda^{(15)}, S_\Lambda^{(10)}+\sigma_2+\sigma_3) \not\approx (, S_\Lambda^{(8)}) \not\approx (, S_\Lambda^{(10)})$, and $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_2) \not\approx (, S_\Lambda^{(10)}+\sigma_2+\sigma_3)$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (, S_\Lambda^{(10)})$. By $(4.10)(vi)$, we have $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_2) \not\approx (, S_\Lambda^{(10)})$, and by $(4.10)(v)$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_3) \not\approx (, S_\Lambda^{(10)}+\sigma_2+\sigma_3)$. So by (4.3) and 4.8, we get Table 4.2 for index $(5,4)$.

$\underline{i,j = 15,15:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(15)}+\sigma_2)\varphi_1(,\star+\sigma_3)\overset{\varphi_{29}}{\approx}(,\star+\sigma_2+\sigma_3)$. It is trivial that $(S_\Lambda^{(15)}, S_\Lambda^{(15)}) \not\approx (,\star+\sigma_2)$.

$\underline{i,j = 15,16:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(16)})\varphi_2(,\star+\sigma_1)$, $(S_\Lambda^{(15)}, S_\Lambda^{(16)}+\sigma_3)\overset{id}{\approx}(,\star+\sigma_1+\sigma_3)$. By $(4.10)(i)$, $(S_\Lambda^{(15)}, S_\Lambda^{(16)}) \not\approx (,\star+\sigma_3)$, and $(S_\Lambda^{(15)}, S_\Lambda^{(16)}) \not\approx (, S_\Lambda^{(15)}) \not\approx (, S_\Lambda^{(16)}+\sigma_3)$, and $(S_\Lambda^{(15)}, S_\Lambda^{(15)}+\sigma_2) \not\approx (, S_\Lambda^{(16)})$. Also by $(4.10)(iii)$ $(S_\Lambda^{(15)}, S_\Lambda^{(15)}+\sigma_2) \not\approx (, S_\Lambda^{(16)}+\sigma_3)$. So by (4.3) and (4.8), we get Table 4.2 for index $(5,5)$.

$\underline{i,j = 15,22:}$ We have $(S_\Lambda^{(15)}, S_\Lambda^{(22)})\varphi_{10}(,\star+\sigma_1)$, and $(S_\Lambda^{(15)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2+\sigma_3)\overset{id}{\approx}(,\star+\sigma_2+\sigma_3)\overset{\varphi_4}{\approx}(,\star+\sigma_2)\varphi_8(,\star+\sigma_1+\sigma_2)\varphi_1(,\star+\sigma_1+\sigma_3)\varphi_8(,\star+\sigma_3)$. From $(4.10)(i)$ we see that $(S_\Lambda^{(15)}, S_\Lambda^{(22)}) \not\approx (,\star+\sigma_2+\sigma_3)$. So by (4.3), we get Table 4.2 for index $(5,6)$.

$\underline{i,j = 22,1:}$ We have $(S_\Lambda^{(22)}, S_\Lambda^{(1)}+\sigma_2)\varphi_7(,\star)\varphi_8(,\star+\sigma_1)\varphi_{24}(,\star+\sigma_3)$. We also have $(S_\Lambda^{(22)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2+\sigma_3)\varphi_{23}^{-1}(,\star+\sigma_1+\sigma_2)\varphi_1(,\star+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3)$, but from $(4.10)(i)$ we get $(S_\Lambda^{(22)}, S_\Lambda^{(1)}) \not\approx (,\star+\sigma_1+\sigma_2)$.

$\underline{i,j = 22,8:}$ We have $(S_\Lambda^{(22)}, S_\Lambda^{(8)})\varphi_8(,\star+\sigma_1)\varphi_5(,\star+\sigma_2)$, and $(S_\Lambda^{(22)}, S_\Lambda^{(8)}+\sigma_3)\overset{\varphi_{31}}{\approx}(,\star+\sigma_1+\sigma_2+\sigma_3)\varphi_7(,\star+\sigma_1+\sigma_3)\varphi_5(,\star+\sigma_2+\sigma_3)$. ¿From $(4.10)(i),(iv)$, we have

$$(S_\Lambda^{(22)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2) \not\approx (,\star) \not\approx (,\star+\sigma_1+\sigma_3),\ (S_\Lambda^{(22)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2) \not\approx (,\star+\sigma_1+\sigma_3).$$

$\underline{i,j = 22,15:}$ We have $(S_\Lambda^{(22)}, S_\Lambda^{(15)}+\sigma_2+\sigma_3)\overset{\varphi_4}{\approx}(,\star+\sigma_2)\varphi_1(,\star+\sigma_3)$, however by $(4.10)(i)$, $(S_\Lambda^{(22)}, S_\Lambda^{(15)}) \not\approx (,\star+\sigma_2)$.

$\underline{i,j = 22,22:}$ We have

$$(S_\Lambda^{(22)}, S_\Lambda^{(22)}+\sigma_2+\sigma_3)\varphi_5(,\star+\sigma_1+\sigma_3)\varphi_1(,\star+\sigma_1+\sigma_2)\varphi_7(,\star+\sigma_1),$$

and $(S_\Lambda^{(22)}, S_\Lambda^{(22)}+\sigma_1)\varphi_5(,\star+\sigma_2)\varphi_1(,\star+\sigma_3)\overset{\varphi_{25}}{\approx}(,\star+\sigma_1+\sigma_2+\sigma_3)$. Trivially $(S_\Lambda^{(22)}, S_\Lambda^{(22)}) \not\approx (,\star+\sigma_1)$. So by (4.3) and (4.9), we get Table 4.2 for index $(6,j)$, $3 \le j \le 7$.

$\underline{i = 23, j,\ 1 \le j \le 23:}$ We know that if $\mathrm{ind}(S) = 3,4,5,6$ or 7, up to similarity $S = S_\Lambda^{(1)}, S_\Lambda^{(8)}, S_\Lambda^{(15)}, S_\Lambda^{(22)}$ or Λ, respectively, so we get Table 4.2 for index $(7,j)$, $3 \le j \le 7$. This completes the proof of the proposition. □

5 Classification of BC-triples

In this section we classify EARS of type BC for $\nu = 1,2,3$ by classifying the corresponding BC-triples and BC-pairs. We give the details for $\nu = 3$ and list the results for $\nu = 1$ and $\nu = 2$. So assume that $\nu = 3$.

For a fixed triple (S, L, F), we set

$$(F)_L = \{F + \eta \mid \eta \in L, L + \eta \subseteq L\}.$$

For $i = 1,2$, let $\tilde{} : \Lambda^{(i)} \to \Lambda^{(i)}/2\Lambda^{(i)}$ be the canonical map.

Theorem 5.1. **(Classification Theorem)** *Up to isomorphism there are n extended affine root systems of type BC_ℓ and nullity $\nu = 1,2,3$, where n is given in Table (5.2) according to either $\ell = 1$ or $\ell \geq 2$.*

	$\nu = 1$	$\nu = 2$	$\nu = 3$
BC_ℓ $(\ell \geq 2)$	*5*	*27*	*230*
BC_1	*4*	*21*	*226*

(5.2)

The classification list of corresponding BC-triples and BC-pairs are given in Tables 6.3–6.6.

Before starting the proof, let us record a few useful lemmas which will be crucial for our computations. The proof of each lemma can be obtained by inducing an automorphism in $GL(\Lambda^{(i)})$ to an automorphism of the finite vector space $\widetilde{(\Lambda^{(i)})}$.

Lemma 5.3. *Let $(\Lambda, \Lambda^{(1)}, E), (\Lambda, \Lambda^{(1)}, E') \in BC(0,1,1)$. If $\#(E \cap 2\Lambda)^\sim \neq \#(E' \cap 2\Lambda)^\sim$, then $(\Lambda, \Lambda^{(1)}, E) \not\cong (\Lambda, \Lambda^{(1)}, E')$.*

Lemma 5.4. *Let E, E' be two translated semilattices with $\langle E \rangle = \langle E' \rangle = \Lambda^{(2)}$ and (S, S') be a Λ-pair. Then $(S, \Lambda^{(2)}, E) \cong (S', \Lambda^{(2)}, E')$ implies:*

(i) $\sum_{\tilde\tau \in \tilde E} \tilde\tau = 0 \Rightarrow \sum_{\tilde\tau' \in \tilde E'} \tilde\tau' = 0$ *and* $\sum_{\tilde\tau \in \tilde E} \tilde\tau \in (2\Lambda)^\sim \Rightarrow \sum_{\tau' \in E'} \tilde\tau' \in (2\Lambda)^\sim$,

(ii) $\#((2(\frac{E \cap 2\Lambda}{2} \cap (S)))^\sim) = \#((2(\frac{E' \cap 2\Lambda}{2} \cap (S')))^\sim)$,

(iii) $\sum_{\tilde\sigma \in (E \setminus 2\Lambda)^\sim} \tilde\sigma \in (2(\frac{E \cap 2\Lambda}{2} \cap S))^\sim \Rightarrow \sum_{\tilde\sigma' \in (E' \setminus 2\Lambda)^\sim} \tilde\sigma' \in (2(\frac{E' \cap 2\Lambda}{2} \cap S'))^\sim$.

Condition (i) holds, replacing $\Lambda^{(2)}$ by $\Lambda^{(1)}$.

Lemma 5.5. *Let (S_1, S_2) be a Λ-pair. Then $(S_1, \Lambda^{(2)}, E_1) \cong (S_2, \Lambda^{(2)}, E_2)$ implies $\#((E_1 \cap 2\Lambda)^\sim) = \#((E_2 \cap 2\Lambda)^\sim)$, where $E_i = F_i + \tau_i$ for $F_i \in \cup_{j=3}^{7} \mathrm{ind}_{\Lambda(0,2,2)}(j)$ and $\tau_i \in \Lambda^{(2)}$ for $i = 1,2$.*

Proof of Proposition. According to duality, we only need to consider twist triples listed at the end of Section 2:

<u>$(0,0,0)$</u>**:** $(S, L, F) = (\Lambda, \Lambda, S_0)$ where S_0 is a semilattice that $\langle S_0 \rangle = \Lambda$. So

$$[[\mathcal{T}(0,0,0)]] = \{[[\Lambda,\Lambda,\Lambda]], [[\Lambda,\Lambda,F_1]], [[\Lambda,\Lambda,F_2]], [[\Lambda,\Lambda,F_3]], [[\Lambda,\Lambda,F_4]]\},$$

where $F_1 = S_\Lambda^{(1)}, F_2 = S_\Lambda^{(8)}, F_3 = S_\Lambda^{(15)}, F_4 = S_\Lambda^{(22)}$. Now let $(\Lambda, \Lambda, F), (\Lambda, \Lambda, F') \in \mathcal{T}(0,0,0)$ and $\eta, \eta' \in \Lambda$. Set $E = F + \eta$ and $E' = F' + \eta'$. If $(\Lambda, \Lambda, E) \overset{\varphi}{\sim} (\Lambda, \Lambda, E')$, then $\varphi(\Lambda) = \Lambda$, $\varphi(E) = E' + 2\sigma$, for some $\sigma \in \Lambda$. But since $\langle E' \rangle = \Lambda$, so $E' + 2\sigma = E'$, $(\Lambda, \Lambda, E) \sim (\Lambda, \Lambda, E')$ if and only if $E \cong E'$. Now using Proposition 3.4, we get Table 6.5 for twist triple $(0,0,0)$.

$\underline{(0,0,1)}$**:** $(S,L,F)=(\Lambda,\Lambda,2\mathbb{Z}\sigma_1+S_0)$ where $\langle S_0\rangle=\mathbb{Z}\sigma_2+\mathbb{Z}\sigma_3$. So

$$[[\mathcal{T}(0,0,1)]]=\{[[(\Lambda,\Lambda,F_1:=2\mathbb{Z}\sigma_1+S(0,\tilde{\sigma}_2,\tilde{\sigma}_3)]],[[(\Lambda,\Lambda,\Lambda^{(1)})]]\},$$

and $(\Lambda^{(1)})_\Lambda=\{\Lambda^{(1)},\Lambda^{(1)}+\sigma_1\}$. Also we have

$$(F_1)_\Lambda=\{F_1,F_1+\sigma_1,F_1+\sigma_2,F_1+\sigma_3,F_1+\sigma_1+\sigma_2,F_1+\sigma_1+\sigma_3,F_1+\sigma_2+\sigma_3,F_1+\sigma_1+\sigma_2+\sigma_3\}.$$

We have $(\Lambda,\Lambda,F_1)\varphi_4(,,\star+\sigma_3)\varphi_1(,,\star+\sigma_2)$ and by [6] we have $(\Lambda,\Lambda,F_1+\sigma_1)\varphi_9(,,\star+\sigma_1+\sigma_2)\varphi_1(,,\star+\sigma_1+\sigma_3)\varphi_9(,,\star+\sigma_1+\sigma_2+\sigma_3)$ and $(\Lambda,\Lambda,F_1+\sigma_1)\not\sim(,,\star+\sigma_2+\sigma_3)$. Since $0\in F_1$, one can see easily that $(\Lambda,\Lambda,F_1+\sigma_2+\sigma_3)\not\sim(,,\star)\not\sim(,,\star+\sigma_1)$.

It is trivial that $(\Lambda,\Lambda,\Lambda^{(1)})\not\sim(\Lambda,\Lambda,\Lambda^{(1)}+\sigma_1)$, so we get Table 6.5 for twist triple $(0,0,1)$.

$\underline{(0,0,2)}$**:** $(S,L,F)=(\Lambda,S_5\oplus\mathbb{Z}\sigma_3,\Lambda^{(2)})$ where $\langle S_5\rangle=\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_2$. Thus

$$[[\mathcal{T}(0,0,2)]]=\{[[(\Lambda,\Lambda,\Lambda^{(2)})]],[[(\Lambda,\overbrace{S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)\oplus\mathbb{Z}\sigma_3}^{L_1},\Lambda^{(2)})]]\},$$

with $(\Lambda^{(2)})_{L_1}=\{\Lambda^{(2)}\}$ and $(\Lambda^{(2)})_\Lambda=\{\Lambda^{(2)}+\sigma_1,\star+\sigma_2,\star+\sigma_1+\sigma_2,\star\}$. Now, we have, $(\Lambda,\Lambda,\Lambda^{(2)}+\sigma_1)\varphi_5(,,\star+\sigma_2)\varphi_{10}(,,\star+\sigma_1+\sigma_2)$. Note that $(\Lambda,\Lambda,\Lambda^{(2)})\not\sim(,,\star+\sigma_1)$ as $\Lambda^{(2)}$ is a semilattice and $\Lambda^{(2)}+\sigma_1$ is not. So, we get Table 6.5 for twist triple $(0,0,2)$.

$\underline{(0,0,3)}$: We have $(S,L,F)=(\Lambda,S_5,2\Lambda)$ where $\langle S_5\rangle=\Lambda$. From [6], we have

$$[[\mathcal{T}(0,0,3)]]=\{[[(\Lambda,\Lambda,2\Lambda)]],[[(\star,L_1,\star)]],[[(\star,L_2,\star)]],[[(\star,L_3,\star)]],[[(\star,L_4,\star)]]\}.$$

where $L_1=S_\Lambda^{(1)}$, $L_2=S_\Lambda^{(8)}$, $L_3=S_\Lambda^{(15)}$, $L_4=S_\Lambda^{(22)}$. One can check that

$$\begin{gathered}(2\Lambda)_\Lambda=\{2\Lambda+\sigma\mid\sigma=0,\sigma_1,\sigma_2,\sigma_3,\sigma_1+\sigma_2,\sigma_1+\sigma_3,\sigma_2+\sigma_3,\sigma_1+\sigma_2+\sigma_3\},\\(2\Lambda)_{L_1}=(2\Lambda)_{L_2}=(2\Lambda)_{L_4}=\{2\Lambda\},\ \ (2\Lambda)_{L_3}=\{2\Lambda,\sigma_1+2\Lambda\}.\end{gathered}$$

It is easy to see that the triples $(\Lambda,\Lambda,\sigma+2\Lambda)$, for $\sigma\in\{\sigma_1,\sigma_2,\sigma_3,\sigma_1+\sigma_2,\sigma_1+\sigma_3,\sigma_2+\sigma_3,\sigma_1+\sigma_2+\sigma_3\}$, are similar and that $(\Lambda,\Lambda,2\Lambda)\not\sim(,,\star+\sigma_1)$, $(\Lambda,L_3,2\Lambda)\not\sim(,,\star+\sigma_1)$. Thus we have Table 6.5 for twist triple $(0,0,3)$.

$\underline{(1,1,1)}$: $(S,L,F)=(\Lambda,\Lambda^{(1)},4\mathbb{Z}\sigma_1\oplus S_0)$ where $\langle S_0\rangle=\mathbb{Z}\sigma_2\oplus\mathbb{Z}\sigma_3$. From [6], we have

$$[[\mathcal{T}(1,1,1)]]=\{[[(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,1)})]],[[(\Lambda,\Lambda^{(1)},F_1)]]\},$$

where $F_1=4\mathbb{Z}\sigma_1+S(0,\tilde{\sigma}_2,\tilde{\sigma}_3)$. Now $(\Lambda^{(1,1,1)})_{\Lambda^{(1)}}=\{\Lambda^{(1,1,1)},2\sigma_1+\Lambda^{(1,1,1)}\}$ and

$$(F_1)_{\Lambda^{(1)}}=\{F_1+\sigma\mid\sigma=0,2\sigma_1,\sigma_2,\sigma_3,2\sigma_1+\sigma_2,2\sigma_1+\sigma_3,\sigma_2+\sigma_3,2\sigma_1+\sigma_2+\sigma_3\}.$$

It is easy to see that $(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,1)})\sim(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},F_1+\sigma_2+\sigma_3)\sim(,,\star+\sigma_2+\sigma_3+2\sigma_1)$, and that the triples $(\Lambda,\Lambda^{(1)},F_1+\sigma)$ for $\sigma\in\{0,2\sigma_1,\sigma_2,\sigma_3,2\sigma_1+\sigma_2,2\sigma_1+\sigma_3\}$ are also similar. Now since $0\in F_1$ and $0\notin F_1+\sigma_2+\sigma_3+2\sigma$ for $\sigma\in\Lambda$, one can see that $(\Lambda,\Lambda^{(1)},F_1)\not\sim(,,\star+\sigma_2+\sigma_3)$. Thus we have Table 6.5 for twist triple $(1,1,1)$.

$\underline{(1,1,2)}$: Since $(S,L,F)=(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,2)})$, we have from [6], $[[\mathcal{T}(1,1,2)]]=\{[[(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,2)})]]\}$. Now

$$(\Lambda^{(1,1,2)})_{\Lambda^{(1)}}=\{\Lambda^{(1,1,2)}+\sigma\mid\sigma=0,2\sigma_1,\sigma_2,2\sigma_1+\sigma_2\}.$$

Also we have $(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,2)})\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},\sigma_2+\Lambda^{(1,1,2)})\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1+\sigma_2)$, and $(\Lambda,\Lambda^{(1)},\Lambda^{(1,1,2)})\not\sim(,,\star+\sigma_2)$, where this last one follows from the fact that $\sigma_2+\Lambda^{(1,1,2)}+2\Lambda$ is not a semilattice. Hence we have obtained Table 6.5 for twist triple $(1,1,2)$.

$\underline{(0,1,3):}$ $(S,L,F)=(\Lambda,S_5\oplus 2\mathbb{Z}\sigma_1,2\Lambda)$ where $\langle S_5\rangle=\mathbb{Z}\sigma_2\oplus\mathbb{Z}\sigma_3$. By [5, Corollary 3.20(i)] we have

$$[[\mathcal{T}(0,1,3)]]=\{[[(\Lambda,\Lambda^{(1)},2\Lambda)]],[[(\Lambda,L_1,2\Lambda)]]\},$$

where $L_1=S(0,\tilde\sigma_2,\tilde\sigma_3)\oplus 2\mathbb{Z}\sigma_1$. It is easy to check that

$$(2\Lambda)_{\Lambda^{(1)}}=\{2\Lambda,2\Lambda+\sigma_2,2\Lambda+\sigma_3,2\Lambda+\sigma_2+\sigma_3\}\quad\text{and}\quad(2\Lambda)_{L_1}=\{2\Lambda\}.$$

Now

$$(\Lambda,\Lambda^{(1)},2\Lambda+\sigma_2)\varphi_1(,,\star+\sigma_3)\varphi_3(,,\star+\sigma_2+\sigma_3).$$

However, $(\Lambda,\Lambda^{(1)},2\Lambda)\not\sim(\Lambda,\Lambda^{(1)},\sigma_2+2\Lambda)$ as $\sigma_2+2\Lambda$ is not a semilattice. Thus we obtain Table 6.5 for twist triple $(0,1,3)$.

$\underline{(0,1,2):}$ $(S,L,F)=(\Lambda,\Lambda^{(1)},2\mathbb{Z}\sigma_2\oplus S_0)$ where $\langle S_0\rangle=\Lambda_0$, and $\Lambda_0=2\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_3$. We have that S_0 is one of the semilattices $2\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_3$, $S(0,2\tilde\sigma_1,\tilde\sigma_3)$, $S(0,2\tilde\sigma_1,\tilde\sigma_3)+2\sigma_1$, $S(0,2\tilde\sigma_1,\tilde\sigma_3)+\sigma_3$. From [6] we have

$$[[\mathcal{T}(0,1,2)]]=\{[[(\Lambda,\Lambda^{(1)},\Lambda^{(2)})]],[[(\Lambda,\Lambda^{(1)},F_1)]]\},$$

where $F_1=2\mathbb{Z}\sigma_2+S(0,2\tilde\sigma_1,\tilde\sigma_3)$. We have

$$(\Lambda^{(2)})_{\Lambda^{(1)}}=\{\Lambda^{(2)},\sigma_2+\Lambda^{(2)}\},$$

$$(F_1)_{\Lambda^{(1)}}=\{\sigma+F_1\mid\sigma=0,2\sigma_1,\sigma_2,\sigma_3,2\sigma_1+\sigma_2,2\sigma_1+\sigma_3,\sigma_2+\sigma_3,2\sigma_1+\sigma_2+\sigma_3\}.$$

Clearly, $(\Lambda,\Lambda^{(1)},\Lambda^{(2)})\not\sim(,,\star+\sigma_2)$. Now $(\Lambda,\Lambda^{(1)},F_1)\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},\sigma_2+F_1)\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1+\sigma_2)$, $(\Lambda,\Lambda^{(1)},\sigma_3+F_1)\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1+\sigma_3)$ and $(\Lambda,\Lambda^{(1)},\sigma_2+\sigma_3+F_1)\overset{\varphi_{id}}{\sim}(,,\star+2\sigma_1+\sigma_2+\sigma_3)$. Also $(\Lambda,\Lambda^{(1)},\sigma_2+F_1)$ is isomorphic to $(\Lambda,\Lambda^{(1)},\sigma_2+\sigma_3+F_1)$ (using φ_4). By looking at the cardinal of images of $(F_1+2\sigma)\cap 2\Lambda,(F_1+\sigma_2+2\sigma)\cap 2\Lambda$ and $(F_1+\sigma_3+2\sigma)\cap 2\Lambda$ in $\Lambda^{(1)}/2\Lambda^{(1)}$ (where $\sigma\in\Lambda$), one can easily see that

$$(\Lambda,\Lambda^{(1)},F_1+\sigma_2)\not\sim(,,\star)\not\sim(,,\star+\sigma_3)\not\sim(,,\star+\sigma_2).$$

Hence we obtain Table 6.5 for twist triple $(0,1,2)$.

$\underline{(0,2,3):}$ $(S,L,F)=(S_1+\mathbb{Z}\sigma_3,\Lambda^{(2)},S_0\oplus 2\mathbb{Z}\sigma_3)$ where $\langle S_0\rangle=2\mathbb{Z}\sigma_1+2\mathbb{Z}\sigma_2$ and $\langle S_1\rangle=\mathbb{Z}\sigma_1+2\mathbb{Z}\sigma_2$. ¿From [6], we have

$$[[T(0,2,3)]]=\{[[(\Lambda,\Lambda^{(2)},2\Lambda)]],[[\Lambda,\star,F_1)]],[[(S_2,\star,2\Lambda)]],[[(S_2,\star,F_1)]]\},$$

where $F_1=S(0,2\tilde\sigma_1,2\tilde\sigma_2)\oplus 2\mathbb{Z}\sigma_3$ and $S_2=S(0,\tilde\sigma_1,\tilde\sigma_2)\oplus\mathbb{Z}\sigma_3$. Now $(2\Lambda)_{\Lambda^{(2)}}=\{2\Lambda,\sigma_3+2\Lambda\}$, and

$$\begin{aligned}(F_1)_{\Lambda^{(2)}}=\{F_1+\sigma\mid\sigma=0,2\sigma_1,2\sigma_2,\sigma_3,2\sigma_1+2\sigma_2,2\sigma_1+\sigma_3,\\2\sigma_2+\sigma_3,2\sigma_1+2\sigma_2+\sigma_3\}.\end{aligned}$$

Clearly $(\Lambda,\Lambda^{(2)},2\Lambda)\not\sim(,,\star+\sigma_3)$ and $(S_2,\Lambda^{(2)},2\Lambda)\not\sim(,,\star+\sigma_3)$. Now the triples $(\Lambda,\Lambda^{(2)},F_1+\sigma)$, for $\sigma\in\{0,2\sigma_1,2\sigma_2,2\sigma_1+2\sigma_2\}$, are similar. This is also true for triples $(\Lambda,\Lambda^{(2)},F_1+\sigma)$ for $\sigma\in\{\sigma_3,\sigma_3+2\sigma_1,\sigma_3+2\sigma_2,\sigma_3+2\sigma_1+2\sigma_2\}$. It is clear that $(\Lambda,\Lambda^{(2)},F_1)\not\sim(,,\star+\sigma_3),(S_2,\Lambda^{(2)},F_1)\not\sim(,,\star+\sigma_3)$, $(S_2,\Lambda^{(2)},F_1+2\sigma_1)\not\sim(,,\star+\sigma_3)$ and $(S_2,\Lambda^{(2)},F_1)\not\sim(,,\star+2\sigma_1)$. Also we have $(S_2,\Lambda^{(2)},F_1+\sigma_3)\overset{\varphi_{13}}{\sim}(,,\star+2\sigma_1+\sigma_3)\overset{\varphi_9}{\sim}(,,\star+2\sigma_1+2\sigma_2+\sigma_3)$, $(S_2,\Lambda^{(2)},F_1+2\sigma_1+\sigma_3)\varphi_5(,,\star+2\sigma_2+\sigma_3)$, and $(S_2,\Lambda^{(2)},F_1+2\sigma_2)\varphi_5(,,\star+2\sigma_1)\overset{\varphi_9}{\sim}(,,\star+2\sigma_1+2\sigma_2)$. Thus we obtain Table 6.5 for twist triple $(0,2,3)$.

$\underline{(1,2,2):}$ $(S,L,F)=(S_1\oplus\mathbb{Z}\sigma_3,\Lambda^{(2)},S_0\oplus 4\mathbb{Z}\sigma_1)$ where $\langle S_0\rangle=2\mathbb{Z}\sigma_2\oplus\mathbb{Z}\sigma_3$ and $\langle S_1\rangle=\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_2$. From [6], we have

$$[[\mathcal{T}(1,2,2)]]=\{[[(\Lambda,\Lambda^{(2)},\Lambda^{(1,2,2)})]],[[(S_2,\star,\Lambda^{(1,2,2)})]],[[(\Lambda,\Lambda^{(2)},F_1)]],[[(S_2,\star,F_1)]]\},$$

where $S_2 = S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) \oplus \mathbb{Z}\sigma_3$ and $F_1 = S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) \oplus 4\mathbb{Z}\sigma_1$. Now $(\Lambda^{(1,2,2)})_{\Lambda^{(2)}} = \{\Lambda^{(1,2,2)}, 2\sigma_1 + \Lambda^{(1,2,2)}\}$ and

$$(F_1)_{\Lambda^{(2)}} = \{F_1 + \sigma \mid \sigma = 0, 2\sigma_1, 2\sigma_2, \sigma_3, 2\sigma_1 + 2\sigma_2, 2\sigma_1 + \sigma_3, 2\sigma_2 + \sigma_3, 2\sigma_1 + 2\sigma_2 + \sigma_3\}.$$

The following similarities are immediate using the identity map,

$$\begin{gathered}(\Lambda, \Lambda^{(2)}, \Lambda^{(1,2,2)}) \sim (,, \star + 2\sigma_1),\\ (\Lambda, \Lambda^{(2)}, F_1) \sim (,, \star + 2\sigma_1) \sim (,, \star + 2\sigma_1 + 2\sigma_2) \sim (,, \star + 2\sigma_2),\\ (\Lambda, \Lambda^{(2)}, \sigma_3 + F_1) \sim (,, \star + 2\sigma_1 + \sigma_3) \sim (,, \star + 2\sigma_1 + 2\sigma_2 + \sigma_3) \sim (,, \star + 2\sigma_2 + \sigma_3).\end{gathered}$$

Now if $(S_2, \Lambda^{(2)}, \Lambda^{(1,2,2)}) \overset{\varphi}{\sim} (S_2, \Lambda^{(2)}, \Lambda^{(1,2,2)} + 2\sigma_1)$, then there exists $\sigma \in S_2$ such that $\varphi(S_2) = S_2 + \sigma, \varphi(\Lambda^{(1,2,2)}) = \Lambda^{(1,2,2)} + 2\sigma_1 + 2\sigma$. Without loss of generality, we may assume that $\sigma \in \{0, \sigma_1, \sigma_2\}$, but since $\Lambda^{(1,2,2)}$ is a semilattice, we can take $\sigma = \sigma_1$. In this case, $\varphi(\sigma_2) \in (\sigma_1 + \Lambda^{(2)}) \cup (\sigma_1 + \sigma_2 + \Lambda^{(2)})$, but $((2\sigma_1 + 2\Lambda^{(2)}) \cup (2\sigma_1 + 2\sigma_2 + 2\Lambda^{(2)})) \cap (4\mathbb{Z}\sigma_1 + 2\mathbb{Z}\sigma_2 + \mathbb{Z}\sigma_3) = \emptyset$. This gives a contradiction.

By looking at the images of $(F_1 + 2\sigma) \cap 2\Lambda, (F_1 + 2\sigma_1 + 2\sigma) \cap 2\Lambda, (F_1 + \sigma_3 + 2\sigma) \cap 2\Lambda, (F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3 + 2\sigma) \cap 2\Lambda$ in $\Lambda^{(2)}/2\Lambda^{(2)}$ (where $\sigma \in \Lambda$), one can see that $(\Lambda, \Lambda^{(2)}, F_1) \not\sim (,, \star + \sigma_3)$, $(S_2, \Lambda^{(2)}, F_1) \not\sim (,, \star + \sigma_3) \not\sim (,, \star + 2\sigma_1)$ and $(S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + \sigma_3) \not\sim (,, E) \not\sim (,, \star + 2\sigma_1 + 2\sigma_2 + \sigma_3)$ (where $E = F_1$ or $F_1 + 2\sigma_1$). Also by noting that $\#(2(1/2(F_1 \cap 2\Lambda) \cap S_2))^\sim \neq \#(2(1/2((F_1 + 2\sigma_1 + 2\sigma) \cap 2\Lambda) \cap (S_2 + \sigma)))^\sim$ (where $\sigma \in S_2$) we have $(S_2, \Lambda^{(2)}, F_1) \not\sim (,, \star + 2\sigma_1)$. Now if $(S_2, \Lambda^{(2)}, F_1 + \sigma_3) \overset{\varphi}{\sim} (S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + \sigma_3)$ then $(S_2, \Lambda^{(2)}, F_1 + \sigma_3) \overset{\varphi}{\cong} (S_2 + \sigma_1, \Lambda^{(2)}, F_1 + \sigma_3)$, then by considering image of $F_1 + \sigma_3$ in $\Lambda^{(2)}/2\Lambda^{(2)}$ we have $\varphi(\sigma_2) \in \sigma_2 + \Lambda^{(2)}$, so $\varphi(S_2) \neq S_2 + \sigma_1$, this gives a contradiction. Also if $(S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + \sigma_3) \overset{\varphi}{\cong} (S_2 + \sigma, \Lambda^{(2)}, F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3 + 2\sigma)$, then by considering images of $F_1 + \sigma_3$ and $F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3$ in $\Lambda^{(2)}/2\Lambda^{(2)}$, we have $\varphi(\sigma_1) \in \sigma_1 + \sigma_2 + \sigma + \Lambda^{(2)}$ and $\varphi(\sigma_2) \in \sigma_2 + \Lambda^{(2)}$ so, as before, $\varphi(S_2) \neq S_2 + \sigma$. Hence $(S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + \sigma_3) \not\sim (S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3)$ and since for each $\sigma \in S_2$, $0 \in F_1 + \sigma_3$ and $0 \notin F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3 + 2\sigma$, we have $(S_2, \Lambda^{(2)}, F_1 + \sigma_3) \not\sim (S_2, \Lambda^{(2)}, F_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3)$.

Also we have

$$\begin{gathered}(S_2, \Lambda^{(2)}, F_1)\varphi_{16}(,, \star + 2\sigma_2), (S_2, \Lambda^{(2)}, F_1 + 2\sigma_1)\varphi_{16}(,, \star + 2\sigma_2 + 2\sigma_1),\\ (S_2, \Lambda^{(2)}, F_1 + \sigma_3) \overset{\varphi_9}{\approx} (,, \star + 2\sigma_2 + \sigma_3).\end{gathered}$$

Thus we obtain Table 6.5 for twist triple $(1, 2, 2)$.

<u>$(0,1,1)$</u>: We have $(S, L, F) = (\Lambda, \Lambda^{(1)}, S_0)$ where $\langle S_0 \rangle = \Lambda^{(1)}$. Also

$$\begin{gathered}(\Lambda, \Lambda^{(1)}, S^{(3)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(4)}_{\Lambda^{(0,1,1)}})\varphi_4^{-1}(,, S^{(1)}_{\Lambda^{(0,1,1)}})\varphi_{15}\\ (,, S^{(2)}_{\Lambda^{(0,1,1)}})\varphi_3(,, S^{(5)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(6)}_{\Lambda^{(0,1,1)}})\end{gathered},$$

and $(\Lambda, \Lambda^{(1)}, S^{(8)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(9)}_{\Lambda^{(0,1,1)}})\varphi_3(,, S^{(13)}_{\Lambda^{(0,1,1)}} + \sigma_2)$. Also

$$\begin{gathered}(\Lambda, \Lambda^{(1)}, S^{(10)}_{\Lambda^{(0,1,1)}} + 2\sigma_1)\varphi_{11}^{-1}(,, S^{(11)}_{\Lambda^{(0,1,1)}})\varphi_{15}(,, S^{(12)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(14)}_{\Lambda^{(0,1,1)}}),\\ (\Lambda, \Lambda^{(1)}, S^{(16)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(17)}_{\Lambda^{(0,1,1)}}),\ (\Lambda, \Lambda^{(1)}, S^{(18)}_{\Lambda^{(0,1,1)}})\varphi_1(,, S^{(19)}_{\Lambda^{(0,1,1)}}),\\ (\Lambda, \Lambda^{(1)}, S^{(20)}_{\Lambda^{(0,1,1)}})\varphi_{15}(,, S^{(21)}_{\Lambda^{(0,1,1)}}),\ (\Lambda, \Lambda^{(1)}, S^{(19)}_{\Lambda^{(0,1,1)}})\varphi_{12}(,, S^{(20)}_{\Lambda^{(0,1,1)}}),\\ \text{and } (\Lambda, \Lambda^{(1)}, S^{(16)}_{\Lambda^{(0,1,1)}})\varphi_4(,, S^{(20)}_{\Lambda^{(0,1,1)}}).\end{gathered}$$

Therefore,

$$[[\mathcal{T}(0,1,1)]] = \{[[(\Lambda, \Lambda^{(1)}, S_0)]] \mid S_0 \in \{S^{(i)}_{\Lambda^{(0,1,1)}} \mid i \in \{1, 7, 8, 11, 15, 16, 22, 23\}\}\}. \quad (5.6)$$

Since, $\Lambda^{(1)}$ is a lattice we have $(S_0)_{\Lambda^{(1)}}=\{S_0+\eta\mid\eta\in\Lambda^{(1)}\}$. Therefore we must find similarity classes of BC-triples $(\Lambda,\Lambda^{(1)},E)$ where E runs through $\cup_{i=3}^{7}(\mathrm{ind}_{\Lambda(0,1,1)}(i)+\Lambda^{(1)})$. If $(\Lambda,\Lambda^{(1)},E)\sim(\Lambda,\Lambda^{(1)},E')$, then there exists $\varphi\in\mathrm{Aut}(\mathcal{V}^0)$ such that $\varphi(\Lambda)=\Lambda$, $\varphi(\Lambda^{(1)})=\Lambda^{(1)}$ and $\varphi(E)=E'+2\sigma$ for some $\sigma\in\Lambda$. But since $\langle E\rangle=\langle E'\rangle=\Lambda^{(1)}$ we may assume that $\sigma\in\{0,\sigma_1\}$.

$\mathrm{ind}_{\Lambda(0,1,1)}(3)$: We have $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)\varphi_1(,,\star+\sigma_3)$. Clearly the following pairs of triples are similar under the identity map: $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)})$ and $(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)$ and $(,,\star+\sigma_2+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_3)$ and $(,,\star+\sigma_3+2\sigma_1)$, as well as $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)$ and $(,,\star+\sigma_2+\sigma_3+2\sigma_1)$.

We claim that $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)})\not\sim(\Lambda,\Lambda^{(1)},\star+\sigma_2)$. Otherwise there exists $\varphi\in GL(\Lambda)$ such that $\varphi(\Lambda)=\Lambda$, $\varphi(\Lambda^{(1)})=\Lambda^{(1)}$, $\varphi(S^{(1)}_{\Lambda(0,1,1)})=S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+2\tau$ where $\tau\in\{0,\sigma_1\}$. Since $S^{(1)}_{\Lambda(0,1,1)}$ is a semilattice, $\tau=0$. On the other hand $2\sigma_1\notin 2\Lambda^{(1)}, 2\sigma_1\in S^{(1)}_{\Lambda(0,1,1)}\cap 2\Lambda$ and $(S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)\cap 2\Lambda=2\Lambda^{(1)}$, so by Lemma 5.3 our claim is proved.

Note that $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\in\mathcal{T}_2$, so it follows from Remark 2.5 that $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_2+\sigma_3)$, $(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)\not\sim(,,\star+\sigma_2+\sigma_3)$. Therefore,

$$\begin{aligned}&\{[(\Lambda,\Lambda^{(1)},S_0+\tau)]\in\mathrm{BC}(0,1,1)\mid\tau\in\Lambda^{(1)},\mathrm{ind}(S_0)=3,S_0\neq S^{(7)}_{\Lambda(0,1,1)}\}=\\&\{[(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)})],[(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)],[(\Lambda,\Lambda^{(1)},S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)]\}.\end{aligned}\tag{5.7}$$

Note that $(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\overset{id}{\sim}(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)}+\sigma_2)\overset{id}{\sim}(,,\star+\sigma_2+2\sigma_1)$,

$$\begin{gathered}(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)}+\sigma_3)\overset{id}{\sim}(,,\star+\sigma_3+2\sigma_1),\\((\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\overset{id}{\sim}(,,\star+\sigma_2+\sigma_3+2\sigma_1),\\((\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\varphi_{11}(,,\star+\sigma_2+\sigma_3+2\sigma_1),\ (\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\varphi_4(,,\star+\sigma_3),\\(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)}+\sigma_2)\varphi_1(,,\star+\sigma_3).\end{gathered}$$

We have $(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\not\sim(,,S^{(1)}_{\Lambda(0,1,1)})$, and $(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\not\sim(,,S^{(1)}_{\Lambda(0,1,1)}+\sigma_2)$ (by Lemma 5.4(i)). Since $0\in S^{(7)}_{\Lambda(0,1,1)}$ and $0\notin S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3+2\sigma$ for $\sigma\in\{0,\sigma_1\}$, we have $(\Lambda,\Lambda^{(1)},S^{(7)}_{\Lambda(0,1,1)})\not\sim(,,S^{(1)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)$. Hence by (5.7) and (5.6) we obtain Table 6.5 for $(S,L,E)\in BC(0,1,1)$ with $\mathrm{ind}(E-\eta)=3$, $\eta\in E$.

$\mathrm{ind}_{\Lambda(0,1,1)}(4)$: We have $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\overset{id}{\sim}(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_2)\overset{id}{\sim}(,,\star+\sigma_2+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_3)\overset{id}{\sim}(,,\star+\sigma_3+2\sigma_1)$ and

$$(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\overset{id}{\sim}(,,\star+\sigma_2+\sigma_3+2\sigma_1),\ (\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\varphi_3(,,\star+\sigma_2).$$

Since $S^{(8)}_{\Lambda(0,1,1)}$ is a semilattice, by Lemma 5.3, $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_3)$. Also if $\sigma\in\{0,\sigma_1\}$, then $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3+2\sigma)\in\mathcal{T}_2$. So by Remark 2.5, $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_2+\sigma_3)$ and $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_3)\not\sim(,,\star+\sigma_2+\sigma_3)$. Therefore for $\tau\in\Lambda^{(1)}$ we have

$$\{[(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\tau)]\in\mathrm{BC}(0,1,1)\}=\{[(,,\star)],[(,,\star+\sigma_2+\sigma_3)],[(,,\star+\sigma_3)]\}.\tag{5.8}$$

Also, we have $(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)})\overset{id}{\sim}(,,\star+2\sigma_1)$, and

$$\begin{aligned}(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)&\overset{id}{\sim}(,,\star+\sigma_2+\sigma_3+2\sigma_1)\varphi_{12}(,,\star+\sigma_3)\varphi_1(,,\star+\sigma_2)\overset{id}{\sim}\\&(,,\star+\sigma_2+2\sigma_1)\varphi_1(,,\star+\sigma_3+2\sigma_1).\end{aligned}$$

Note that $S^{(11)}_{\Lambda(0,1,1)}$ is a semilattice and $S^{(11)}_{\Lambda(0,1,1)}+\sigma_2+2\sigma_1$ does not contain zero, so the triples $(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)})$ and $(,,\star+\sigma_2+2\sigma_1)$ are not isomorphic. Also, by Lemma 5.3, $(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)})\not\cong(,,\star+\sigma_2)$. So $(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_2)$. Therefore

$$\{[(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)}+\tau)]\in \mathrm{BC}(0,1,1)\mid \tau\in\Lambda^{(1)}\}=\{[(,,\star)],[(,,\star+\sigma_2)]\}. \tag{5.9}$$

If $\sigma\in\{0,\sigma_1\}$, then $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+2\sigma)$ and $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_3)\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+\sigma_2+2\sigma)$ (by Lemma 5.4(i)). Since $0\notin S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3$ and $S^{(11)}_{\Lambda(0,1,1)},S^{(11)}_{\Lambda(0,1,1)}+\sigma_2$ are semilattices, we get that $(\Lambda,\Lambda^{(1)},S^{(11)}_{\Lambda(0,1,1)}+\sigma_2)\not\cong(,,S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\not\cong(,,S^{(11)}_{\Lambda(0,1,1)})$. Similarly, we have $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+2\sigma_1)$, and also $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+2\sigma_1+\sigma_2)$. From Lemma 5.3 we have $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_3)\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+2\sigma)$, $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)})\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+\sigma_2)$, and $(\Lambda,\Lambda^{(1)},S^{(8)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\not\cong(,,S^{(11)}_{\Lambda(0,1,1)}+2\sigma_1+\sigma_2)$. So by (5.6), (5.8) and (5.9), we obtain Table 6.5 for $(S,L,E)\in BC(0,1,1)$ with $\mathrm{ind}(E-\eta)=4$, $\eta\in E$.

$\mathrm{ind}_{\Lambda(0,1,1)}(5)$: By (5.6), we must check similarity classes which have the form $[((\Lambda,\Lambda^{(1)},S^{(i)}_{\Lambda(0,1,1)}+\tau)]$ where $i=15,16$ and $\tau\in\Lambda^{(1)}$. Note that $S^{(15)}_{\Lambda(0,1,1)}+\sigma+2\sigma_1=S^{(15)}_{\Lambda(0,1,1)}+\sigma$ for each $\sigma\in\Lambda^{(1)}$. So the notions of similarity and isomorphism for the classes of the form $[((\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)}+\sigma)]$ with $\sigma\in\Lambda^{(1)}$ coincide. Also note that $S^{(16)}_{\Lambda(0,1,1)}+\sigma+\sigma_2=S^{(16)}_{\Lambda(0,1,1)}+\sigma$ for each $\sigma\in\Lambda^{(1)}$.

We have $(\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)}+\sigma_2)\varphi_1(,,S^{(15)}_{\Lambda(0,1,1)}+\sigma_3)$ and $(\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)})\varphi_4(,,\star+\sigma_3)$. Since $(\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)\in\mathcal{T}_2$, by Remark 2.5, we have $(\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_2+\sigma_3)$. Also we have, $(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)})\overset{id}{\sim}(,,\star+2\sigma_1)$, $(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)}+\sigma_3)\overset{id}{\sim}(,,\star+\sigma_3+2\sigma_1)$. On the other hand, by Lemma 5.3, $(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)})\not\cong(,,\star+\sigma_3)$. Since $0\in S^{(16)}_{\Lambda(0,1,1)}$ and $S^{(16)}_{\Lambda(0,1,1)}+\sigma_3+2\sigma_1$ does not contain zero we obtain $(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)})\not\cong(,,\star+\sigma_3+2\sigma_1)$. Thus $(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)})\not\sim(,,\star+\sigma_3)$. By Lemma 5.4($i$),

$$(\Lambda,\Lambda^{(1)},S^{(16)}_{\Lambda(0,1,1)})\not\sim(,,S^{(15)}_{\Lambda(0,1,1)})\not\sim(,,S^{(16)}_{\Lambda(0,1,1)}+\sigma_3).$$

Since $0\in S^{(16)}_{\Lambda(0,1,1)}\cap(S^{(16)}_{\Lambda(0,1,1)}+\sigma_3)$ but $S^{(15)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3$ does not contain zero, we have that $(\Lambda,\Lambda^{(1)},S^{(15)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3)$ is not similar to either $(,,S^{(16)}_{\Lambda(0,1,1)})$ or $(,,S^{(16)}_{\Lambda(0,1,1)}+\sigma_3)$. Therefore we get Table 6.5 for triples $(S,L,E)\in BC(0,1,1)$ with $\mathrm{ind}(E-\eta)=5$, $\eta\in E$.

$\mathrm{ind}_{\Lambda(0,1,1)}(6)$: It is easy to check that $(\Lambda,\Lambda^{(1)},S^{(22)}_{\Lambda(0,1,1)}+\sigma)\overset{id}{\sim}(,,\star+\sigma+2\sigma_1)$ for $\sigma=0,\sigma_2,\sigma_3$ and $\sigma_2+\sigma_3$. Also we have, $(\Lambda,\Lambda^{(1)},S^{(22)}_{\Lambda(0,1,1)})\varphi_4(,,\star+\sigma_3)\varphi_1(,,\star+\sigma_2)$. But by Lemma 5.3, $(\Lambda,\Lambda^{(1)},S^{(22)}_{\Lambda(0,1,1)})\not\cong(,,\star+\sigma_2+\sigma_3)$. Since $0\in S^{(22)}_{\Lambda(0,1,1)}$ and $S^{(22)}_{\Lambda(0,1,1)}+\sigma_2+\sigma_3+2\sigma_1$ does not contain zero, $(\Lambda,\Lambda^{(1)},S^{(22)}_{\Lambda(0,1,1)})\not\cong(,,\star+\sigma_2+\sigma_3+2\sigma_1)$. Hence we get Table 6.5 for triples $(S,L,E)\in BC(0,1,1)$ with $\mathrm{ind}(E-\eta)=6$, $\eta\in E$.

$\underline{(0,2,2):}$ We have $t_1=0,t=2$ and $t_2=2$. So $\Lambda_3=0$, $\Lambda_4=\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_2$, $\Lambda_5=0$, $\Lambda_6=\mathbb{Z}\sigma_3$, $\langle S_1\rangle=\Lambda_4$, $S_5=0$, $\langle S_0\rangle=2\Lambda_4+\Lambda_6=\Lambda^{(2)}=\Lambda^{(0,2,2)}$, $S=S_1\oplus\Lambda_5\oplus\Lambda_6=S_1\oplus\mathbb{Z}\sigma_3$, $L=2\Lambda_3\oplus2\Lambda_4\oplus S_5\oplus\Lambda_6=\Lambda^{(2)}$ and $F=4\Lambda_3\oplus2\Lambda_5\oplus S_0=S_0$. Now let S_1 be one of the semilattices

$$\mathbb{Z}\sigma_1\oplus\mathbb{Z}\sigma_2,\quad S(0,\sigma_1,\sigma_2):=2\Lambda_4\cup(\sigma_1+2\Lambda_4)\cup(\sigma_2+2\Lambda_4). \tag{5.10}$$

Then $(S_1{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)})\varphi_{10}(S_1+\mathbb{Z}\sigma_3+\sigma_1,,S^{(6)}_{\Lambda(0,2,2)})$. Also

$$\begin{gathered}(S_1+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\varphi_{15}(,,S^{(2)}_{\Lambda(0,2,2)})\varphi_5(,\Lambda^{(2)},S^{(3)}_{\Lambda(0,2,2)})\varphi_{15}(,,S^{(5)}_{\Lambda(0,2,2)}),\\ (S_1{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(6)}_{\Lambda(0,2,2)})\varphi_5(,,S^{(7)}_{\Lambda(0,2,2)}),\\ (S_1{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2)\varphi_5(,,S^{(10)}_{\Lambda(0,2,2)}+2\sigma_1)\varphi_{15}(,,S^{(14)}_{\Lambda(0,2,2)})\varphi_5(,,S^{(13)}_{\Lambda(0,2,2)})\varphi_{14}\\ (S_1+\mathbb{Z}\sigma_3+\sigma_1,\Lambda^{(2)},S^{(12)}_{\Lambda(0,2,2)})\varphi_{15}(,,S^{(11)}_{\Lambda(0,2,2)}),\\ (S_1{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)})\ \varphi_5(,,S^{(16)}_{\Lambda(0,2,2)})\varphi_{10}(S_1+\mathbb{Z}\sigma_3+\sigma_1,\Lambda^{(2)},S^{(18)}_{\Lambda(0,2,2)}),\\ (S_1{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)})\varphi_{15}(,,S^{(19)}_{\Lambda(0,2,2)})\varphi_5(,,S^{(20)}_{\Lambda(0,2,2)})\varphi_{15}(,,S^{(21)}_{\Lambda(0,2,2)}).\end{gathered}$$

So $[[(\mathcal{T}(0,2,2))]]$ is the set

$$\begin{gathered}\{[[(S_1+\mathbb{Z}\sigma_3,\Lambda^{(2)},S_0)]] \text{ where } S_1 \text{ runs through (5.10) and } S_0=S^{(i)}_{\Lambda(0,2,2)}\\ \text{for } i=1,4,8,9,15,17,22,23\}.\end{gathered}\tag{5.11}$$

We now want to find similarity classes of the form $[(S_1+\mathbb{Z}\sigma_3,\Lambda^{(2)},F+\tau)]$, where F runs through $\cup_{i=3}^{7}\mathrm{ind}_{\Lambda(0,2,2)}(i)$, $\tau\in\Lambda^{(2)}$ and S_1 is a semilattice that $\langle S_1\rangle=\Lambda_4$. But we know that two isomorphic semilattices have the same index, so for each $i\in\{3,4,5,6,7\}$, it is enough to find similarity classes of the form $[(S_1+\mathbb{Z}\sigma_3,\Lambda^{(2)},F+\tau)]$ where F runs through $\mathrm{ind}_{\Lambda(0,2,2)}(i),\tau\in\Lambda^{(2)}$ and S_1 is a semilattice with $\langle S_1\rangle=\Lambda_4$. Also it is trivial that we may assume $\tau=m\sigma_1+n\sigma_2+p\sigma_3$, where $m,n\in\{0,2\}$ and $p\in\{0,1\}$. First let $S_1\neq\Lambda_4$. So consider $S=S_1=S(0,\tilde\sigma_1,\tilde\sigma_2)$.

$\mathrm{ind}_{\Lambda(0,2,2)}(3)$: It follows from (5.11) that we must check similarity classes of the form $[(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(i)}_{(0,2,2)}+\tau)]$ where $i=1,4$.

$i=1$: We have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+2\sigma_2)\varphi_5(,,\star+2\sigma_1)\overset{\varphi_{14}}{\sim}(,,\star+2\sigma_1+2\sigma_2)$ and $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3)\varphi_5(,,\star+2\sigma_1+\sigma_3)\overset{\varphi_{13}}{\sim}(,,\star+\sigma_3)$. We claim that

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_1).$$

Note that $(S^{(1)}_{\Lambda(0,2,2)}\cap 2\Lambda)/2=S+\mathbb{Z}\sigma_3$ and $((S^{(1)}_{\Lambda(0,2,2)}+2\sigma_1)\cap 2\Lambda)/2=S+\mathbb{Z}\sigma_3+\sigma_1$, so we have from Lemma 5.4(ii) that $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\ncong(,,\star+2\sigma_1)$. In the same way, $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\ncong(S+\mathbb{Z}\sigma_3+\sigma_1,,S^{(1)}_{\Lambda(0,2,2)})$. But $S^{(1)}_{\Lambda(0,2,2)}$ is a semilattice, so we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_1)$ and, from Lemma 5.5, we obtain that $(S{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\not\sim(,,\star{+}\sigma_3)$. Now $(S{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}),(,,\star{+}\sigma_3),(,,\star{+}2\sigma_1)\notin\mathcal{T}_2$ and $(,,\star+2\sigma_1+2\sigma_2+\sigma_3)\in\mathcal{T}_2$, so $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_1+2\sigma_2+\sigma_3)$,

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_1+2\sigma_2+\sigma_3)$$

and $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+2\sigma_1)\not\sim(,,\star+2\sigma_1+2\sigma_2+\sigma_3)$. We can use Lemma 5.5, thus we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_1)$.

$i=4$: We have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)}+\sigma_3)\varphi_5(,,\star)\varphi_{15}(,,\star+2\sigma_1)\varphi_5(,,\star+2\sigma_2+\sigma_3)$,

$$\begin{aligned}(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)}+2\sigma_2)\varphi_{15}(,,\star+2\sigma_1+2\sigma_2)\varphi_{17}^{-1}&\\ (,,\star+2\sigma_1+\sigma_3)\varphi_{16}(,,\star+2\sigma_1+2\sigma_2+\sigma_3).&\end{aligned}$$

We claim that $(S{+}\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)})\not\sim(,,\star{+}2\sigma_1{+}2\sigma_2{+}\sigma_3)$. Suppose to the contrary that $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)})\overset{\varphi}{\sim}(,,\star+2\sigma_1+2\sigma_2+\sigma_3)$. Then $\varphi(S+\mathbb{Z}\sigma_3)=S+\mathbb{Z}\sigma_3+\sigma$ and $\varphi(S^{(4)}_{\Lambda(0,2,2)})=S^{(4)}_{\Lambda(0,2,2)}+2\sigma_1+2\sigma_2+\sigma_3+2\sigma$, for some $\sigma\in S+\mathbb{Z}\sigma_3$. Since

$S^{(4)}_{\Lambda(0,2,2)}$ is a semilattice, we may assume that $\sigma=\sigma_1$. Then it follows from Lemma 5.4(ii) that our claim holds. On the other hand, we have $\sum_{\tau\in S^{(4)}_{\Lambda(0,2,2)}}\tilde{\tau}\in(2\Lambda)^{\sim}$ but $\sum_{\tau'\in S^{(1)}_{\Lambda(0,2,2)}}\tilde{\tau}'\notin(2\Lambda)^{\sim}$. Since $\operatorname{ind}(S^{(1)}_{\Lambda(0,2,2)})=\operatorname{ind}(S^{(4)}_{\Lambda(0,2,2)})=3$, for each $\sigma,\sigma'\in\Lambda^{(2)}$ we have $\sum_{\tau\in S^{(4)}_{\Lambda(0,2,2)}+\sigma}\tilde{\tau}\in(2\Lambda)^{\sim}$ and $\sum_{\tau'\in S^{(1)}_{\Lambda(0,2,2)}+\sigma'}\tilde{\tau}'\notin(2\Lambda)^{\sim}$. Thus by Lemma 5.4$(i)$, for each $\sigma,\sigma'\in\Lambda^{(2)}$, the following holds $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+\sigma)\not\sim(,,S^{(4)}_{\Lambda(0,2,2)}+\sigma')$. So, we obtain Table 6.5 for triples $(S,L,E)\in BC(0,2,2)$ with $\operatorname{ind}(E-\eta)=3$, $\eta\in E$.

$\operatorname{ind}_{\Lambda(0,2,2)}(4)$: According to (5.11) we must find similarity classes of the form $[(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)}+\tau)]$ with $i\in\{8,9\}$.

$i=8$: $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+2\sigma_1+2\sigma_2)\varphi_{18}(,,\star)\overset{\varphi_{10}}{\sim}(,,\star+2\sigma_1)\varphi_5(,,\star+2\sigma_2)$. Also we have, $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma_3)\overset{\varphi_{13}}{\sim}(,,\star+2\sigma_1+\sigma_3)\varphi_5(,,\star+2\sigma_2+\sigma_3)$. On the other hand, $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+2\sigma_1+2\sigma_2+\sigma_3)\in\mathcal{T}_2$ but $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}),(,,\star+\sigma_3)\notin\mathcal{T}_2$ so by Remark 2.5,

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_1+2\sigma_2+\sigma_3)\text{ and}$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_1+2\sigma_2+\sigma_3).$$

Next note that $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)})\not\sim(,,\star+\sigma_3)$. Otherwise there exists $\varphi\in GL(\Lambda)$ such that $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)})\overset{\varphi}{\sim}(,,\star+\sigma_3)$. So there exists $\sigma\in\{0,\sigma_1,\sigma_2\}$ such that $\varphi(S+\mathbb{Z}\sigma_3)=S+\mathbb{Z}\sigma_3+\sigma$, $\varphi(S^{(8)}_{\Lambda(0,2,2)})=S^{(8)}_{\Lambda(0,2,2)}+\sigma_3+2\sigma$. But for $\sigma\in\{\sigma_1,\sigma_2\}$, $0\notin S^{(8)}_{\Lambda(0,2,2)}+\sigma_3+2\sigma$ so $\sigma=0$. Thus $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)})\cong(,,\star+\sigma_3)$. This contradicts Lemma 5.5.

$i=9$: We have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2)\overset{\varphi_{10}}{\sim}(,,\star+2\sigma_1+2\sigma_2)$ and

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1+\sigma_3)\overset{\varphi_{13}}{\sim}(,,\star+\sigma_3),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3)\varphi_{15}(,,\star+2\sigma_1+2\sigma_2+\sigma_3).$$

By Lemma 5.5, we have $(S+\mathbb{Z}\sigma_3,S^{(9)}_{\Lambda(0,2,2)})\not\sim(,,\star+\sigma_3)$ and

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_2+\sigma_3),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_1),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1)\not\sim(,,\star+2\sigma_2+\sigma_3),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_2),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3)\not\sim(,,\star+2\sigma_2).$$

Applying Lemma 5.4(ii), we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_1)$ and

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_2+\sigma_3),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2)\not\sim(,,\star).$$

Also by Lemma 5.4(iii), we have

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1)\ncong(,,\star+2\sigma_2),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1)\ncong(S+\mathbb{Z}\sigma_3+\sigma_2,,\star).$$

So, $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1)\not\sim(,,\star+2\sigma_2)$. By Lemma 5.5, if $\sigma\in\{0,\sigma_3,2\sigma_1+2\sigma_2+\sigma_3\}$ and $\tau\in\{0,\sigma_3,2\sigma_1,2\sigma_2,2\sigma_2+\sigma_3\}$ we obtain $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma)\not\sim$

$(,,S^{(9)}_{\Lambda(0,2,2)}+\tau)$. So for $\sigma,\tau\in\Lambda^{(2)}$ we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma)\not\sim(,,S^{(9)}_{\Lambda(0,2,2)}+\tau)$. Therefore, we get Table 6.5 for triples $(S,L,E)\in BC(0,2,2)$ with $\text{ind}(E-\eta)=4$, $\eta\in E$.

$\text{ind}_{\Lambda(0,2,2)}(5)$: By 5.11, it is enough to check similarity classes of the form $[(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)}+\tau)]$ for $i=15,17$.

$i=15$: Using Lemma 5.5, we see that $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)})\not\sim(,,\star+\sigma_3)$ and similarly $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)})\not\sim(,,\star+2\sigma_2+\sigma_3)$. But $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3)\not\sim(,,\star+\sigma_3)$, by Lemma 5.4$(ii)$. On the other hand, $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)})\overset{\varphi_{20}}{\sim}(,,\star+2\sigma_2)$.

$i=17$: By Lemma 5.4(ii), $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)})\not\sim(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},\star+2\sigma_1)$. Also $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)}+2\sigma_2)\varphi_5(,,\star+2\sigma_1)\overset{\varphi_9}{\sim}(,,\star+2\sigma_1+2\sigma_2)$, but

$$\sum_{\tau\in S^{(15)}_{\Lambda(0,2,2)}}\tilde{\tau}\in(2\Lambda)^{\sim},\qquad\sum_{\tau'\in S^{(17)}_{\Lambda(0,2,2)}}\tilde{\tau}'\notin(2\Lambda)^{\sim}\text{ and }\text{ind}(S^{(15)}_{\Lambda(0,2,2)})=\text{ind}(S^{(17)}_{\Lambda(0,2,2)})=5.$$

So for each $\sigma\in\Lambda^{(2)}$, we have $\sum_{\tau\in S^{(15)}_{\Lambda(0,2,2)}+\sigma}\tilde{\tau}\in(2\Lambda)^{\sim}$ and $\sum_{\tau'\in S^{(17)}_{\Lambda(0,2,2)}+\sigma}\tilde{\tau}'\notin(2\Lambda)^{\sim}$. Hence by Lemma 5.4(i), for each $\eta\in\{0,\sigma_1,\sigma_2\}$ and $\sigma\in\Lambda^{(2)}$, we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+\sigma)\not\cong(S+\mathbb{Z}\sigma_3+\eta,,S^{(17)}_{\Lambda(0,2,2)}+2\eta)$. So for each $\sigma,\tau\in\Lambda^{(2)}$ we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+\sigma)\not\sim(,,S^{(17)}_{\Lambda(0,2,2)}+\tau)$. Therefore, we get Table 6.5 for triples $(S,L,E)\in BC(0,2,2)$ with $\text{ind}(E-\eta)=5$, $\eta\in E$.

$\text{ind}_{\Lambda(0,2,2)}(6)$: We have

$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)})\overset{\varphi_{14}}{\sim}(,,\star+2\sigma_1)\varphi_5(,,\star+2\sigma_2)\varphi_{15}(,,\star+2\sigma_1+2\sigma_2),$$
$$(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+2\sigma_1+\sigma_3)\varphi_5(,,\star+2\sigma_2+\sigma_3)\overset{\varphi_{13}}{\sim}(,,\star+2\sigma_1+2\sigma_2+\sigma_3).$$

But by Lemma 5.5, we have $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star)\not\sim(,,\star+2\sigma_1+\sigma_3)$, and by Lemma 5.4$(ii)$ $(S+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+\sigma_3)\not\sim(,,\star+2\sigma_1+\sigma_3)$. Therefore, we get Table 6.5 for triples $(S,L,E)\in BC(0,2,2)$ with $\text{ind}(E-\eta)=6$, $\eta\in E$.

Now let $S_1=\Lambda$. Since for each $1\le i\le 23$ and $\sigma\in\{0,\sigma_3\}$ we have $(\Lambda,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)}+\sigma+2\sigma_2)\overset{id}{\sim}(,,\star+\sigma)\overset{id}{\sim}(,,\star+\sigma+2\sigma_1)\overset{id}{\sim}(,,\star+\sigma+2\sigma_1+2\sigma_2)$ so by 5.11 for each $i\in\{1,4,8,9,15,17,22,23\}$, we must check if $(\Lambda,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)})$ and $(\Lambda,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)}+\sigma_3)$ are similar. But by Lemma 5.5, for $i\in\{1,8,9,15,22\}$, we have $(\Lambda,\Lambda^{(2)},S^{(i)}_{\Lambda(0,2,2)})\not\sim(,,\star+\sigma_3)$, and for $\sigma,\tau\in\{0,\sigma_3\}$ and $(r,s)\in\{(1,4),(8,9),(15,17)\}$, we have

$$((\Lambda,\Lambda^{(2)},S^{(r)}_{\Lambda(0,2,2)}+\sigma)\not\sim(,,S^{(s)}_{\Lambda(0,2,2)}+\tau)$$

On the other hand, $S^{(17)}_{\Lambda(0,2,2)}=S^{(17)}_{\Lambda(0,2,2)}+\sigma_3$ and $(\Lambda,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)})\varphi_5(,,\star+\sigma_3)$, so we get Table 6.5 for triples $(S,L,S_0+\tau)\in BC(0,2,2)$ with $\langle S_0\rangle=\Lambda^{(2)}$, $\tau\in\Lambda^{(2)}$.

<u>$(0,3,3):$</u> $(S,L,F)=(S_1,2\Lambda,S_0)$ where $\langle S_1\rangle=\Lambda$ and $\langle S_0\rangle=2\Lambda$. So there exists a one to one correspondence between the set of Λ-pairs and $\mathcal{T}(0,3,3)$ and also there exists a one to one correspondence between the set of translated Λ-pairs and $BC(0,3,3)$. It is easy to see that there exists a one to one correspondence between the set of directed similarity classes of translated Λ-pairs and $[BC(0,3,3)]$. So by Proposition 4.2 and Table 6.2, we have $[BC(0,3,3)]$.

This completes the proof of proposition for the twist triples listed at the end of Section 2. The result is 182 non-similar BC-triples which are listed in Table 6.5

(the last column shows the number of BC-triples for that particular twist triple). The classification for dual triples can be obtained routinely as in [6]. Since the twist-triples $(0,0,3)$, $(1,1,2)$, $(0,1,3)$, $(0,2,3)$, $(1,2,2)$ and $(0,3,3)$ are self dual, it follows that we have 114 more non-similar BC-triples, so the total is 230. □

6 Classification tables

The isomorphisms used in the proofs

$$\varphi_1=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\ \varphi_2=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix},\ \varphi_3=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix},\ \varphi_4=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix},$$
$$\varphi_5=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ \varphi_6=\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix},\ \varphi_7=\begin{bmatrix}1&0&0\\1&1&1\\0&0&1\end{bmatrix},\ \varphi_8=\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix},$$
$$\varphi_9=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix},\varphi_{10}=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\varphi_{11}=\begin{bmatrix}1&2&2\\0&1&0\\0&0&1\end{bmatrix},\varphi_{12}=\begin{bmatrix}1&0&2\\0&1&1\\0&0&1\end{bmatrix},$$
$$\varphi_{13}=\begin{bmatrix}1&1&2\\0&1&0\\0&0&1\end{bmatrix},\varphi_{14}=\begin{bmatrix}1&1&2\\1&0&0\\0&0&1\end{bmatrix},\varphi_{15}=\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix},\varphi_{16}=\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix},$$
$$\varphi_{17}=\begin{bmatrix}0&1&2\\1&0&0\\0&0&1\end{bmatrix},\varphi_{18}=\begin{bmatrix}0&1&2\\1&0&2\\0&0&1\end{bmatrix},\varphi_{19}=\begin{bmatrix}1&0&0\\1&1&2\\0&0&1\end{bmatrix},\varphi_{20}=\begin{bmatrix}1&1&0\\0&1&2\\0&0&1\end{bmatrix}, \tag{6.1}$$
$$\varphi_{21}=\begin{bmatrix}1&1&2\\0&1&2\\0&0&1\end{bmatrix},\varphi_{22}=\begin{bmatrix}1&1&1\\1&0&0\\0&0&1\end{bmatrix},\varphi_{23}=\begin{bmatrix}1&0&0\\1&1&1\\0&1&0\end{bmatrix},\varphi_{24}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},$$
$$\varphi_{25}=\begin{bmatrix}0&0&1\\1&1&1\\0&1&1\end{bmatrix},\varphi_{26}=\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix},\varphi_{27}=\begin{bmatrix}1&1&1\\0&0&1\\0&1&0\end{bmatrix},\varphi_{28}=\begin{bmatrix}1&1&1\\1&0&0\\0&0&1\end{bmatrix},$$
$$\varphi_{29}=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix},\varphi_{30}=\begin{bmatrix}1&1&1\\0&1&0\\1&0&0\end{bmatrix},\varphi_{31}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix},\varphi_{32}=\begin{bmatrix}1&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

Table 6.2. Translated Λ-pairs up to directed similarity

index	$(S, F+\eta)$	#
(3,3)	$(S_\Lambda^{(1)}, S_\Lambda^{(1)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_2)$, $(S_\Lambda^{(1)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2+\sigma_3)$, $(S_\Lambda^{(1)}, S_\Lambda^{(2)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(2)}+\sigma_2)$, $(S_\Lambda^{(1)}, S_\Lambda^{(2)}+\sigma_2+\sigma_3)$	6
(3,4)	$(S_\Lambda^{(1)}, S_\Lambda^{(8)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1)$, $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_3)$, $(S_\Lambda^{(1)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2+\sigma_3)$, $(S_\Lambda^{(1)}, S_\Lambda^{(12)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(12)}+\sigma_1+\sigma_3)$	6
(3,5)	$(S_\Lambda^{(1)}, S_\Lambda^{(15)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(15)}+\sigma_2)$, $(S_\Lambda^{(1)}, S_\Lambda^{(15)}+\sigma_2+\sigma_3)$, $(S_\Lambda^{(1)}, S_\Lambda^{(21)})$	4
(3,6)	$(S_\Lambda^{(1)}, S_\Lambda^{(22)})$, $(S_\Lambda^{(1)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2+\sigma_3)$	2
(3,7)	$(S_\Lambda^{(1)}, \Lambda)$	1
(4,3)	$(S_\Lambda^{(8)}, S_\Lambda^{(1)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1)$, $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(1)}+\sigma_1+\sigma_2+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(4)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(4)}+\sigma_1)$	6
(4,4)	$(S_\Lambda^{(8)}, S_\Lambda^{(8)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_1)$, $(S_\Lambda^{(8)}, S_\Lambda^{(8)}+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(9)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_2)$, $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_2+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(9)}+\sigma_1+\sigma_2+\sigma_3)$	8
(4,5)	$(S_\Lambda^{(8)}, S_\Lambda^{(15)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(15)}+\sigma_2)$, $(S_\Lambda^{(8)}, S_\Lambda^{(15)}+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(17)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(17)}+\sigma_1+\sigma_2)$	5
(4,6)	$(S_\Lambda^{(8)}, S_\Lambda^{(22)})$, $(S_\Lambda^{(8)}, S_\Lambda^{(22)}+\sigma_3)$, $(S_\Lambda^{(8)}, S_\Lambda^{(22)}+\sigma_1+\sigma_2)$	3
(4,7)	$(S_\Lambda^{(8)}, \Lambda)$	1
(5,3)	$(S_\Lambda^{(15)}, S_\Lambda^{(1)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(1)}+\sigma_2)$, $(S_\Lambda^{(15)}, S_\Lambda^{(1)}+\sigma_2+\sigma_3)$, $(S_\Lambda^{(15)}, S_\Lambda^{(7)})$	4
(5,4)	$(S_\Lambda^{(15)}, S_\Lambda^{(8)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_3)$, $(S_\Lambda^{(15)}, S_\Lambda^{(8)}+\sigma_2)$, $(S_\Lambda^{(15)}, S_\Lambda^{(10)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(10)}+\sigma_2+\sigma_3)$	5
(5,5)	$(S_\Lambda^{(15)}, S_\Lambda^{(15)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(15)}+\sigma_2)$, $(S_\Lambda^{(15)}, S_\Lambda^{(16)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(16)}+\sigma_3)$	4
(5,6)	$(S_\Lambda^{(15)}, S_\Lambda^{(22)})$, $(S_\Lambda^{(15)}, S_\Lambda^{(22)}+\sigma_2+\sigma_3)$	2
(5,7)	$(S_\Lambda^{(15)}, \Lambda)$	1
(6,3)	$(S_\Lambda^{(22)}, S_\Lambda^{(1)})$, $(S_\Lambda^{(22)}, S_\Lambda^{(1)}+\sigma_2+\sigma_3)$	2
(6,4)	$(S_\Lambda^{(22)}, S_\Lambda^{(8)})$, $(S_\Lambda^{(22)}, S_\Lambda^{(8)}+\sigma_1+\sigma_2)$, $(S_\Lambda^{(22)}, S_\Lambda^{(8)}+\sigma_1+\sigma_3)$	3
(6,5)	$(S_\Lambda^{(22)}, S_\Lambda^{(15)})$, $(S_\Lambda^{(22)}, S_\Lambda^{(15)}+\sigma_2)$	2
(6,6)	$(S_\Lambda^{(22)}, S_\Lambda^{(22)})$, $(S_\Lambda^{(22)}, S_\Lambda^{(22)}+\sigma_1)$	2
(6,7)	$(S_\Lambda^{(22)}, \Lambda)$	1
(7,3)	$(\Lambda, S_\Lambda^{(1)})$	1
(7,4)	$(\Lambda, S_\Lambda^{(8)})$	1
(7,5)	$(\Lambda, S_\Lambda^{(15)})$	1
(7,6)	$(\Lambda, S_\Lambda^{(22)})$	1
(7,7)	(Λ, Λ)	1
Total		73

Table 6.3. BC-triples, up to similarity for $\nu = 1, 2$

ν	(t_1, t, t_2)	(S, L, E)	#
1	(0,0,0)	$(\Lambda, \Lambda, \Lambda)$	1
1	(0,0,1)	$(\Lambda, \Lambda, 2\Lambda), (\Lambda, \Lambda, 2\Lambda+\sigma_1)$	2
1	(0,1,1)	$(\Lambda, 2\Lambda, 2\Lambda)$	1
1	(1,1,1)	$(\Lambda, 2\Lambda, 4\Lambda)$	1
2	(1,2,2)	$(\Lambda, 2\Lambda, 2\Lambda^{(1)}), (S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 2\Lambda^{(1)}),$ $(S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 2\Lambda^{(1)}+2\sigma_1)$	3
2	(0,1,2)	$(\Lambda, \Lambda^{(1)}, 2\Lambda), (\Lambda, \Lambda^{(1)}, 2\Lambda+\sigma_2)$	2
2	(0,0,0)	$(\Lambda, \Lambda, \Lambda),$ $(\Lambda, \Lambda, S(0,\tilde\sigma_1,\tilde\sigma_2)), (\Lambda, \Lambda, S(0,\tilde\sigma_1,\tilde\sigma_2)+\sigma_1+\sigma_2)$	3
2	(2,2,2)	$(\Lambda, 2\Lambda, 4\Lambda),$ $(S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 4\Lambda), (S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 4\Lambda+2\sigma_1+2\sigma_2)$	3
2	(0,2,2)	$(\Lambda, 2\Lambda, 2\Lambda), (S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 2S(0,\tilde\sigma_1,\tilde\sigma_2)),$ $(S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 2S(0,\tilde\sigma_1,\tilde\sigma_2)+2\sigma_1),$ $(\Lambda, 2\Lambda, 2S(0,\tilde\sigma_1,\tilde\sigma_2)), (S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda, 2\Lambda)$	5
2	(1,1,2)	$(\Lambda, \Lambda^{(1)}, 2\Lambda^{(1)}), (\Lambda, \Lambda^{(1)}, 2\Lambda^{(1)}+\sigma_2)$	2
2	(0,0,2)	$(\Lambda, \Lambda, 2\Lambda), (\Lambda, \Lambda, 2\Lambda+\sigma_1),$ $(\Lambda, S(0,\tilde\sigma_1,\tilde\sigma_2), 2\Lambda)$	3
2	(1,1,1)	$(\Lambda, \Lambda^{(1)}, 4\mathbb{Z}\sigma_1+\mathbb{Z}\sigma_2)$	1
2	(0,1,1)	$(\Lambda, \Lambda^{(1)}, S(0,2\tilde\sigma_1,\tilde\sigma_2)),$ $(\Lambda, \Lambda^{(1)}, S(0,2\tilde\sigma_1,\tilde\sigma_2)+\sigma_2), (\Lambda, \Lambda^{(1)}, \Lambda^{(1)})$	3
2	(0,0,1)	$(\Lambda, \Lambda, \Lambda^{(1)}), (\Lambda, \Lambda, \Lambda^{(1)}+\sigma_1)$	2
Total			32

Table 6.4. BC-pairs, up to similarity for $\nu = 1, 2$

ν	(t_1, t, t_2)	(S, E)	#
1	(0,0,0)	(Λ, Λ)	1
1	(0,0,1)	$(\Lambda, 2\Lambda + \sigma_1)$	1
1	(0,1,1)	$(\Lambda, 2\Lambda)$	1
1	(1,1,1)	$(\Lambda, 4\Lambda)$	1
2	(1,2,2)	$(\Lambda, 2\Lambda^{(1)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 2\Lambda^{(1)})$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 2\Lambda^{(1)} + 2\sigma_1)$	3
2	(0,1,2)	$(\Lambda, 2\Lambda + \sigma_2),$	1
2	(0,0,0)	$(\Lambda, \Lambda),$ $(\Lambda, S(0, \tilde{\sigma}_1, \tilde{\sigma}_2)), (\Lambda, S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \sigma_1 + \sigma_2)$	3
2	(2,2,2)	$(\Lambda, 4\Lambda),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 4\Lambda), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 4\Lambda + 2\sigma_1 + 2\sigma_2)$	3
2	(0,2,2)	$(\Lambda, 2\Lambda), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 2S(0, \tilde{\sigma}_1, \tilde{\sigma}_2)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 2S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + 2\sigma_1),$ $(\Lambda, 2S(0, \tilde{\sigma}_1, \tilde{\sigma}_2)), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2), 2\Lambda)$	5
2	(1,1,2)	$(\Lambda, 2\Lambda^{(1)} + \sigma_2)$	1
2	(1,1,1)	$(\Lambda, 4\mathbb{Z}\sigma_1 + \mathbb{Z}\sigma_2)$	1
2	(0,1,1)	$(\Lambda, S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_2)),$ $(\Lambda, S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_2) + \sigma_2), (\Lambda, \Lambda^{(1)})$	3
2	(0,0,1)	$(\Lambda, \Lambda^{(1)} + \sigma_1)$	1
Total			25

Table 6.5. BC-triples, up to similarity and duality, for $\nu = 3$

(t_1, t, t_2)	(S, L, E)	#
(0,0,0)	$(\Lambda, \Lambda, S_\Lambda^{(1)}), (\Lambda, \Lambda, S_\Lambda^{(1)} + \sigma_1 + \sigma_2 + \sigma_3),$ $(\Lambda, \Lambda, S_\Lambda^{(8)}), (\Lambda, \Lambda, S_\Lambda^{(8)} + \sigma_1 + \sigma_2 + \sigma_3), (\Lambda, \Lambda, S_\Lambda^{(11)}),$ $(\Lambda, \Lambda, S_\Lambda^{(15)}), (\Lambda, \Lambda, S_\Lambda^{(15)} + \sigma_1 + \sigma_2 + \sigma_3),$ $(\Lambda, \Lambda, S_\Lambda^{(22)}), (\Lambda, \Lambda, S_\Lambda^{(22)} + \sigma_1 + \sigma_2 + \sigma_3),$ $(\Lambda, \Lambda, \Lambda)$	10
(0,0,1)	$(\Lambda, \Lambda, \Lambda^{(1)}), (\Lambda, \Lambda, \Lambda^{(1)} + \sigma_1),$ $(\Lambda, \Lambda, 2\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3)), (\Lambda, \Lambda, 2\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3) + \sigma_1),$ $(\Lambda, \Lambda, 2\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3) + \sigma_2 + \sigma_3)$	5
(0,0,2)	$(\Lambda, \Lambda, \Lambda^{(2)}), (\Lambda, \Lambda, \Lambda^{(2)} + \sigma_2), (\Lambda, S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) \oplus \mathbb{Z}\sigma_3, \Lambda^{(2)})$	3
(0,0,3)	$(\Lambda, \Lambda, 2\Lambda), (\Lambda, \Lambda, \sigma_1 + 2\Lambda), (\Lambda, S_\Lambda^{(1)}, 2\Lambda), (\Lambda, S_\Lambda^{(8)}, 2\Lambda),$ $(\Lambda, S_\Lambda^{(15)}, 2\Lambda), (\Lambda, S_\Lambda^{(22)}, 2\Lambda), (\Lambda, S_\Lambda^{(15)}, \sigma_1 + 2\Lambda)$	7
(1,1,1)	$(\Lambda, \Lambda^{(1)}, \Lambda^{(1,1,1)}), (\Lambda, \Lambda^{(1)}, 4\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3)), (\Lambda, \Lambda^{(1)}, 4\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3) + \sigma_2 + \sigma_3)$	3
(1,1,2)	$(\Lambda, \Lambda^{(1)}, \Lambda^{(1,1,2)}), (\Lambda, \Lambda^{(1)}, \sigma_2 + \Lambda^{(1,1,2)})$	2
(0,1,3)	$(\Lambda, \Lambda^{(1)}, 2\Lambda), (\Lambda, \Lambda^{(1)}, \sigma_2 + 2\Lambda), (\Lambda, 2\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3), 2\Lambda)$	3
(0,1,2)	$(\Lambda, \Lambda^{(1)}, \Lambda^{(2)}), (\Lambda, \Lambda^{(1)}, \sigma_2 + \Lambda^{(2)}), (\Lambda, \Lambda^{(1)}, 2\mathbb{Z}\sigma_2 + S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_3)),$ $(\Lambda, \Lambda^{(1)}, \sigma_2 + 2\mathbb{Z}\sigma_2 + S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_3)), (\Lambda, \Lambda^{(1)}, 2\mathbb{Z}\sigma_2 + S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_3) + \sigma_3)$	5
(0,2,3)	$(\Lambda, \Lambda^{(2)}, 2\Lambda), (\Lambda, \Lambda^{(2)}, \sigma_3 + 2\Lambda), (\Lambda, \Lambda^{(2)}, 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3)),$ $(\Lambda, \Lambda^{(2)}, \sigma_3 + 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3)), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, 2\Lambda),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \sigma_3 + 2\Lambda), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \sigma_3 + 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, 2\sigma_1 + 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3))$	9
(1,2,2)	$(\Lambda, \Lambda^{(2)}, \Lambda^{(1,2,2)}), (\Lambda, \Lambda^{(2)}, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1),$ $(\Lambda, \Lambda^{(2)}, \sigma_3 + S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \sigma_3 + S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \Lambda^{(1,2,2)}),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \Lambda^{(1,2,2)} + 2\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1 + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3))$	10
(0,1,1)	$(\Lambda, \Lambda^{(1)}, S^{(1)}_{\Lambda^{(0,1,1)}}), (\Lambda, \Lambda^{(1)}, S^{(1)}_{\Lambda^{(0,1,1)}} + \sigma_2),$ $(\Lambda, \Lambda^{(1)}, S^{(1)}_{\Lambda^{(0,1,1)}} + \sigma_2 + \sigma_3), (\Lambda, \Lambda^{(1)}, S^{(7)}_{\Lambda^{(0,1,1)}}), (\Lambda, \Lambda^{(1)}, S^{(11)}_{\Lambda^{(0,1,1)}}),$ $(\Lambda, \Lambda^{(1)}, S^{(11)}_{\Lambda^{(0,1,1)}} + \sigma_2), (\Lambda, \Lambda^{(1)}, S^{(8)}_{\Lambda^{(0,1,1)}}), (\Lambda, \Lambda^{(1)}, S^{(8)}_{\Lambda^{(0,1,1)}} + \sigma_2 + \sigma_3),$ $(\Lambda, \Lambda^{(1)}, S^{(8)}_{\Lambda^{(0,1,1)}} + \sigma_3), (\Lambda, \Lambda^{(1)}, S^{(16)}_{\Lambda^{(0,1,1)}}), (\Lambda, \Lambda^{(1)}, S^{(16)}_{\Lambda^{(0,1,1)}} + \sigma_3),$ $(\Lambda, \Lambda^{(1)}, S^{(15)}_{\Lambda^{(0,1,1)}}), (\Lambda, \Lambda^{(1)}, S^{(15)}_{\Lambda^{(0,1,1)}} + \sigma_2 + \sigma_3), (\Lambda, \Lambda^{(1)}, S^{(22)}_{\Lambda^{(0,1,1)}}),$ $(\Lambda, \Lambda^{(1)}, S^{(22)}_{\Lambda^{(0,1,1)}} + \sigma_2 + \sigma_3), (\Lambda, \Lambda^{(1)}, \Lambda^{(0,1,1)})$	16

(t_1,t,t_2)	(S,L,E)	#
(0,2,2)	$(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+2\sigma_1),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+2\sigma_1+2\sigma_2+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)}),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)}+2\sigma_2),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+2\sigma_1+2\sigma_2+\sigma_3),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_1),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+\sigma_3),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+2\sigma_2+\sigma_3),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)}),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)}+2\sigma_1),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+\sigma_3),(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+2\sigma_1+\sigma_3),$ $(S(0,\tilde{\sigma}_1,\tilde{\sigma}_2)+\mathbb{Z}\sigma_3,\Lambda^{(2)},\Lambda^{(2)}),$ $(\Lambda,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}),(\Lambda,\Lambda^{(2)},S^{(1)}_{\Lambda(0,2,2)}+\sigma_3),(\Lambda,\Lambda^{(2)},S^{(4)}_{\Lambda(0,2,2)}),$ $(\Lambda,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}),(\Lambda,\Lambda^{(2)},S^{(8)}_{\Lambda(0,2,2)}+\sigma_3),(\Lambda,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}),$ $(\Lambda,\Lambda^{(2)},S^{(9)}_{\Lambda(0,2,2)}+\sigma_3),(\Lambda,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}),(\Lambda,\Lambda^{(2)},S^{(15)}_{\Lambda(0,2,2)}+\sigma_3),$ $(\Lambda,\Lambda^{(2)},S^{(17)}_{\Lambda(0,2,2)}),(\Lambda,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}),(\Lambda,\Lambda^{(2)},S^{(22)}_{\Lambda(0,2,2)}+\sigma_3),$ $(\Lambda,\Lambda^{(2)},\Lambda^{(2)})$	36
(0,3,3)	$(S^{(1)}_\Lambda,2\Lambda,2S^{(1)}_\Lambda),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_2)),$ $(S^{(1)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_1+\sigma_2+\sigma_3)),\ (S^{(1)}_\Lambda,2\Lambda,2S^{(2)}_\Lambda),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(2)}_\Lambda+\sigma_2)),$ $(S^{(1)}_\Lambda,2\Lambda,2(S^{(2)}_\Lambda+\sigma_2+\sigma_3)),$ $(S^{(1)}_\Lambda,2\Lambda,2S^{(8)}_\Lambda),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_1)),$ $(S^{(1)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_3)),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_1+\sigma_2+\sigma_3)),$ $(S^{(1)}_\Lambda,2\Lambda,2S^{(12)}_\Lambda),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(12)}_\Lambda+\sigma_1+\sigma_3)),$ $(S^{(1)}_\Lambda,2\Lambda,2S^{(15)}_\Lambda),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_2)),$ $(S^{(1)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_2+\sigma_3)),\ (S^{(1)}_\Lambda,2\Lambda,2S^{(21)}_\Lambda),$ $(S^{(1)}_\Lambda,2\Lambda,2S^{(22)}_\Lambda)),\ (S^{(1)}_\Lambda,2\Lambda,2(S^{(22)}_\Lambda+\sigma_1+\sigma_2+\sigma_3)),$ $(S^{(1)}_\Lambda,2\Lambda,2\Lambda),$ $(S^{(8)}_\Lambda,2\Lambda,2S^{(1)}_\Lambda),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_1)),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_3)),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_1+\sigma_2+\sigma_3)),$ $(S^{(8)}_\Lambda,2\Lambda,2S^{(4)}_\Lambda),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(4)}_\Lambda+\sigma_1),$ $(S^{(8)}_\Lambda,2\Lambda,2S^{(8)}_\Lambda),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_1)),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_3)),\ (S^{(8)}_\Lambda,2\Lambda,2S^{(9)}_\Lambda),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(9)}_\Lambda+\sigma_2)),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(9)}_\Lambda+\sigma_3)),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(9)}_\Lambda+\sigma_2+\sigma_3)),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(9)}_\Lambda+\sigma_1+\sigma_2+\sigma_3)),$ $(S^{(8)}_\Lambda,2\Lambda,2S^{(15)}_\Lambda),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_2)),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_3)),\ (S^{(8)}_\Lambda,2\Lambda,2S^{(17)}_\Lambda),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(17)}_\Lambda+\sigma_1+\sigma_2),\ (S^{(8)}_\Lambda,2\Lambda,2S^{(22)}_\Lambda),$ $(S^{(8)}_\Lambda,2\Lambda,2(S^{(22)}_\Lambda+\sigma_3)),\ (S^{(8)}_\Lambda,2\Lambda,2(S^{(22)}_\Lambda+\sigma_1+\sigma_2)),$ $(S^{(8)}_\Lambda,2\Lambda,2\Lambda),\ (S^{(15)}_\Lambda,2\Lambda,2S^{(1)}_\Lambda),$ $(S^{(15)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_2)),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(1)}_\Lambda+\sigma_2+\sigma_3)),$ $(S^{(15)}_\Lambda,2\Lambda,2S^{(7)}_\Lambda)\ ,(S^{(15)}_\Lambda,2\Lambda,2S^{(8)}_\Lambda),$ $(S^{(15)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_3)),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_2)),$ $(S^{(15)}_\Lambda,2\Lambda,2S^{(10)}_\Lambda),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(10)}_\Lambda+\sigma_2+\sigma_3)),$ $(S^{(15)}_\Lambda,2\Lambda,2S^{(15)}_\Lambda),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_2)),$ $(S^{(15)}_\Lambda,2\Lambda,2S^{(16)}_\Lambda),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(16)}_\Lambda+\sigma_3)),$ $(S^{(15)}_\Lambda,2\Lambda,2S^{(22)}_\Lambda),\ (S^{(15)}_\Lambda,2\Lambda,2(S^{(22)}_\Lambda+\sigma_2+\sigma_3)),$ $(S^{(15)}_\Lambda,2\Lambda,2\Lambda),\ (S^{(22)}_\Lambda,2\Lambda,2S^{(1)}_\Lambda),$ $(S^{(22)}_\Lambda,2\Lambda,(2S^{(1)}_\Lambda+\sigma_2+\sigma_3)),\ (S^{(22)}_\Lambda,2\Lambda,2S^{(8)}_\Lambda),$ $(S^{(22)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_1+\sigma_2)),\ (S^{(22)}_\Lambda,2\Lambda,2(S^{(8)}_\Lambda+\sigma_1+\sigma_3)),$ $(S^{(22)}_\Lambda,2\Lambda,2S^{(15)}_\Lambda),\ (S^{(22)}_\Lambda,2\Lambda,2(S^{(15)}_\Lambda+\sigma_2)),$ $(S^{(22)}_\Lambda,2\Lambda,2S^{(22)}_\Lambda),\ (S^{(22)}_\Lambda,2\Lambda,2(S^{(22)}_\Lambda+\sigma_1)),$ $(S^{(22)}_\Lambda,2\Lambda,2\Lambda),$ $(\Lambda,2\Lambda,2S^{(1)}_\Lambda),\ (\Lambda,2\Lambda,2S^{(8)}_\Lambda),\ (\Lambda,2\Lambda,2S^{(15)}_\Lambda),$ $(\Lambda,2\Lambda,2S^{(22)}_\Lambda),\ (\Lambda,2\Lambda,2\Lambda)$	73
Total		182
+Dual		230

Table 6.6. BC-pairs, up to similarity and duality for $\nu = 3$

ν	(t_1, t, t_2)	(S, E)	#
3	(0,0,0)	$(\Lambda, S_\Lambda^{(1)}), (\Lambda, S_\Lambda^{(1)} + \sigma_1 + \sigma_2 + \sigma_3),$ $(\Lambda, S_\Lambda^{(8)}), (\Lambda, S_\Lambda^{(8)} + \sigma_1 + \sigma_2 + \sigma_3), (\Lambda, S_\Lambda^{(11)}),$ $(\Lambda, S_\Lambda^{(15)}), (\Lambda, S_\Lambda^{(15)} + \sigma_1 + \sigma_2 + \sigma_3),$ $(\Lambda, S_\Lambda^{(22)}), (\Lambda, S_\Lambda^{(22)} + \sigma_1 + \sigma_2 + \sigma_3),$ (Λ, Λ)	10
	(0,0,1)	$(\Lambda, \Lambda^{(1)} + \sigma_1), (\Lambda, 2\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3) + \sigma_1)$	2
	(1,1,1)	$(\Lambda, \Lambda^{(1,1,1)}), (\Lambda, 4\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3)), (\Lambda, 4\mathbb{Z}\sigma_1 + S(0, \tilde{\sigma}_2, \tilde{\sigma}_3) + \sigma_2 + \sigma_3)$	3
	(1,1,2)	$(\Lambda, \sigma_2 + \Lambda^{(1,1,2)})$	1
	(0,1,2)	$(\Lambda, \sigma_2 + \Lambda^{(2)}),$ $(\Lambda, \sigma_2 + 2\mathbb{Z}\sigma_2 + S(0, 2\tilde{\sigma}_1, \tilde{\sigma}_3))$	2
	(0,2,3)	$(\Lambda, 2\Lambda + \sigma_3), (\Lambda, \sigma_3 + 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, 2\Lambda + \sigma_3)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}, \sigma_3 + 2(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3))$	4
	(1,2,2)	$(\Lambda, \Lambda^{(1,2,2)}), (\Lambda, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1),$ $(\Lambda, \sigma_3 + S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \sigma_3 + S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1)), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(1,2,2)}),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, 2\sigma_1 + \Lambda^{(1,2,2)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1 + \sigma_3)),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S(0, 2\tilde{\sigma}_2, \tilde{\sigma}_3) + 4\mathbb{Z}\sigma_1 + 2\sigma_1 + 2\sigma_2 + \sigma_3))$	10
	(0,1,1)	$(\Lambda, S^{(1)}_{\Lambda(0,1,1)}), (\Lambda, S^{(1)}_{\Lambda(0,1,1)} + \sigma_2),$ $(\Lambda, S^{(1)}_{\Lambda(0,1,1)} + \sigma_2 + \sigma_3), (\Lambda, S^{(7)}_{\Lambda(0,1,1)}), (\Lambda, S^{(11)}_{\Lambda(0,1,1)}),$ $(\Lambda, S^{(11)}_{\Lambda(0,1,1)} + \sigma_2), (\Lambda, S^{(8)}_{\Lambda(0,1,1)}), (\Lambda, S^{(8)}_{\Lambda(0,1,1)} + \sigma_2 + \sigma_3),$ $(\Lambda, S^{(8)}_{\Lambda(0,1,1)} + \sigma_3), (\Lambda, S^{(16)}_{\Lambda(0,1,1)}), (\Lambda, S^{(16)}_{\Lambda(0,1,1)} + \sigma_3),$ $(\Lambda, S^{(15)}_{\Lambda(0,1,1)}), (\Lambda, S^{(15)}_{\Lambda(0,1,1)} + \sigma_2 + \sigma_3), (\Lambda, S^{(22)}_{\Lambda(0,1,1)}),$ $(\Lambda, S^{(22)}_{\Lambda(0,1,1)} + \sigma_2 + \sigma_3), (\Lambda, \Lambda^{(0,1,1)})$	16
	(0,2,2)	$(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(1)}_{\Lambda(0,2,2)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(1)}_{\Lambda(0,2,2)} + 2\sigma_1),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(1)}_{\Lambda(0,2,2)} + \sigma_3), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(1)}_{\Lambda(0,2,2)} + 2\sigma_1 + 2\sigma_2 + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(4)}_{\Lambda(0,2,2)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(4)}_{\Lambda(0,2,2)} + 2\sigma_2),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(8)}_{\Lambda(0,2,2)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(8)}_{\Lambda(0,2,2)} + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(8)}_{\Lambda(0,2,2)} + 2\sigma_1 + 2\sigma_2 + \sigma_3), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(9)}_{\Lambda(0,2,2)}),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(9)}_{\Lambda(0,2,2)} + 2\sigma_1), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(9)}_{\Lambda(0,2,2)} + 2\sigma_2),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(9)}_{\Lambda(0,2,2)} + \sigma_3), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(9)}_{\Lambda(0,2,2)} + 2\sigma_2 + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(15)}_{\Lambda(0,2,2)}), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(15)}_{\Lambda(0,2,2)} + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(15)}_{\Lambda(0,2,2)} + 2\sigma_2 + \sigma_3), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(17)}_{\Lambda(0,2,2)}),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(17)}_{\Lambda(0,2,2)} + 2\sigma_1), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(22)}_{\Lambda(0,2,2)}),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(22)}_{\Lambda(0,2,2)} + \sigma_3), (S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, S^{(22)}_{\Lambda(0,2,2)} + 2\sigma_1 + \sigma_3),$ $(S(0, \tilde{\sigma}_1, \tilde{\sigma}_2) + \mathbb{Z}\sigma_3, \Lambda^{(2)}),\ (\Lambda, S^{(1)}_{\Lambda(0,2,2)}),$ $(\Lambda, S^{(1)}_{\Lambda(0,2,2)} + \sigma_3), (\Lambda, S^{(4)}_{\Lambda(0,2,2)}),\ (\Lambda, S^{(8)}_{\Lambda(0,2,2)}),$ $(\Lambda, S^{(8)}_{\Lambda(0,2,2)} + \sigma_3), (\Lambda, S^{(9)}_{\Lambda(0,2,2)}),\ (\Lambda, S^{(9)}_{\Lambda(0,2,2)} + \sigma_3),$ $(\Lambda, S^{(15)}_{\Lambda(0,2,2)}), (\Lambda, S^{(15)}_{\Lambda(0,2,2)} + \sigma_3),\ (\Lambda, S^{(17)}_{\Lambda(0,2,2)}),$ $(\Lambda, S^{(22)}_{\Lambda(0,2,2)}), (\Lambda, S^{(22)}_{\Lambda(0,2,2)} + \sigma_3),\ (\Lambda, \Lambda^{(2)})$	36
	(0,3,3)	$(S_\Lambda^{(1)}, 2S_\Lambda^{(1)}),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(1)} + \sigma_2)),$ $(S_\Lambda^{(1)}, 2(S_\Lambda^{(1)} + \sigma_1 + \sigma_2 + \sigma_3)),\ (S_\Lambda^{(1)}, 2S_\Lambda^{(2)}),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(2)} + \sigma_2)),$ $(S_\Lambda^{(1)}, 2(S_\Lambda^{(1)} + \sigma_2 + \sigma_3))\ (S_\Lambda^{(1)}, 2S_\Lambda^{(8)}),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(8)} + \sigma_1)),$ $(S_\Lambda^{(1)}, 2(S_\Lambda^{(8)} + \sigma_3)),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(8)} + \sigma_1 + \sigma_2 + \sigma_3)),$ $(S_\Lambda^{(1)}, 2S_\Lambda^{(12)})\ ,(S_\Lambda^{(1)}, 2(S_\Lambda^{(12)} + \sigma_1 + \sigma_3)),$ $(S_\Lambda^{(1)}, 2S_\Lambda^{(15)}),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(15)} + \sigma_2)),$ $(S_\Lambda^{(1)}, 2(S_\Lambda^{(15)} + \sigma_2 + \sigma_3)),\ (S_\Lambda^{(1)}, 2S_\Lambda^{(21)}),$ $(S_\Lambda^{(1)}, 2S_\Lambda^{(22)}),\ (S_\Lambda^{(1)}, 2(S_\Lambda^{(22)} + \sigma_1 + \sigma_2 + \sigma_3)),$ $(S_\Lambda^{(1)}, 2\Lambda),$ $(S_\Lambda^{(8)}, 2S_\Lambda^{(1)}),\ (S_\Lambda^{(8)}, 2(S_\Lambda^{(1)} + \sigma_1)),$ $(S_\Lambda^{(8)}, 2(S_\Lambda^{(1)} + \sigma_3)),\ (S_\Lambda^{(8)}, 2(S_\Lambda^{(1)} + \sigma_1 + \sigma_2 + \sigma_3)),$ $(S_\Lambda^{(8)}, 2S_\Lambda^{(4)}),\ (S_\Lambda^{(8)}, 2(S_\Lambda^{(4)} + \sigma_1))$ $(S_\Lambda^{(8)}, 2S_\Lambda^{(8)}),\ (S_\Lambda^{(8)}, 2(S_\Lambda^{(8)} + \sigma_1)),$ $(S_\Lambda^{(8)}, 2(S_\Lambda^{(8)} + \sigma_3)),\ (S_\Lambda^{(8)}, 2S_\Lambda^{(9)}),$ $(S_\Lambda^{(8)}, 2(S_\Lambda^{(9)} + \sigma_2)),\ (S_\Lambda^{(8)}, 2(S_\Lambda^{(9)} + \sigma_3)),$	73

ν	(t_1,t,t_2)	(S,E)	#
3	(0,3,3)	$(S_\Lambda^{(8)},2(S_\Lambda^{(9)}+\sigma_2+\sigma_3))$, $(S_\Lambda^{(8)},2(S_\Lambda^{(9)}+\sigma_1+\sigma_2+\sigma_3))$, $(S_\Lambda^{(8)},2S_\Lambda^{(15)})$, $(S_\Lambda^{(8)},2(S_\Lambda^{(15)}+\sigma_2))$, $(S_\Lambda^{(8)},2(S_\Lambda^{(15)}+\sigma_3))$, $(S_\Lambda^{(8)},2S_\Lambda^{(17)})$, $(S_\Lambda^{(8)},2(S_\Lambda^{(17)}+\sigma_1+\sigma_2))$, $(S_\Lambda^{(8)},2S_\Lambda^{(22)})$, $(S_\Lambda^{(8)},2(S_\Lambda^{(22)}+\sigma_3))$, $(S_\Lambda^{(8)},2(S_\Lambda^{(22)}+\sigma_1+\sigma_2))$, $(S_\Lambda^{(8)},2\Lambda)$, $(S_\Lambda^{(15)},2S_\Lambda^{(1)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(1)}+\sigma_2))$, $(S_\Lambda^{(15)},2(S_\Lambda^{(1)}+\sigma_2+\sigma_3))$, $(S_\Lambda^{(15)},2(S_\Lambda^{(7)}))$,$(S_\Lambda^{(15)},2S_\Lambda^{(8)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(8)}+\sigma_3))$, $(S_\Lambda^{(15)},2(S_\Lambda^{(8)}+\sigma_2))$, $(S_\Lambda^{(15)},2S_\Lambda^{(10)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(10)}+\sigma_2+\sigma_3))$, $(S_\Lambda^{(15)},2S_\Lambda^{(15)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(15)}+\sigma_2))$, $(S_\Lambda^{(15)},2S_\Lambda^{(16)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(16)}+\sigma_3))$, $(S_\Lambda^{(15)},2S_\Lambda^{(22)})$, $(S_\Lambda^{(15)},2(S_\Lambda^{(22)}+\sigma_2+\sigma_3))$, $(S_\Lambda^{(15)},2\Lambda)$, $(S_\Lambda^{(22)},2S_\Lambda^{(1)})$, $(S_\Lambda^{(22)},(2S_\Lambda^{(1)}+\sigma_2+\sigma_3))$, $(S_\Lambda^{(22)},2S_\Lambda^{(8)})$, $(S_\Lambda^{(22)},2(S_\Lambda^{(8)}+\sigma_1+\sigma_2))$, $(S_\Lambda^{(22)},2(S_\Lambda^{(8)}+\sigma_1+\sigma_3))$, $(S_\Lambda^{(22)},2S_\Lambda^{(15)})$, $(S_\Lambda^{(22)},2(S_\Lambda^{(15)}+\sigma_2))$, $(S_\Lambda^{(22)},2S_\Lambda^{(22)})$, $(S_\Lambda^{(22)},2(S_\Lambda^{(22)}+\sigma_1))$, $(S_\Lambda^{(22)},2\Lambda)$,$(\Lambda,2S_\Lambda^{(1)})$, $(\Lambda,2S_\Lambda^{(8)})$, $(\Lambda,2S_\Lambda^{(15)})$, $(\Lambda,2S_\Lambda^{(22)})$, $(\Lambda,2\Lambda)$	
	Total		157
	+Dual		226

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