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Berezin-Toeplitz Quantization on the Schwartz Space of Bounded Symmetric Domains

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Abstract. Borthwick, Lesniewski and Upmeier ["Nonperturbative deformation quantization of Cartan domains," J. Funct. Anal. 113 (1993), 153–176] proved that on any bounded symmetric domain (Hermitian symmetric space of non-compact type), for any compactly supported smooth functions f and g, the product of the Toeplitz operators $T_f T_g$ on the standard weighted Bergman spaces can be asymptotically expanded into a series of another Toeplitz operators multiplied by decreasing powers of the Wallach parameter ν. This is the Berezin-Toeplitz quantization. In this paper, we remove the hypothesis of compact support and show that their result can be extended to functions f, g in a certain algebra which contains both the space of all smooth functions whose derivatives of all orders are bounded and the Schwartz space. Applications to deformation quantization are also given.
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1. Introduction

Let (Ω, ω) be an irreducible bounded symmetric domain in $\mathbf{C}^d$ in its Harish-Chandra realization (i.e. Ω is circular and convex), r its rank, p its genus, and $K_\Omega(x,y)$ its Bergman kernel. It is then known that

$$K_\Omega(x,y) = \Lambda_p h(x,y)^{-p}, \tag{1.1}$$

where $1/\Lambda_p$ is the volume of Ω and $h(x,y)$ is a certain irreducible polynomial, called the *Jordan triple determinant,* holomorphic in x and anti-holomorphic in y, and such that $h(x,0) = 1 \ \forall x \in \mathbf{C}^d$. Further, for any $\nu > p-1$, $h(x,x)^{\nu-p}$ is integrable over Ω, and if we choose normalizing constants Λ_ν so that

$$d\mu_\nu(z) := \Lambda_\nu \, h(z,z)^{\nu-p} \, dm(z)$$

(dm being the Lebesgue volume on $\mathbf{C}^d$) are probability measures, then the weighted Bergman spaces

$$A^2_\nu(\Omega) := \{f \in L^2(\Omega, d\mu_\nu) : \ f \text{ holomorphic on } \Omega\}$$

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have reproducing kernels given by

$$K_\nu(x,y) := h(x,y)^{-\nu}.$$

For any $f \in L^\infty(\Omega)$, the *Toeplitz operator* $T_f^{(\nu)}$ on A^2_ν is defined as

$$T_f^{(\nu)}\phi := P_\nu(f\phi), \qquad \phi \in A^2_\nu, \tag{1.2}$$

where P_ν is the orthogonal projection in $L^2(\Omega, d\mu_\nu)$ onto A^2_ν. Explicitly,

$$T_f^{(\nu)}\phi(x) = \int_\Omega f(y)\phi(y)K_\nu(x,y)\,d\mu_\nu(y).$$

It is immediate from (1.2) that $T_f^{(\nu)}$ is bounded on A^2_ν and $\|T_f^{(\nu)}\| \le \|f\|_\infty$.

In [3], Borthwick, Lesniewski and Upmeier proved the following theorem.

Theorem A. *Let $f,g \in C^\infty(\Omega)$ have compact support. Then*

$$\left\| T_f^{(\nu)} T_g^{(\nu)} - T^{(\nu)}_{C_0(f,g)} - \nu^{-1} T^{(\nu)}_{C_1(f,g)} \right\| \le \frac{C_{f,g}}{\nu^2} \qquad \text{as } \nu \to +\infty, \tag{1.3}$$

where the norm is the operator norm in A^2_ν, $C_{f,g}$ is a constant (depending on f and g), and

$$C_0(f,g) = fg, \quad C_1(f,g) = \frac{i}{2\pi} \sum_{j,k=1}^d \omega^{jk}(z)\,\frac{\partial f}{\partial z_j}\,\frac{\partial g}{\partial \overline{z}_k},$$

where $[\omega^{jk}]_{j,k=1}^d$ is the inverse matrix to

$$\omega_{kl} := \frac{-\partial \log h(z,z)}{\partial z_k \partial \overline{z}_l}. \tag{1.4}$$

Note that (1.4) means precisely that

$$ds^2 = \sum_{k,l=1}^d \omega_{kl}\, dz_k\, d\overline{z}_l \tag{1.5}$$

is the (suitably normalized) invariant metric on Ω; thus

$$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\},$$

where $\{f,g\}$ is the invariant Poisson bracket on Ω. This is the starting point for using Theorem A for carrying out the *Berezin-Toeplitz quantization* on Ω; see e.g. [3], [4] for details.

The aim of the present paper is to extend Theorem A in two ways: first, to get also the higher order terms (i.e. at ν^{-k}, $k \ge 2$) in (1.3); and, second, to remove the hypothesis of compact support of f and g. While the first part is easy (and to some extent already implicit in [3]), the second seems to require more effort. To state our result, we need a few more definitions.

Recall that the identity component G of the group of all biholomorphic self-maps of Ω is a semi-simple Lie group with finite center, and denoting by K the stabilizer of the origin $0 \in \Omega$ in G we may (and will) identify Ω with the coset space G/K. Any function f on Ω can thus be lifted to a function $f^\#$ on G by composing with the canonical projection $G \to G/K = \Omega$, i.e.

$$f^\#(g) := f(g0), \qquad g \in G,\ g0 \in \Omega. \tag{1.6}$$

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra, and for $P \in \mathfrak{U}(\mathfrak{g})$ let L_P be the left-invariant differential operator on G induced by P. (That is, if $P = P_1 \cdots P_m$, with $P_1, \dots, P_m \in \mathfrak{g}$, and f is a function on G, then

$$L_P f(g) := \frac{\partial^m}{\partial t_1 \dots \partial t_m} f(g e^{t_1 P_1} \dots e^{t_m P_m}) \Big|_{t_1 = \dots = t_m = 0};$$

and for general elements $P \in \mathfrak{U}(\mathfrak{g})$, L_P is defined by linearity.)

Definition. The space IBC^∞ is defined as

$$IBC^\infty(\Omega) := \{ f \in C^\infty(\Omega) : \ L_P f^\# \text{ is bounded on } G,\ \forall P \in \mathfrak{U}(\mathfrak{g}) \}.$$

It will be shown below that IBC^∞ is an algebra which contains both BC^∞ (the space of all functions in $C^\infty(\Omega)$ whose derivatives of all orders are bounded) and the Schwartz space $\mathcal{S}(\Omega)$ (whose definition will also be recalled below). Our main result is then the following.

Main Theorem. *There exist bidifferential operators C_j $(j = 0, 1, 2, \dots)$ such that, for any $g \in IBC^\infty(\Omega)$ and $f \in IBC^\infty(\Omega) \cap L^2(\Omega, d\mu)$,*

(i) $C_j(f, g) \in L^\infty(\Omega)$ $\forall j$*; and*

(ii) *for any integer $N \ge 0$,*

$$\Big\| T_f^{(\nu)} T_g^{(\nu)} - \sum_{j=0}^{N} \nu^{-j} T_{C_j(f,g)}^{(\nu)} \Big\| = O(\nu^{-N-1}) \qquad \text{as } \nu \to +\infty. \tag{1.7}$$

Here

$$d\mu(z) := h(z, z)^{-p}\, dm(z)$$

stands for the G-invariant measure on Ω (this is the volume element associated to the metric (1.5)), and by a (linear) *bidifferential operator* we mean that

$$C_j(f, g) = \sum_{\alpha, \beta \text{ multiindices}} c_{j\alpha\beta} \cdot D^\alpha f \cdot D^\beta g \qquad \Big(D^\alpha := \frac{\partial^{\alpha_1 + \dots + \alpha_m}}{\partial x^{\alpha_1} \dots \partial x^{\alpha_m}} \Big), \tag{1.8}$$

with some coefficient functions $c_{j\alpha\beta}$ (which must then belong to $C^\infty(\Omega)$).

We will actually prove a somewhat more refined version of (1.7) (see Theorem 8 below), and it will also turn out that C_j involve only holomorphic derivatives of f and anti-holomorphic derivatives of g, so that even

$$C_j(f, g) = \sum_{\alpha, \beta} c_{j\alpha\beta} \cdot \partial^\alpha f \cdot \overline{\partial}^\beta g$$

(with the obvious multiindex notation). Further, the operators C_j will also be shown to be G-invariant, i.e.

$$C_j(f \circ \phi, g \circ \phi) = C_j(f,g) \circ \phi, \qquad \forall \phi \in G.$$

The paper is organized as follows. In Section 2., we recall some additional prerequisites on bounded symmetric domains which will be needed. In Section 3., we establish some technical lemmas. Section 4. introduces certain invariant bidifferential operators which will play an important role. The proof of the Main Theorem appears in Section 5.. The last Section 6. discusses some applications to quantization.

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Notation. We use the symbol ∂_j as an abbreviation for the operator of the holomorphic differentiation $\partial/\partial z_j$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multiindex, then $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. Analogously for $\overline{\partial}$. Similarly, the symbol D^α denotes differentiation with respect to a real variable (as in (1.8) above). Subscripts like D_x^α, ∂_z^β indicate the differentiated variable in cases where there is a danger of confusion. When g is an element of G and $x \in \Omega$, we will often write just gx instead of $g(x)$ (including, in particular, $g0$ instead of $g(0)$). Finally, for typographic reasons, we will sometimes denote the Toeplitz operators by $T_\nu[f]$ instead of $T_f^{(\nu)}$.

2. Bounded symmetric domains

Throughout the rest of this paper, let thus $\Omega = G/K$ be a Cartan (i.e. irreducible bounded symmetric) domain in $\mathbf{C}^d$ in its Harish-Chandra realization, with G a semi-simple Lie group with finite center and K its maximal compact subgroup of all elements stabilizing the origin $0 \in \Omega$. Fix an Iwasawa decomposition $G = NAK$, denote by $\mathfrak{g}, \mathfrak{n}, \mathfrak{a}, \mathfrak{k}$ the corresponding Lie algebras, and for $g \in G$ let $A(g) \in \mathfrak{a}$ be the (unique) element of $\mathfrak{a}$ such that $g \in N \exp A(g)\, K$. Let further M be the centralizer of A in K. Introduce the function Ξ on G by

$$\Xi(g) := \int_K e^{\rho(A(k^{-1}g))}\, dk,$$

where dk stands for the normalized Haar measure on K and $\rho \in \mathfrak{a}^*$ is the half-sum of the positive roots. Similarly, using the Bruhat decomposition $G = K\overline{\exp \mathfrak{a}^+}K$, with $\mathfrak{a}^+$ a fixed positive Weyl chamber in $\mathfrak{a}$ (and the bar denoting closure), the function σ on G is defined by

$$\sigma(k_1 e^H k_2) := \|H\|, \qquad k_1, k_2 \in K,\ H \in \mathfrak{a}^+,$$

where $\|\cdot\|$ stands for the Euclidean norm on $\mathfrak{a} \simeq \mathbf{R}^r$ (with some normalization). It is known that Ξ extends continuously to the closure of Ω in $\mathbf{C}^d$, satisfies $0 < \Xi \le 1$ on Ω, and vanishes on the topological boundary $\partial\Omega$ of Ω in $\mathbf{C}^d$. On the other hand, $\sigma(x) \to +\infty$ as $x \to \partial\Omega$ in $\mathbf{C}^d$.

The (L^2-)*Schwartz space* $\mathcal{S}$ *on* Ω consists, by definition, of all functions $f \in C^\infty(\Omega)$ such that, for any left-invariant differential operator L and any right-invariant differential operator R on G, and for any nonnegative integer k,

$$\sup_{g \in G} |(LRf^\#)(g)|\,(1+\sigma(g))^k\, \Xi(g)^{-1} =: \|f\|_{k,L,R} < \infty. \tag{2.1}$$

We topologize $\mathcal{S}$ using this family of seminorms. (In fact, it is enough to take only the seminorms $\|f\|_{k,L,I}$, i.e. the right-invariant differential operators R can be omitted from the definition.)

A (linear) differential operator L on Ω is called *G-invariant* (or just *invariant* for short) if

$$L(f \circ \phi) = (Lf) \circ \phi \qquad \forall \phi \in G.$$

It is known that any such operator maps $\mathcal{S}$ into itself (continuously).

Let $\mathcal{P}$ be the vector space of all (holomorphic) polynomials on $\mathbf{C}^d$, equipped with the *Fock-Fischer inner product*

$$\begin{aligned} \langle f, g \rangle_F :&= \int_{\mathbf{C}^d} f(z)\overline{g(z)} e^{-|z|^2} \, dm(z) \\ &= g^*(\partial) f(0) = f(\partial) g^*(0), \end{aligned}$$

where

$$g^*(x) := \overline{g(\overline{x})}$$

and e.g. $g^*(\partial)$ is the (constant coefficient linear) differential operator obtained from $g^*(z)$ upon substituting ∂ for z. Under the action $f \mapsto f \circ k$ of the maximal compact subgroup K, the space $\mathcal{P}$ decomposes, with multiplicity one, into the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}, \tag{2.2}$$

where the summation extends over all *signatures* $\mathbf{m}$, i.e. r-tuples $\mathbf{m} = (m_1, \dots, m_r)$ of integers satisfying $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, where r is the *rank* of Ω; further, each $\mathcal{P}_{\mathbf{m}}$ consists of homogeneous polynomials of total degree $|\mathbf{m}| := m_1 + \cdots + m_r$. See e.g. [5] for more details on this matter, as well as on several further properties of the spaces $\mathcal{P}_{\mathbf{m}}$ which we use below.

For any K-invariant inner product on $\mathcal{P}$, it is immediate from the Schur Lemma that $\mathcal{P}_{\mathbf{m}}$ and $\mathcal{P}_{\mathbf{n}}$ are orthogonal for $\mathbf{m} \neq \mathbf{n}$, while on each $\mathcal{P}_{\mathbf{m}}$ any such inner product is proportional to $\langle \cdot, \cdot \rangle_F$. This applies, in particular, to the inner products of A^2_ν: namely,

$$\langle f, g \rangle_\nu = \frac{\langle f, g \rangle_F}{(\nu)_{\mathbf{m}}} \qquad \forall f, g \in \mathcal{P}_{\mathbf{m}}, \tag{2.3}$$

where $(\nu)_{\mathbf{m}}$ is the generalized Pochhammer symbol

$$(\nu)_{\mathbf{m}} := \prod_{j=1}^{r} \prod_{k=0}^{m_j - 1} \Big(\nu - \frac{j-1}{2} a + k \Big). \tag{2.4}$$

Here a is the first of the so-called *characteristic multiplicities* a, b of Ω, which are related to the genus p, the rank r and the dimension d by the formulas

$$p = (r-1)a + b + 2, \qquad d = r(r-1)\frac{a}{2} + rb + r.$$

Each $\mathcal{P}_{\mathbf{m}}$ equipped with the Fock-Fischer scalar product is a finite-dimensional Hilbert space of functions on $\mathbf{C}^d$, hence has a reproducing kernel $K_{\mathbf{m}}(x, y)$. An important consequence of (2.3) is the Faraut-Koranyi formula

$$h(x, y)^{-\nu} = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(x, y), \tag{2.5}$$

which holds for any $\nu \in \mathbf{C}$, uniformly for x, y in compact subsets of Ω.

For any $g \in G$, the familiar transformation property of the Bergman kernel

$$K_\Omega(z,w) = K_\Omega(gz,gw) \cdot Jg(z) \cdot \overline{Jg(w)}$$

(where Jg stands for the complex Jacobian of g) together with (1.1) implies that

$$h(gz,gw)^p = h(z,w)^p \cdot Jg(z) \cdot \overline{Jg(w)}. \tag{2.6}$$

Taking in particular $w = g^{-1}0 =: a$ and $z = w = a$, we obtain

$$Jg(z) = \epsilon_g \frac{h(a,a)^{p/2}}{h(z,a)^p}$$

for some unimodular constant ϵ_g. Substituting this back into (2.6) gives the important relation

$$h(gz,gw) = \frac{h(a,a)h(z,w)}{h(z,a)h(a,w)}, \qquad a := g^{-1}(0) \tag{2.7}$$

valid for all $z, w \in \Omega$ and $g \in G$.

Consequently, we have the change-of-variable formula

$$d\mu_\nu(gz) = \frac{|h(gz,g0)|^{2\nu}}{h(g0,g0)^\nu}\, d\mu_\nu(z). \tag{2.8}$$

It follows that the operators

$$U_g^{(\nu)}\phi(x) := \frac{h(gx,g0)^\nu}{h(g0,g0)^{\nu/2}}\, \phi(gx) \tag{2.9}$$

act unitarily on A^2_ν, and thus give a projective unitary representation of G on this space. The same is true for $L^2(\Omega, d\mu_\nu)$, and it follows that

$$U_g^{(\nu)} T_f^{(\nu)} U_g^{(\nu)*} = T_{f\circ g}^{(\nu)} \qquad \forall g \in G,\ \forall f \in L^\infty(\Omega). \tag{2.10}$$

Finally, we recall some facts from Jordan theory, see e.g. [7] or [1] for details and notation. In particular, we let $\{xyz\}$ stand for the Jordan triple product on $\mathbf{C}^d$ for which Ω is the unit ball, $D(x,y)$ for the multiplication operator $z \mapsto \{xyz\}$, $Q(x)$ for the quadratic operator $z \mapsto \{xzx\}$, and $B(x,y)$ for the Bergman operator

$$B(x,y)z := z - 2D(x,y)z + Q(x)Q(y)z.$$

The Bergman operator satisfies

$$\det B(x,y) = h(x,y)^p.$$

For each $z \in \Omega$, the mapping (see [6], pp. 513–515)

$$\begin{aligned} \phi_a(z) :&= a - B(a,a)^{1/2}(I - D(z,a))^{-1} z \\ &= a - B(a,a)^{1/2} B(z,a)^{-1}(z - \{zaz\}) \end{aligned} \tag{2.11}$$

is an element of G which interchanges a and the origin. Following [3], we will instead use the related mapping

$$\begin{aligned}\gamma_a(z) := \phi_a(-z) &= a + B(a,a)^{1/2}(I + D(z,a))^{-1}z \\ &= a + B(a,a)^{1/2}B(z,-a)^{-1}(z + \{zaz\})\end{aligned} \tag{2.12}$$

which sends 0 into a.

An element $v \in \mathbf{C}^d$ is a *tripotent* if $\{vvv\} = v$; two tripotents u, v are *orthogonal* if $D(u,v) = 0$. The cardinality of any maximal set of nonzero, pairwise orthogonal tripotents is equal to the rank r; such sets are called *Jordan frames.* For any Jordan frame $e_1, \dots, e_r$, each element $z \in \mathbf{C}^d$ has a *polar decomposition*

$$z = k(t_1 e_1 + \dots + t_r e_r), \tag{2.13}$$

where $k \in K$ and $t_1 \geq t_2 \geq \dots \geq t_r \geq 0$; the numbers t_j are uniquely determined (but k need not be), and z belongs to Ω, $\partial\Omega$ or the exterior of Ω according as $t_1 < 1$, $t_1 = 1$ or $t_1 > 1$. Further, for z as in (2.13),

$$h(z,z) = \prod_{j=1}^{r}(1 - t_j^2). \tag{2.14}$$

There is the following relation between the Lie-theoretic and the Jordan-theoretic formalisms: for any Jordan frame, one can choose the maximal Abelian subgroup A in the Iwasawa decomposition of G in such a way that there are $E_j \in \mathfrak{a}$ $(j = 1, \dots, r)$ for which

$$(\exp \sum_{j=1}^{r} \tau_j E_j)\, 0 = \sum_{j=1}^{r} (\tanh \tau_j)\, e_j, \qquad \forall \tau_1, \dots, \tau_r \in \mathbf{R}. \tag{2.15}$$

See e.g. Lemmas 2.3 and 4.3 in [7].

Finally, for any Jordan frame $e_1, \dots, e_r$, the Shilov boundary of Ω coincides with the set

$$\{ke;\ k \in K\}$$

where e is a *maximal tripotent* given by $e = e_1 + \dots + e_r$.

3. The algebra IBC^∞

Let $BC^\infty(\Omega)$ denote the space of all functions in $C^\infty(\Omega)$ whose derivatives of all orders are bounded on Ω, i.e. $\|f\|_{m,\infty} < \infty$ $\forall m$, where

$$\|f\|_{m,\infty} := \sup\{|D^\alpha f(x)| :\ x \in \Omega,\ |\alpha| \leq m\}.$$

Note that in contrast to the Euclidean situation, none of the spaces $\mathcal{S}$ and $BC^\infty(\Omega)$ is contained in the other: an example of a function in $\mathcal{S} \setminus BC^\infty(\Omega)$ is $(1 - |z|^2)^\alpha$, with $\alpha > 1/2$ and not an integer, on the unit disc.

Recall that we have defined

$$IBC^\infty(\Omega) := \{f \in C^\infty(\Omega) :\ L_P f^\# \text{ is bounded on } G,\ \forall P \in \mathfrak{U}(\mathfrak{g})\}.$$

(The letters IBC are supposed to stand for "invariant $BC^\infty(\Omega)$".) We topologize IBC^∞ using the family of seminorms $|\!|\!| f |\!|\!|_P := \|L_P f^\#\|_\infty$, $P \in \mathfrak{U}(\mathfrak{g})$. In this section we establish some facts about the space IBC^∞, as well as several auxiliary results which will be needed later.

Let us introduce also the space

$$\begin{aligned}\mathcal{X}(\Omega) := \{f \in C^\infty(\Omega) :\ &\text{for any multiindex } \alpha \text{ there exists } r_\alpha \ge 0 \\ &\text{such that } \sup\nolimits_{x\in\Omega} |D^\alpha f(x)|\, h(x,x)^{r_\alpha} < \infty\}.\end{aligned} \tag{3.1}$$

Here h stands, as before, for the Jordan triple determinant. Let γ_z be the mapping (2.12).

Lemma 1. *For any multiindices α, β, there are constants $C_{\alpha,\beta} < \infty$ such that*

$$|D_z^\beta \partial_x^\alpha (\gamma_z(x))_i| \le C_{\alpha,\beta}\, h(x,x)^{-(|\alpha|+|\beta|+1)pr}\, h(z,z)^{-c(\beta)} \qquad \forall i = 1,\dots,d\ \forall x, z \in \Omega,$$

where r and p are the rank and the genus of Ω, respectively, and

$$c(\beta) = \begin{cases} 0 & \text{if } |\beta| = 0, \\ 2(|\beta|+1) & \text{if } |\beta| > 0. \end{cases}$$

Proof. The complex derivative of $\gamma_z(x)$ with respect to x satisfies (cf. (4.27) in [3])

$$\gamma_z'(x) = B(z,z)^{1/2} B(x,-z)^{-1}. \tag{3.2}$$

Since $\{xzx\}$ is quadratic in x and conjugate-linear in z (hence, in particular, C^∞ on $\mathbf{C}^d \times \mathbf{C}^d$), it follows that there exist constants $C_{\beta\alpha} < \infty$ such that

$$|D_z^\beta \partial_x^\alpha (\gamma_z(x))_i| \le C_{\beta\alpha} \sup_{|\delta| \le |\beta|} \|D_z^\delta (B(z,z)^{1/2})\| \cdot \|B(x,-z)^{-1}\|^{|\alpha|+1+|\beta|}$$

$\forall x, z \in \Omega$ and $\forall i = 1,\dots,d$ (the norms are the operator norms on $\mathbf{C}^d$).

The inverse of an arbitrary matrix $A = (a_{ij})$ can be expressed as $A^{-1} = (b_{ij})/\det A$, where b_{ij} are polynomials (with universal coefficients) in a_{ij} (they are determinants of certain minors of A). As $\det B(x,-z)^{-1} = h(x,-z)^{-p}$ and $B(x,-z)$ is continuous on all of $\mathbf{C}^d \times \mathbf{C}^d$ (hence its entries are bounded on $\Omega \times \Omega$), it follows that

$$\|B(x,-z)^{-1}\| \le C_\Omega |h(x,-z)^{-p}|.$$

Since $h(x,\cdot)^{-p}$ is a conjugate-holomorphic function on Ω, it attains its maximum on the Shilov boundary, which coincides with the orbit $\{ke, k \in K\}$ of any given maximal tripotent e under the maximal compact subgroup K. Thus

$$|h(x,-z)^{-p}| \le \sup_{k\in K} |h(x,ke)^{-p}| = \sup_{k\in K} |h(k\textstyle\sum_j t_j e_j, e)^{-p}|,$$

where $x = k_1 \sum_j t_j e_j$ ($k_1 \in K$, $1 > t_1 \ge t_2 \ge \dots \ge t_r \ge 0$) is the polar decomposition of x with respect to some system of minimal orthogonal tripotents $e_1,\dots,e_r$, which we may choose so that $e_1 + \dots + e_r = e$. Clearly always $k \sum t_j e_j \in t_1 \Omega$, and using again the above Shilov boundary argument thus gives

$$\sup_{k\in K} |h(k\textstyle\sum_j t_j e_j, e)^{-p}| \le \sup_{k\in K} |h(kt_1 e, e)^{-p}|.$$

Recall now that by the Faraut-Koranyi formula

$$h(x,y)^{-p}=\sum_{\mathbf{m}}(p)_{\mathbf{m}}K_{\mathbf{m}}(x,y).$$

Using the Schwarz inequality, and the homogeneity and the K-invariance of $K_{\mathbf{m}}$, we have

$$\begin{aligned}|K_{\mathbf{m}}(kt_1e,e)|&=t_1^{|\mathbf{m}|}|K_{\mathbf{m}}(ke,e)|\\&\le t_1^{|\mathbf{m}|}K_{\mathbf{m}}(ke,ke)^{1/2}K_{\mathbf{m}}(e,e)^{1/2}\\&=t_1^{|\mathbf{m}|}K_{\mathbf{m}}(e,e)\\&=K_{\mathbf{m}}(t_1e,e).\end{aligned}$$

Substituting this into the Faraut-Koranyi formula gives

$$\sup_{k\in K}|h(kt_1e,e)^{-p}|\le h(t_1e,e)^{-p}=(1-t_1)^{-pr}.$$

The right-hand side can be estimated from above by

$$2^{pr}(1-t_1^2)^{-pr}\le 2^{pr}\Big[\prod_{j=1}^{r}(1-t_j^2)\Big]^{-pr}=2^{pr}h(x,x)^{-pr}.$$

We thus obtain

$$\|B(x,z)^{-1}\|\le C_\Omega\, h(x,x)^{-pr}\qquad \forall x,z\in\Omega.$$

To estimate $\|D_z^\delta(B(z,z)^{1/2})\|$, we use the Riesz-Dunford functional calculus (in the space of operators on $\mathbf{C}^d$) to write

$$B(z,z)^{1/2}=\int_\Gamma\sqrt{\lambda}\,\big(B(z,z)-\lambda I\big)^{-1}\,d\lambda \tag{3.3}$$

with some contour Γ in the right half-plane enclosing the spectrum of $B(z,z)$. Now for any invertible operator-valued function $X(z)$,

$$(X^{-1})'=-X^{-1}X'X^{-1}.$$

By iteration, it follows that any derivative of X^{-1} is a polynomial in X^{-1} and the derivatives of X. Applying this to $X(z)=B(z,z)-\lambda I$, and noting that all derivatives of $B(z,z)$ are bounded on Ω (since $B(z,z)$ is a quadratic polynomial in z and $\overline{z}$), it follows that

$$\|D_z^\delta(B(z,z)-\lambda I)^{-1}\|\le C_\delta\,\|(B(z,z)-\lambda I)^{-1}\|^{|\delta|+1}.$$

Differentiating under the integral sign in (3.3) (which is easily justified), we therefore get

$$\|D_z^\delta B(z,z)^{1/2}\|\le C_\delta|\Gamma|\sup_{\lambda\in\Gamma}|\sqrt{\lambda}|\|(B(z,z)-\lambda I)^{-1}\|^{|\delta|+1}.$$

If $z=k\sum_j t_je_j$ is the polar decomposition of z, then $B(z,z)$ is a diagonal operator with eigenvalues $s_{ij}:=(1-t_i^2)(1-t_j^2)$, $0\le i\le j\le r$, $i+j>0$ ($t_0:=0$). Denote

$\sigma := \min_{i,j} s_{ij} = (1-t_1^2)^2$, $\tau := \max_{i,j} s_{ij} \le 1$, and take Γ to be the contour consisting of the two segments $[\sigma + \frac{i}{2}\sigma, \tau + \frac{i}{2}\sigma]$, $[\sigma - \frac{i}{2}\sigma, \tau - \frac{i}{2}\sigma]$ and the two half-circles of radius $\sigma/2$ centered at σ and τ, respectively. Then for $\lambda \in \Gamma$, $|\sqrt{\lambda}| < \sqrt{2}$ and

$$\|(B(z,z)-\lambda I)^{-1}\| \le \frac{2}{\sigma} = \frac{2}{(1-t_1^2)^2} \le \frac{2}{\prod_{j=1}^r (1-t_j^2)^2} = 2h(z,z)^{-2}.$$

Thus

$$\|D_z^\delta B(z,z)^{1/2}\| \le C'_\delta\, h(z,z)^{-2(|\delta|+1)}.$$

Finally, it is clear that for $|\delta| = 0$, one can in fact replace the exponent -2 by zero, since $B(z,z)$ is bounded on Ω. Combining everything together, the assertion of the lemma follows. This completes the proof. ■

Theorem 2. *IBC^∞ is an algebra containing properly both $BC^\infty(\Omega)$ and $\mathcal{S}$, and contained in L^∞ (all these inclusions being continuous).*

Proof. Since $0 < \Xi \le 1$, we have $|\!|\!|g|\!|\!|_L \le \|g\|_{0,L,I}$, and thus the inclusion $\mathcal{S} \subset IBC^\infty$ is obvious; similarly, since $|\!|\!|g|\!|\!|_I = \|g\|_\infty$, so is $IBC^\infty \subset L^\infty$. To show that $BC^\infty \subset IBC^\infty$ continuously, observe that any left-invariant differential operator L on G satisfies

$$Lf^\#(\phi) = \sum_{\nu \text{ multiindex}} c_\nu\, D^\nu(f\circ\phi)(0) \tag{3.4}$$

for some constant coefficients c_ν. (Indeed, it is enough to check this for $L = L_P$ with P of the form $P = P_1 \cdots P_m$, $P_1, \dots, P_m \in \mathfrak{g}$; but then

$$\begin{aligned} L_P f^\#(\phi) &= \frac{\partial^m}{\partial t_1 \dots \partial t_m} f^\#(\phi e^{t_1P_1} \dots e^{t_mP_m})\Big|_{t_1=\dots=t_m=0} \\ &= \frac{\partial^m}{\partial t_1 \dots \partial t_m} f(\phi e^{t_1P_1} \dots e^{t_mP_m}0)\Big|_{t_1=\dots=t_m=0} \\ &= \frac{\partial^m}{\partial t_1 \dots \partial t_m} (f\circ\phi)(e^{t_1P_1} \dots e^{t_mP_m}0)\Big|_{t_1=\dots=t_m=0}, \end{aligned}$$

which must coincide with the right-hand side of (3.4) for some c_ν.) It therefore suffices to show that $\phi \mapsto D^\nu(f\circ\phi)(0)$ is bounded for any $f \in BC^\infty$ and any multiindex ν. Since any $\phi \in G$ is of the form $\phi = \gamma_z \circ k$ with some $z \in \Omega$ and $k \in K$, and

$$\|\nabla^m(f\circ\gamma_z\circ k)(0)\| = \|\nabla^m(f\circ\gamma_z)(0)\|$$

in view of the fact that k is a unitary map, it is enough to consider $\phi = \gamma_z$. But by an easy induction argument,

$$\begin{aligned} \partial^\alpha\overline{\partial}^\beta(f\circ\gamma_z)(0) = \sum_{q,\iota,\alpha_1,\dots,\alpha_q} \sum_{s,\upsilon,\beta_1,\dots,\beta_s} & \kappa_{\iota,\alpha_1,\dots,\alpha_q;\upsilon,\beta_1,\dots,\beta_s} \\ & \cdot \big(\partial^{\alpha_1}(\gamma_z)_{\iota_1}(0)\big) \cdot \dots \cdot \big(\partial^{\alpha_q}(\gamma_z)_{\iota_q}(0)\big) \cdot \\ & \cdot \overline{\partial^{\beta_1}(\gamma_z)_{\upsilon_1}(0)} \cdot \dots \cdot \overline{\partial^{\beta_s}(\gamma_z)_{\upsilon_s}(0)} \cdot \\ & \cdot \partial_{\iota_1} \dots \partial_{\iota_q} \overline{\partial}_{\upsilon_1} \dots \overline{\partial}_{\upsilon_s} f(z), \end{aligned} \tag{3.5}$$

where the first summation extends over all q-tuples $\iota = (\iota_1, \dots, \iota_q)$, $0 \le q \le |\alpha|$, $1 \le \iota_j \le d$, and multiindices $\alpha_1, \dots, \alpha_q$ such that $|\alpha_1|, \dots, |\alpha_q| \ge 1$, $|\alpha_1| + \dots + |\alpha_q| = |\alpha|$, and similarly for the second summation; and $\kappa_{\iota,\alpha_1,\dots,\alpha_q;\upsilon,\beta_1,\dots,\beta_s}$ are certain universal constants. By Lemma 1 with $|\gamma| = 0 = x$,

$$|\partial^\nu (\gamma_z)_j(0)| \le C_\nu \qquad \forall z \in \Omega \tag{3.6}$$

for suitable constants $C_\nu < \infty$. Thus

$$|\partial^\alpha \overline{\partial}^\beta (f \circ \gamma_z)(0)| \le C_{\alpha,\beta} \|f\|_{|\alpha|+|\beta|,\infty},$$

and the desired inclusion follows.

An example of a function in IBC^∞ which is not in $BC^\infty \cup \mathcal{S}$ is $(1-|z|^2)^\alpha$ on the unit disc, with $0 < \alpha \le 1/2$.

Finally, the fact that IBC^∞ is an algebra is immediate from the Leibniz rule. ∎

Lemma 3. *The space $\mathcal{X}$ (defined by (3.1)) has the following properties:*

(i) *it is an algebra and is closed under differentiation;*

(ii) *$h(x,x)^{-1} \in \mathcal{X}$;*

(iii) *if $g \in \mathcal{X}$ and α, β are multiindices, then the function $z \mapsto \partial^\alpha \overline{\partial}^\beta (g \circ \gamma_z)(0)$ belongs to $\mathcal{X}$;*

(iv) *$IBC^\infty \subset \mathcal{X}$ (hence, in particular, $\mathcal{S} \subset \mathcal{X}$).*

Proof. Property (i) is immediate from the Leibniz rule, and (ii) from the chain rule. Property (iii) is a consequence of (3.5) and Lemma 1 (and the Leibniz rule again). It remains to prove (iv). Thus let $f \in IBC^\infty$ and let α be a multiindex. For each $z \in \Omega$, define $X_{\alpha z} \in \mathfrak{U}(\mathfrak{g})$ by

$$X_{\alpha z} f(0) := D_x^\alpha (f \circ \gamma_z^{-1})(x)\Big|_{x=z} \qquad \forall f \in C^\infty(\Omega). \tag{3.7}$$

(Since $\gamma_z^{-1}(z) = 0$, the right-hand side indeed depends only on the germ of f at 0, so the definition makes sense.) By a similar argument as (3.5), we have

$$\begin{aligned} X_{\alpha z} f(0) = \sum_{q,\iota,\alpha_1,\dots,\alpha_q} \kappa_{\iota,\alpha_1,\dots,\alpha_q} \cdot D_x^{\alpha_1}(\gamma_z^{-1})_{\iota_1}(x) \cdot \ldots \cdot D_x^{\alpha_q}(\gamma_z^{-1})_{\iota_q}(x)\Big|_{x=z} \\ \cdot D_{\iota_1} \dots D_{\iota_q} f(0) \\ =: \sum_{|\iota| \le |\alpha|} Q_\iota(z)\, D^\iota f(0). \end{aligned}$$

Now by the formula for the derivative of an inverse function and by Cramer's rule,

$$\begin{aligned} [\partial^j (\gamma_z^{-1})_i(x)]_{ij} &= \big([\partial^j (\gamma_z)_i(\gamma_z^{-1}x)]_{ij}\big)^{-1} \\ &= \frac{[\text{a polynomial in } \partial^k (\gamma_z)_m(\gamma_z^{-1}x),\ k,m = 1,\dots,d]_{ij}}{(\det[\partial^j (\gamma_z)_i(\gamma_z^{-1}x)]_{ij})}; \end{aligned}$$

therefore

$$\partial^\alpha(\gamma_z^{-1})_i(x) = \frac{(\text{a polynomial in } \partial^\beta(\gamma_z)_k(\gamma_z^{-1}x),\ |\beta| \le |\alpha|,\ k = 1, \dots, d)}{(\det[\partial^j(\gamma_z)_i(\gamma_z^{-1}x)]_{ij})^{|\alpha|}}$$

for any $|\alpha| \ge 1$. Evaluating this at $x = z$ gives

$$\partial^\alpha(\gamma_z^{-1})_i(x)\Big|_{x=z} = \frac{(\text{a polynomial in } \partial^\beta(\gamma_z)_k(0),\ |\beta| \le |\alpha|,\ k = 1, \dots, d)}{(\det[\partial^j(\gamma_z)_i(0)]_{ij})^{|\alpha|}}.$$

But the numerator is bounded on Ω by (3.6), while the determinant downstairs equals $h(z,z)^{p/2}$ by (3.2). Consequently,

$$\left|\partial_x^{\alpha_j}(\gamma_z^{-1})_{\iota_j}(x)|_{x=z}\right| \le C_{\alpha_j} h(z,z)^{-p|\alpha_j|/2},$$

so $|Q_\iota(z)| \le C_\iota h(z,z)^{-p|\alpha|/2}$. On the other hand, replacing f by $f \circ \gamma_z$ in (3.7) shows that $X_{\alpha z}(f \circ \gamma_z)(0) = D^\alpha f(z)$. Thus finally

$$\begin{aligned} |D^\alpha f(z)| &= \Big|\sum_{|\iota| \le |\alpha|} Q_\iota(z)\, D^\iota(f \circ \gamma_z)(0)\Big| \le \sum_{|\iota| \le |\alpha|} C_\iota h(z,z)^{-p|\alpha|/2}\, |D^\iota(f \circ \gamma_z)(0)| \\ &= \sum_{|\iota| \le |\alpha|} C_\iota h(z,z)^{-p|\alpha|/2}\, |L_{P_\iota} f^\#(\gamma_z)| \le C\, h(z,z)^{-p|\alpha|/2} \sum_{|\iota| \le |\alpha|} \|\!|f|\!\|_{P_\iota}, \end{aligned}$$

where $P_\iota \in \mathfrak{U}(\mathfrak{g})$ are such that $P_\iota f(0) = D^\iota f(0)\ \ \forall f \in C^\infty(\Omega)$. Since α can be arbitrary, the inclusion $IBC^\infty \subset \mathcal{X}$ follows. ∎

For IBC^∞ replaced by the Schwartz space $\mathcal{S}$, the analogue of the next proposition was established in [4]; it turns out that the same proof works also here.

Proposition 4. *Let C be any bidifferential operator which is invariant in the sense that*

$$C(f \circ \phi, g \circ \phi) = C(f,g) \circ \phi \qquad \forall \phi \in G. \tag{3.8}$$

Then C maps $IBC^\infty \times IBC^\infty$ continuously into IBC^∞.

Proof. It follows from (3.8) that

$$C(f,g)(\phi 0) = C(f \circ \phi, g \circ \phi)(0) = \sum_{\alpha,\beta} c_{\alpha\beta}(0) \cdot D^\alpha(f \circ \phi)(0) \cdot D^\beta(g \circ \phi)(0) \tag{3.9}$$

where we have used the notation from (1.8).

Recalling the standard identification of differential operators on a Lie group with elements of its universal enveloping algebra, let $P_1, \dots, P_m$ be some elements of $\mathfrak{a} + \mathfrak{n}$ (the Lie algebra of the Levy subgroup $L = AN$, which acts simply transitively on Ω) such that $P_1 \cdots P_m =: P_\alpha \in \mathfrak{U}(\mathfrak{a} + \mathfrak{n})$ induces the operator D^α at the origin, i.e.

$$D^\alpha f(0) = \frac{\partial^m}{\partial t_1 \dots \partial t_m} f(e^{t_1 P_1} \dots e^{t_m P_m} 0)\Big|_{t_1 = \dots = t_m = 0}$$

for all functions f on Ω. As in the proof of Theorem 2, we then see that for any $\phi \in G$,

$$D^\alpha (f \circ \phi)(0) = L_{P_\alpha} f^\#(\phi),$$

where L_{P_α} is the left-invariant differential operator on G induced by $P_\alpha \in \mathfrak{U}(\mathfrak{a} + \mathfrak{n}) \subset \mathfrak{U}(\mathfrak{g})$, and $f^\#$ is associated to f via (1.6). Applying a similar argument also to $D^\beta g$ and substituting both outcomes into (3.9), we thus get

$$C(f,g)^\#(\phi) = \sum_{\alpha,\beta} c_{\alpha\beta}(0) \cdot L_{P_\alpha} f^\#(\phi) \cdot L_{P_\beta} g^\#(\phi).$$

Let now $Q_1, \dots, Q_q \in \mathfrak{g}$ and let us apply to the last equality the left-invariant differential operator L_Q on G corresponding to the element $Q := Q_1 \cdots Q_q$ of $\mathfrak{U}(\mathfrak{g})$. We obtain, using the Leibniz rule,

$$L_Q C(f,g)^\#(\phi) = \sum_{\alpha,\beta} \sum_{Q' \subset Q} c_{\alpha\beta}(0)\, L_{Q' P_\alpha} f^\#(\phi) \cdot L_{(Q\setminus Q') P_\beta} g^\#(\phi). \tag{3.10}$$

Thus if $f, g \in IBC^\infty$, then for any integer $k \geq 0$,

$$\|C(f,g)\|_Q \leq \sum_{\alpha,\beta} \sum_{Q' \subset Q} |c_{\alpha\beta}(0)|\, \|f\|_{Q' P_\alpha} \|g\|_{(Q\setminus Q') P_\beta},$$

showing that $\|C(f,g)\|_Q$ is finite whenever $f, g \in IBC^\infty$. ■

The last result in this section will not be needed in the sequel, but we include it for completeness. It is well known that any invariant differential operator maps the Schwartz space into itself. It turns out that IBC^∞ enjoys the same property.

Proposition 5. *Any invariant differential operator L maps IBC^∞ continuously into itself.*

Proof. Invariant differential operators on $\Omega = G/K$ are precisely the left-invariant operators on G which preserve the space of right K-invariant functions (i.e. map any function which is constant on each coset gK, $g \in G$, into another such function). In particular, there exists $Q \in \mathfrak{U}(\mathfrak{g})$ such that

$$(Lf)^\# = L_Q f^\# \qquad \forall f \in C^\infty(\Omega).$$

It follows that for any $P \in \mathfrak{U}(\mathfrak{g})$,

$$L_P (Lf)^\# = L_P L_Q f^\# = L_{PQ} f^\#,$$

so that $\|Lf\|_P = \|f\|_{PQ}$. The assertion follows. ■

4. Some invariant bidifferential operators

For each signature $\mathbf{m}$, let $K_\mathbf{m}(\partial, \partial)$ be the differential operator (with constant coefficients) obtained from $K_\mathbf{m}(x, y)$ upon substituting ∂ and $\overline{\partial}$ for x and $\overline{y}$, respectively. Let further $\mathcal{K}_\mathbf{m}$ be the G-invariant differential operator coinciding with the (K-invariant) operator $K_\mathbf{m}(\partial, \partial)$ at the origin; that is,

$$\mathcal{K}_\mathbf{m} f(z) := K_\mathbf{m}(\partial, \partial)(f \circ \phi)(0)$$

for some (equivalently, any) $\phi \in G$ such that $\phi(0) = z$. As before with ∂ and D, we will again write $\mathcal{K}_{\mathbf{m},z}$ to indicate that $\mathcal{K}_{\mathbf{m}}$ applies to the variable z, if there is a danger of confusion.

By the Leibniz rule, we have for any holomorphic function F on Ω and any $g \in C^\infty(\Omega)$,

$$\mathcal{K}_{\mathbf{m}}(gF) = \sum_{|\gamma| \le |\mathbf{m}|} R_{\mathbf{m}\gamma} g \cdot \partial^\gamma F \tag{4.1}$$

for some (non-invariant) differential operators $R_{\mathbf{m}\gamma}$ on Ω with C^∞ coefficients. Define the bidifferential operators $A_{\mathbf{m}}(f,g)$ on Ω by

$$A_{\mathbf{m}}(f,g)(z) := \frac{1}{K_\Omega(z,z)} \sum_{|\gamma| \le |\mathbf{m}|} (-1)^{|\gamma|} \partial^\gamma \Big[f(z)\, K_\Omega(z,z)\, R_{\mathbf{m}\gamma} g(z) \Big]. \tag{4.2}$$

Surprisingly, these operators turn out to be invariant.

Proposition 6. *The following assertions hold:*

(i) *if $f, g \in \mathcal{X}$ (the space defined by (3.1)) and ϕ, ψ are holomorphic in a neighbourhood of $\overline{\Omega}$, then for all ν sufficiently large,*

$$\begin{aligned} &\int_\Omega \phi(z)\, \overline{\psi(z)}\, A_{\mathbf{m}}(f,g)(z)\, d\mu_\nu(z) \\ &\qquad = \int_\Omega f(z) \overline{\psi(z)} h(z,z)^{-\nu}\, \mathcal{K}_{\mathbf{m},z} \left(\frac{g(z)\phi(z)}{h(z,x)^{-\nu}} \right) \Bigg|_{x=z} d\mu_\nu(z); \end{aligned} \tag{4.3}$$

(ii) *the bidifferential operator $A_{\mathbf{m}}(\cdot\,,\cdot)$ is invariant, i.e.*

$$A_{\mathbf{m}}(f \circ \phi, g \circ \phi) = A_{\mathbf{m}}(f,g) \circ \phi \qquad \forall \phi \in G;$$

(iii) $\overline{A_{\mathbf{m}}(f,g)} = A_{\mathbf{m}}(\overline{g}, \overline{f})$*;*

(iv) *$A_{\mathbf{m}}(f,g)$ involves only holomorphic derivatives of f and anti-holomorphic derivatives of g.*

Proof. (i) By (4.1), the right-hand side of (4.3) equals

$$\begin{aligned} &\int_\Omega f(z) \overline{\psi(z)} h(z,z)^{-\nu} \sum_{|\gamma| \le |\mathbf{m}|} R_{\mathbf{m}\gamma} g(z) \left[\partial_z^\gamma \frac{\phi(z)}{h(z,x)^{-\nu}} \right]_{x=z} d\mu_\nu(z) \\ &\qquad = \Lambda_\nu \sum_{|\gamma| \le |\mathbf{m}|} \int_\Omega f(z) \overline{\psi(z)} h(z,z)^{-p} R_{\mathbf{m}\gamma} g(z)\, \partial_z^\gamma \frac{\phi(z)}{h(z,z)^{-\nu}}\, dz. \end{aligned}$$

On the other hand, if

$$R_{\mathbf{m}\gamma} g(z) =: \sum_{|\delta| \le 2|\mathbf{m}|} c_{\mathbf{m}\gamma\delta}(z) D^\delta g(z),$$

then it follows from the definition of $\mathcal{K}_{\mathbf{m}}$ and the chain rule that the coefficients $c_{\mathbf{m}\gamma\delta}$ are finite sums of finite products of expressions of the form $\partial^\kappa (\gamma_z)_i(0)$,

$|\kappa| \le |\mathbf{m}|$, $i = 1, \dots, d$, and their complex conjugates. But by Lemma 1, the functions γ_z satisfy

$$|D_z^\eta \partial^\kappa (\gamma_z)_i(0)| \le C_{\eta,\kappa} h(z,z)^{-2(|\eta|+1)},$$

thus $c_{\mathbf{m}\gamma\delta} \in \mathcal{X}$. From the hypothesis that $f, g \in \mathcal{X}$ and Lemma 3 it therefore follows that we can find constants $c_{\mathbf{m}} < \infty$ and $r_{\mathbf{m}} \ge 0$ such that

$$|\partial^\gamma [f(z)\, R_{\mathbf{m}\gamma} g(z)\, h(z,z)^{-p}]| \le c_{\mathbf{m}} h(z,z)^{-r_{\mathbf{m}}}$$

for all $|\gamma| \le |\mathbf{m}|$ and $z \in \Omega$. Similarly, since h is a polynomial, we have the estimates

$$|\partial^\eta h(z,z)^\nu| \le C_{\eta,\nu} h(z,z)^{\nu-|\eta|}. \tag{4.4}$$

Since ϕ is holomorphic in a neighbourhood of $\overline{\Omega}$ (hence has all derivatives bounded on $\overline{\Omega}$), it follows easily that

$$|\partial^\gamma [\phi(\,\cdot\,) h(\,\cdot\,, z)^\nu](z)| \le c'_{\mathbf{m},\nu,\phi} h(z,z)^{\nu-|\mathbf{m}|} \qquad \forall |\gamma| \le |\mathbf{m}|,\ \forall z \in \Omega.$$

Consequently, for $\nu > |\mathbf{m}| + r_{\mathbf{m}}$ we can perform the partial integration as in (3.30)–(3.31) in [3]:

$$\begin{aligned} \Lambda_\nu \int_\Omega f(z)\overline{\psi(z)} h(z,z)^{-p} R_{\mathbf{m}\gamma} g(z)\, \partial^\gamma \frac{\phi(z)}{h(z,z)^{-\nu}}\, dz \\ = (-1)^{|\gamma|} \int_\Omega \partial^\gamma \big[f(z) h(z,z)^{-p} R_{\mathbf{m}\gamma} g(z)\big] \phi(z)\overline{\psi(z)}\, \Lambda_\nu h(z,z)^\nu\, dz. \end{aligned}$$

Using (4.2), the assertion follows.

(ii) It suffices to show that

$$\int_\Omega \phi(z)\overline{\psi(z)}\, A_{\mathbf{m}}(f,g)(z)\, d\mu_\nu(z) = \int_\Omega U_\gamma^{(\nu)}\phi(z)\, \overline{U_\gamma^{(\nu)}\psi(z)}\, A_{\mathbf{m}}(f\circ\gamma, g\circ\gamma)(z)\, d\mu_\nu(z)$$

for all ν sufficiently large, for any functions ϕ, ψ holomorphic in a neighbourhood of $\overline{\Omega}$, any $f, g \in \mathcal{D}(\Omega)$, and any $\gamma \in G$, where $U_\gamma^{(\nu)}$ are the unitary operators (2.9). By (4.3), this is equivalent to showing that

$$\begin{aligned} \int_\Omega f(z)\overline{\psi(z)} h(z,z)^{-\nu}\, \mathcal{K}_{\mathbf{m},z} \left[\frac{g(z)\phi(z)}{h(z,x)^{-\nu}}\right]_{x=z} d\mu_\nu(z) \\ = \int_\Omega f(\gamma z)\overline{U_\gamma^{(\nu)}\psi(z)} h(z,z)^{-\nu} \mathcal{K}_{\mathbf{m},z} \left[\frac{g(\gamma z) U_\gamma^{(\nu)}\phi(z)}{h(z,x)^{-\nu}}\right]_{x=z} d\mu_\nu(z). \end{aligned}$$

Substituting (2.9) for the $U_\gamma^{(\nu)}$, the right-hand side becomes

$$\begin{aligned} \int_\Omega f(\gamma z)\overline{\psi(\gamma z)} h(z,z)^{-\nu}\, \mathcal{K}_{\mathbf{m},z} \left(\frac{g(\gamma z)\phi(\gamma z)}{h(z,x)^{-\nu} h(\gamma z, \gamma 0)^{-\nu}}\right)\bigg|_{x=z} \frac{h(\gamma 0, \gamma 0)^{-\nu}}{h(\gamma 0, \gamma z)^{-\nu}}\, d\mu_\nu(z) \\ = \int_\Omega f(\gamma z)\overline{\psi(\gamma z)}\, \mathcal{K}_{\mathbf{m},z} \left(h(x,x)^{-\nu}\, \frac{g(\gamma z)\phi(\gamma z)}{h(z,x)^{-\nu} h(\gamma z, \gamma 0)^{-\nu}}\, \frac{h(\gamma 0, \gamma 0)^{-\nu}}{h(\gamma 0, \gamma x)^{-\nu}}\right)\bigg|_{x=z} d\mu_\nu(z), \end{aligned}$$

while the left-hand side, upon changing the variable z to γz, transforms into (using also the formula (2.8), as well as the invariance of $\mathcal{K}_{\mathbf{m}}$)

$$\begin{aligned}
&\int_\Omega f(\gamma z)\overline{\psi(\gamma z)}h(\gamma z,\gamma z)^{-\nu}\mathcal{K}_{\mathbf{m}}\Big(\frac{g(\cdot)\phi(\cdot)}{h(\cdot,\gamma x)^{-\nu}}\Big)(\gamma z)\Big|_{x=z}\frac{h(\gamma 0,\gamma 0)^{-\nu}}{|h(\gamma z,\gamma 0)^{-\nu}|^2}\,d\mu_\nu(z)\\
&\quad=\int_\Omega f(\gamma z)\overline{\psi(\gamma z)}h(\gamma z,\gamma z)^{-\nu}\,\mathcal{K}_{\mathbf{m},z}\Big(\frac{g(\gamma z)\phi(\gamma z)}{h(\gamma z,\gamma x)^{-\nu}}\Big)\Big|_{x=z}\frac{h(\gamma 0,\gamma 0)^{-\nu}}{|h(\gamma z,\gamma 0)^{-\nu}|^2}\,d\mu_\nu(z)\\
&\quad=\int_\Omega f(\gamma z)\overline{\psi(\gamma z)}\,\mathcal{K}_{\mathbf{m},z}\Big(h(\gamma x,\gamma x)^{-\nu}\,\frac{g(\gamma z)\phi(\gamma z)}{h(\gamma z,\gamma x)^{-\nu}}\,\frac{h(\gamma 0,\gamma 0)^{-\nu}}{|h(\gamma x,\gamma 0)^{-\nu}|^2}\Big)\Big|_{x=z}\,d\mu_\nu(z).
\end{aligned}$$

Thus we will be done if we show that

$$\frac{h(x,x)^{-\nu}}{h(z,x)^{-\nu}h(\gamma z,\gamma 0)^{-\nu}}\,\frac{h(\gamma 0,\gamma 0)^{-\nu}}{h(\gamma 0,\gamma x)^{-\nu}}=\frac{h(\gamma x,\gamma x)^{-\nu}}{h(\gamma z,\gamma x)^{-\nu}}\,\frac{h(\gamma 0,\gamma 0)^{-\nu}}{|h(\gamma x,\gamma 0)^{-\nu}|^2}\tag{4.5}$$

for all $x,z\in\Omega$ and $\gamma\in G$. However, since

$$\frac{h(\gamma z,\gamma 0)h(\gamma 0,\gamma y)}{h(\gamma z,\gamma y)h(\gamma 0,\gamma 0)}=h(z,y)\qquad\forall z,y\in\Omega,\tag{4.6}$$

by (2.7), the right-hand side of (4.5) is equal to

$$\frac{h(x,x)^{-\nu}}{h(\gamma z,\gamma x)^{-\nu}}$$

(just take $y=z=x$ in (4.6)). Thus (4.5) reduces to

$$\frac{h(\gamma 0,\gamma 0)^{-\nu}}{h(z,x)^{-\nu}h(\gamma z,\gamma 0)^{-\nu}h(\gamma 0,\gamma x)^{-\nu}}=\frac{1}{h(\gamma z,\gamma x)^{-\nu}}.$$

But this is just (4.6) with x in the place of y.

(iii) For each $\mathbf{m}$, choose an orthonormal basis (with respect to the Fischer-Fock inner product) $\{\psi_{\mathbf{m}j}\}_{j=1}^{\dim\mathcal{P}_{\mathbf{m}}}$ of $\mathcal{P}_{\mathbf{m}}$, so that

$$h(x,y)^\nu=\sum_{\mathbf{m}}(-\nu)_{\mathbf{m}}K_{\mathbf{m}}(x,y)=\sum_{\mathbf{m},j}(-\nu)_{\mathbf{m}}\psi_{\mathbf{m}j}(x)\overline{\psi_{\mathbf{m}j}(y)}.$$

Then the right-hand side of (4.3) can be rewritten as

$$\begin{aligned}
&\int_\Omega\sum_{\mathbf{m},j}(-\nu)_{\mathbf{m}}\,f(z)\overline{\psi(z)}h(z,z)^{-\nu}\overline{\psi_{\mathbf{m}j}(z)}\,\mathcal{K}_{\mathbf{m}}\big(g(z)\phi(z)\psi_{\mathbf{m}j}(z)\big)\,d\mu_\nu(z)\\
&\qquad=\Lambda_\nu\int_\Omega\sum_{\mathbf{m},j}(-\nu)_{\mathbf{m}}f(z)\overline{\psi(z)\psi_{\mathbf{m}j}(z)}\,\mathcal{K}_{\mathbf{m}}\big(g(z)\phi(z)\psi_{\mathbf{m}j}(z)\big)d\mu(z).
\end{aligned}$$

Since invariant differential operators with real coefficients are formally self-adjoint with respect to the invariant measure $d\mu(z)$, the last expression is equal to

$$\begin{aligned}
&\Lambda_\nu\int_\Omega\sum_{\mathbf{m},j}(-\nu)_{\mathbf{m}}\mathcal{K}_{\mathbf{m}}\big(f(z)\overline{\psi(z)\psi_{\mathbf{m}j}(z)}\big)\,g(z)\phi(z)\psi_{\mathbf{m}j}(z)\,d\mu(z)\\
&\qquad=\int_\Omega\sum_{\mathbf{m},j}(-\nu)_{\mathbf{m}}g(z)\phi(z)h(z,z)^{-\nu}\psi_{\mathbf{m}j}(z)\,\overline{\mathcal{K}_{\mathbf{m}}\big(\overline{f(z)}\psi(z)\psi_{\mathbf{m}j}(z)\big)}\,d\mu_\nu(z)\\
&\qquad=\overline{\int_\Omega\overline{g(z)}\,\overline{\phi(z)}h(z,z)^{-\nu}\,\mathcal{K}_{\mathbf{m},z}\Big(\frac{\overline{f(z)}\psi(z)}{h(z,x)^{-\nu}}\Big)\Big|_{x=z}\,d\mu_\nu(z)},
\end{aligned}$$

whenever $f, g \in \mathcal{D}(\Omega)$. Using (4.3), we thus obtain

$$\int_\Omega \phi(z)\overline{\psi(z)}\, A_{\mathbf{m}}(f,g)(z)\, d\mu_\nu(z) = \overline{\int_\Omega \psi(z)\overline{\phi(z)}\, A_{\mathbf{m}}(\overline{g},\overline{f})(z)\, d\mu_\nu(z)}.$$

Consequently,

$$A_{\mathbf{m}}(f,g) = \overline{A_{\mathbf{m}}(\overline{g},\overline{f})} \qquad \forall f, g \in \mathcal{D}(\Omega),$$

as required.

(iv) It is clear from (4.2) that $A_{\mathbf{m}}(f,g)$ contains only the holomorphic derivatives $\partial^\gamma f$ of f. From (iii) it then follows that it can only contain anti-holomorphic derivatives of g. ∎

The next lemma is (essentially) reproduced here from [2] for convenience.

Lemma 7. *For any polynomial f in z and $\overline{z}$,*

$$\int_\Omega f\, d\mu_\nu = \sum_{\mathbf{m}} \frac{K_{\mathbf{m}}(\partial,\partial) f(0)}{(\nu)_{\mathbf{m}}}. \tag{4.7}$$

Note that the sum on the right-hand side is in fact finite (the summands vanish if $|\mathbf{m}| >$ the degree of f).

Proof. It is enough to prove the assertion for $f = p_{\mathbf{n}}\overline{q}_{\mathbf{k}}$, with $p_{\mathbf{n}} \in \mathcal{P}_{\mathbf{n}}$, $q_{\mathbf{k}} \in \mathcal{P}_{\mathbf{k}}$ for some signatures $\mathbf{n}$ and $\mathbf{k}$. But if $\{\psi_{\mathbf{m}j}\}_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}}$ is any orthonormal basis of $\mathcal{P}_{\mathbf{m}}$ (with respect to the Fischer-Fock norm), then $K_{\mathbf{m}}(x,y) = \sum_j \psi_{\mathbf{m}j}(x)\overline{\psi_{\mathbf{m}j}(y)}$, so

$$\begin{aligned} K_{\mathbf{m}}(\partial,\partial)(p_{\mathbf{n}}\overline{q}_{\mathbf{k}})(0) &= \sum_j \psi_{\mathbf{m}j}(\partial)p_{\mathbf{n}}(0)\, \overline{\psi_{\mathbf{m}j}(\partial)q_{\mathbf{k}}(0)} = \sum_j \langle \psi_{\mathbf{m}j}, p^*_{\mathbf{n}}\rangle_F\, \langle q^*_{\mathbf{k}}, \psi_{\mathbf{m}j}\rangle_F \\ &= \delta_{\mathbf{mn}}\delta_{\mathbf{mk}}\, \langle q^*_{\mathbf{k}}, p^*_{\mathbf{n}}\rangle_F = \delta_{\mathbf{mn}}\delta_{\mathbf{mk}}\, \langle p_{\mathbf{n}}, q_{\mathbf{k}}\rangle_F. \end{aligned}$$

On the other hand,

$$\int_\Omega p_{\mathbf{n}}\overline{q}_{\mathbf{k}}\, d\mu_\nu = \langle p_{\mathbf{n}}, q_{\mathbf{k}}\rangle_\nu = \frac{\delta_{\mathbf{nk}}}{(\nu)_{\mathbf{k}}}\, \langle p_{\mathbf{n}}, q_{\mathbf{k}}\rangle_F,$$

and so (4.7) follows. ∎

5. Proof of the Main Theorem

We are now ready to state the main result of this paper.

Theorem 8. *Let $g \in IBC^\infty(\Omega)$ and $f \in IBC^\infty(\Omega) \cap L^2(\Omega, d\mu)$. Then for any integer $N \geq 0$,*

$$\left\| T_\nu[f]T_\nu[g] - \sum_{|\mathbf{m}| \leq N} \frac{1}{(\nu)_{\mathbf{m}}} T_\nu[A_{\mathbf{m}}(f,g)] \right\| = O(\nu^{-N-1})$$

as $\nu \to +\infty$.

The Main Theorem, as stated in the Introduction, follows from this by noting that, as is immediate from (2.4), there are asymptotic expansions

$$\frac{1}{(\nu)_{\mathbf{m}}} = \sum_{j=0}^{\infty} c_{j\mathbf{m}}\, \nu^{-|\mathbf{m}|-j}$$

as $\nu \to \infty$, with some coefficients $c_{j\mathbf{m}}$; inserting these and grouping together terms with equal powers of ν^{-1} gives (1.7).

Note also that the operators $T_\nu[A_{\mathbf{m}}(f,g)]$ are bounded for any $\mathbf{m}$, since $A_{\mathbf{m}}(f,g) \in IBC^\infty \subset L^\infty$ by Proposition 4.

We denote by $\operatorname{Tayl}_m f$ the Taylor expansion of a function f out to order m at the origin, i.e.

$$\operatorname{Tayl}_m f(z) := \sum_{|\alpha|+|\beta|\le m} \partial^\alpha \overline{\partial}^\beta f(0)\, \frac{z^\alpha \overline{z}^\beta}{\alpha!\beta!}$$

with the usual multiindex notation; and by $\operatorname{Rem}_m f := f - \operatorname{Tayl}_m f$ the corresponding Taylor remainder.

Proof. Let $\phi, \psi \in A^2_\nu$ be holomorphic in a neighbourhood of $\overline{\Omega}$ (such functions are dense in A^2_ν). As in the proof of Theorem 2.3 in [3], we start with (cf. (3.19) there)

$$\langle T_\nu[f] T_\nu[g]\phi, \psi\rangle_\nu = \iint_{\Omega\times\Omega} h(z,y)^{-\nu} f(z) g(y) \phi(y) \overline{\psi(z)}\, d\mu_\nu(z)\, d\mu_\nu(y),$$

which upon the change of variables $y = \gamma_z(x)$ can be rewritten as

$$\begin{aligned} \langle T_\nu[f] T_\nu[g]\phi, \psi\rangle_\nu &= \iint_{\Omega\times\Omega} f(z) g(\gamma_z(x)) \phi(\gamma_z(x)) \overline{\psi(z)} \frac{h(z,z)^{-\nu}}{h(\gamma_z x, z)^{-\nu}}\, d\mu_\nu(z)\, d\mu_\nu(x) \\ &= \iint_{\Omega\times\Omega} f(z) \overline{\psi(z)} h(z,z)^{-\nu/2} g(\gamma_z x) U^{(\nu)}_{\gamma_z} \phi(x)\, d\mu_\nu(x)\, d\mu_\nu(z) \qquad (5.1) \end{aligned}$$

(cf. (3.20) in [3]). We split the inner integrand (with respect to the x variable) as follows:

$$\begin{aligned} g\circ\gamma_z \cdot U^{(\nu)}_{\gamma_z}\phi &= \operatorname{Tayl}_M(g\circ\gamma_z \cdot U^{(\nu)}_{\gamma_z}\phi) \\ &\quad + \Big[\operatorname{Tayl}_M(g\circ\gamma_z)\cdot U^{(\nu)}_{\gamma_z}\phi - \operatorname{Tayl}_M(g\circ\gamma_z \cdot U^{(\nu)}_{\gamma_z}\phi)\Big] + \operatorname{Rem}_M(g\circ\gamma_z)\cdot U^{(\nu)}_{\gamma_z}\phi, \end{aligned}$$

and let G_I, G_{II} and G_{III} be the corresponding contributions to the integral (5.1); here M is an integer which will be specified at the end of the proof.

Let us first deal with G_I. By Lemma 7, we have

$$\begin{aligned} \int_\Omega \operatorname{Tayl}_M(g\circ\gamma_z \cdot U^{(\nu)}_{\gamma_z}\phi)\, d\mu_\nu &= \sum_{|\mathbf{m}|\le M} \frac{1}{(\nu)_{\mathbf{m}}} K_{\mathbf{m}}(\partial,\partial)(g\circ\gamma_z \cdot U^{(\nu)}_{\gamma_z}\phi)(0) \\ &= \sum_{|\mathbf{m}|\le M} \frac{1}{(\nu)_{\mathbf{m}}} K_{\mathbf{m}}(\partial,\partial)\Big(g\circ\gamma_z \cdot \phi\circ\gamma_z \cdot \frac{h(\gamma_z\cdot, z)^\nu}{h(z,z)^{\nu/2}}\Big)(0) \\ &= \sum_{|\mathbf{m}|\le M} \frac{1}{(\nu)_{\mathbf{m}}} \mathcal{K}_{\mathbf{m},x}\Big(\frac{g(x)\phi(x)h(x,z)^\nu}{h(z,z)^{\nu/2}}\Big)\Big|_{x=z} \\ &= \sum_{|\mathbf{m}|\le M} \frac{1}{(\nu)_{\mathbf{m}}} \mathcal{K}_{\mathbf{m},z}\Big(\frac{g(z)\phi(z)h(z,x)^\nu}{h(x,x)^{\nu/2}}\Big)\Big|_{x=z}. \end{aligned}$$

Consequently,

$$G_I=\sum_{|\mathbf{m}|\le M}\frac{1}{(\nu)_{\mathbf{m}}}\int_\Omega f(z)\overline{\psi(z)}h(z,z)^{-\nu}\,\mathcal{K}_{\mathbf{m},z}\left[\frac{g(z)\phi(z)}{h(z,x)^{-\nu}}\right]_{x=z}d\mu_\nu(z).$$

But by (4.3), the last integral is equal to $\langle T_\nu[A_{\mathbf{m}}(f,g)]\phi,\psi\rangle_\nu$, as soon as ν is sufficiently large; that is,

$$G_I=\sum_{|\mathbf{m}|\le M}\frac{1}{(\nu)_{\mathbf{m}}}\langle T_\nu[A_{\mathbf{m}}(f,g)]\phi,\psi\rangle_\nu \qquad \text{if } \nu\gg 0. \tag{5.2}$$

Let us now turn to G_{II}. Note that since $U^{(\nu)}_{\gamma_z}\phi$ is holomorphic, $\operatorname{Tayl}_M(g\circ\gamma_z\cdot U^{(\nu)}_{\gamma_z}\phi)=\operatorname{Tayl}_M([\operatorname{Tayl}_M(g\circ\gamma_z)]\cdot U^{(\nu)}_{\gamma_z}\phi)$. Denoting temporarily, for brevity, $\operatorname{Tayl}_M(g\circ\gamma_z)=:H$ and $U^{(\nu)}_{\gamma_z}\phi=:\Phi$, we are thus lead to study

$$\int_\Omega H\Phi-\operatorname{Tayl}_M(H\Phi)\,d\mu_\nu=\int_\Omega\operatorname{Rem}_M(H\Phi)\,d\mu_\nu.$$

Consider first the case when $H=p_{\mathbf{k}}\overline{q_{\mathbf{n}}}$, with $p_{\mathbf{k}}\in\mathcal{P}_{\mathbf{k}}$, $q_{\mathbf{n}}\in\mathcal{P}_{\mathbf{n}}$, $|\mathbf{k}|+|\mathbf{n}|\le M$. Let $\Phi=\sum_{\mathbf{m}}\Phi_{\mathbf{m}}$ be the Peter-Weyl expansion of Φ. Then

$$\begin{aligned}\int_\Omega\operatorname{Rem}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi)\,d\mu_\nu&=\sum_{\mathbf{m}}\int_\Omega\operatorname{Rem}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi_{\mathbf{m}})\,d\mu_\nu\\&=\sum_{|\mathbf{m}|>M-|\mathbf{k}|-|\mathbf{n}|}\int_\Omega p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi_{\mathbf{m}}\,d\mu_\nu.\end{aligned}$$

Since

$$\mathcal{P}_{\mathbf{k}}\mathcal{P}_{\mathbf{m}}\subset\sum_{\substack{\mathbf{j}\supset\mathbf{m},\mathbf{k}\\|\mathbf{j}|=|\mathbf{m}|+|\mathbf{k}|}}\mathcal{P}_{\mathbf{j}}, \tag{5.3}$$

the last integral can be nonzero only if $\mathbf{k}\subset\mathbf{n}$, $\mathbf{m}\subset\mathbf{n}$ and $|\mathbf{m}|=|\mathbf{n}|-|\mathbf{k}|$; the last implies that the inequality $|\mathbf{m}|>M-|\mathbf{k}|-|\mathbf{n}|$ is equivalent to $|\mathbf{n}|>M/2$. Thus

$$\int_\Omega\operatorname{Rem}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi)\,d\mu_\nu=\begin{cases}0 & \text{if } |\mathbf{n}|\le\frac{M}{2} \text{ or } \mathbf{k}\not\subset\mathbf{n},\\ \sum_{\substack{\mathbf{m}\subset\mathbf{n},\\|\mathbf{m}|=|\mathbf{n}|-|\mathbf{k}|}}\int_\Omega p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi_{\mathbf{m}}\,d\mu_\nu & \text{if } |\mathbf{n}|>\frac{M}{2} \text{ and } \mathbf{k}\subset\mathbf{n}.\end{cases}$$

Similar computation shows that

$$\int_\Omega\operatorname{Tayl}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi)\,d\mu_\nu=\begin{cases}0 & \text{if } |\mathbf{n}|>\frac{M}{2} \text{ or } \mathbf{k}\not\subset\mathbf{n},\\ \sum_{\substack{\mathbf{m}\subset\mathbf{n},\\|\mathbf{m}|=|\mathbf{n}|-|\mathbf{k}|}}\int_\Omega p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi_{\mathbf{m}}\,d\mu_\nu & \text{if } |\mathbf{n}|\le\frac{M}{2} \text{ and } \mathbf{k}\subset\mathbf{n}.\end{cases}$$

Thus

$$\int_\Omega\operatorname{Rem}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi)\,d\mu_\nu=\begin{cases}0 & \text{if } |\mathbf{n}|\le\frac{M}{2} \text{ or } \mathbf{k}\not\subset\mathbf{n},\\ \int_\Omega p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi\,d\mu_\nu & \text{if } |\mathbf{n}|>\frac{M}{2} \text{ and } \mathbf{k}\subset\mathbf{n}.\end{cases}$$

Now

$$\Big|\int_\Omega p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi\,d\mu_\nu\Big|\le\|\Phi\|_\nu\cdot\Big(\int_\Omega|p_{\mathbf{k}}\overline{q_{\mathbf{n}}}|^2\,d\mu_\nu\Big)^{1/2}=\|\Phi\|_\nu\cdot\|p_{\mathbf{k}}q_{\mathbf{n}}\|_\nu.$$

Again by (5.3), we have $p_{\mathbf{k}}q_{\mathbf{n}} \in \sum_{\mathbf{j}} \mathcal{P}_{\mathbf{j}}$ where the summation is only over $\mathbf{j} \supset \mathbf{k}, \mathbf{n}$, $|\mathbf{j}| = |\mathbf{k}| + |\mathbf{n}|$; thus $|\mathbf{j}| \leq M$ and if $|\mathbf{n}| > {}^{M}\!/2$, then $|\mathbf{j}| > {}^{M}\!/2$. It follows that if $|\mathbf{n}| > {}^{M}\!/2$, then there exists a constant c_M such that

$$\frac{1}{(\nu)_{\mathbf{j}}} \leq \frac{c_M^2}{\nu^{M/2}} \qquad \forall \nu > p-1.$$

Hence by (2.3)

$$\|p_{\mathbf{k}}q_{\mathbf{n}}\|_\nu \leq c_M \frac{\|p_{\mathbf{k}}q_{\mathbf{n}}\|_F}{\nu^{M/4}}.$$

Summing up, we see that we always have

$$\Big| \int_\Omega \operatorname{Rem}_M(p_{\mathbf{k}}\overline{q_{\mathbf{n}}}\Phi)\, d\mu_\nu \Big| \leq \|\Phi\|_\nu \cdot \frac{c_M}{\nu^{M/4}} \|p_{\mathbf{k}}q_{\mathbf{n}}\|_F. \tag{5.4}$$

Returning now to the case of general H, choose, for each $\mathbf{m}$, a basis $\{\psi_{\mathbf{m}j}\}_{j=1}^{\dim \mathcal{P}_{\mathbf{m}}}$ of $\mathcal{P}_{\mathbf{m}}$. Then H can be uniquely written in the form

$$H = \sum_{\substack{\mathbf{m},j,\mathbf{n},k;\\ |\mathbf{m}|+|\mathbf{n}|\leq M}} c_{\mathbf{m}j\mathbf{n}k}\, \psi_{\mathbf{m}j}\overline{\psi_{\mathbf{n}k}} \tag{5.5}$$

with some complex coefficients $c_{\mathbf{m}j\mathbf{n}k}$. These coefficients, in turn, can be computed from the derivatives of order $\leq M$ of H at the origin by solving an appropriate system of linear equations (corresponding to the change of basis from the standard monomials z^α to the polynomials $\psi_{\mathbf{m}j}$); hence they satisfy

$$|c_{\mathbf{m}j\mathbf{n}k}| \leq c'_M \sup_{j=1,\dots,M} \|\nabla^j H(0)\|.$$

Applying (5.4) to each summand in (5.5), it therefore follows that

$$\Big| \int_\Omega \operatorname{Rem}_M(H\Phi)\, d\mu_\nu \Big| \leq \|\Phi\|_\nu \cdot \frac{c''_M}{\nu^{M/4}} \sup_{j=1,\dots,M} \|\nabla^j H(0)\|.$$

Recalling what Φ and H stood for, noting that $\nabla^j H = \nabla^j(g\circ\gamma_z)$ for $j \leq M$ and $\|U^{(\nu)}_{\gamma_z}\phi\|_\nu = \|\phi\|_\nu$, and inserting everything into (5.1), we thus obtain

$$|G_{II}| \leq \int_\Omega |f(z)\overline{\psi(z)}|\, h(z,z)^{-\nu/2} \cdot \|\phi\|_\nu \frac{c''_M}{\nu^{M/4}} \sup_{j=1,\dots,M} \|\nabla^j(g\circ\gamma_z)(0)\|\, d\mu_\nu(z).$$

As we have seen in course of the proof of Proposition 4, for any multiindex α we have $D^\alpha(g\circ\gamma_z)(0) = L_{P_\alpha} g^\#(\gamma_z)$ for some $P_\alpha \in \mathfrak{U}(\mathfrak{g})$. Consequently, if $g \in IBC^\infty$, then

$$\begin{aligned} \sup_{j=1,\dots,M} \|\nabla^j(g\circ\gamma_z)(0)\| &= \sup_{|\alpha|\leq M} |D^\alpha(g\circ\gamma_z)(0)| \\ &\leq \sup_{|\alpha|\leq M} \|L_{P_\alpha} g^\#\|_\infty = \sup_{|\alpha|\leq M} \|\!|g|\!\|_{P_\alpha} < \infty \end{aligned}$$

for any $z \in \Omega$. Denoting the last supremum by $[\![g]\!]_M$, we can thus continue with

$$\begin{aligned}
|G_{II}| &\le \|\phi\|_\nu \frac{c''_M}{\nu^{M/4}} \cdot [\![g]\!]_M \int_\Omega |f(z)\overline{\psi(z)}|\, h(z,z)^{-\nu/2}\, d\mu_\nu(z) \\
&\le \|\phi\|_\nu \frac{c''_M}{\nu^{M/4}} \cdot [\![g]\!]_M \cdot \|\psi\|_\nu \cdot \Big(\int_\Omega |f(z)|^2 h(z,z)^{-\nu}\, d\mu_\nu(z) \Big)^{1/2} \\
&= \|\phi\|_\nu \frac{c''_M}{\nu^{M/4}} \cdot [\![g]\!]_M \cdot \|\psi\|_\nu \cdot \Lambda_\nu^{1/2} \|f\|_{L^2(\Omega,d\mu)}.
\end{aligned}$$

As $\Lambda_\nu \le C\,\nu^d$ (cf. Lemma 3.1(i) in [3]), we thus arrive at

$$|G_{II}| \le c'''_M \|\phi\|_\nu \|\psi\|_\nu [\![g]\!]_M \|f\|_{L^2(\Omega,d\mu)}\, \nu^{(2d-M)/4}. \tag{5.6}$$

Finally, let us consider G_{III}. By Taylor's theorem (formula (3.37) in [3])

$$\mathrm{Rem}_M(g\circ\gamma_z)(x) = \frac{1}{M!} \int_0^1 (1-s)^M \frac{d^{M+1}}{ds^{M+1}} (g\circ\gamma_z)(sx)\, ds.$$

Let $x = k_x \sum_{j=1}^r t_j e_j$ be the polar decomposition of x. Recalling (2.15), applying it to $\tau_j = \operatorname{arctanh}(st_j)$, and denoting for brevity

$$u(\tau_1,\dots,\tau_r) := g^\#(\gamma_z k_x e^{\sum_j \tau_j E_j}) = g(\gamma_z(sx)),$$

it follows that (by an inductive argument, similarly as in (3.5) above)

$$\begin{aligned}
\frac{d^{M+1}}{ds^{M+1}} (g\circ\gamma_z)(sx) &= \frac{d^{M+1}}{ds^{M+1}} u(\tau_1,\dots,t_r) \\
&= \sum_{q,\iota,m_1,\dots,m_q} \kappa_{\iota,m_1,\dots,m_q} \frac{d^{m_1}\tau_{\iota_1}}{ds^{m_1}} \dots \frac{d^{m_q}\tau_{\iota_q}}{ds^{m_q}} \cdot \frac{\partial^q u}{\partial\tau_{\iota_1}\dots\partial\tau_{\iota_q}},
\end{aligned}$$

with some universal constants $\kappa_{\iota,m_1,\dots,m_q}$ and the summation extending over all q-tuples $\iota = (\iota_1,\dots,\iota_q)$, $0 \le q \le M+1$, $1 \le \iota_j \le d$, and indices $m_1,\dots,m_q$ such that $m_1,\dots,m_q \ge 1$, $m_1 + \dots + m_q = M+1$. Since for $k \ge 1$

$$\frac{d^k \tau_i}{ds^k} = t_i^k \frac{(\text{a polynomial in } st_i)}{(1-s^2t_i^2)^k},$$

so that

$$\Big|\frac{d^k\tau_i}{ds^k}\Big| \le t_i^k \frac{C_k}{(1-t_i^2)^k} \le |x|^k \frac{C_k}{\prod_{j=1}^r (1-t_j^2)^k} = C_k |x|^k h(x,x)^{-k},$$

while

$$\begin{aligned}
\frac{\partial^q u(\tau_1,\dots,\tau_r)}{\partial\tau_{\iota_1}\dots\partial\tau_{\iota_q}} &= \frac{\partial^q}{\partial t_{\iota_1}\dots\partial t_{\iota_q}} g^\#(\gamma_z k_x e^{\sum \tau_j E_j} e^{\sum t_j E_j})\Big|_{t_1=\dots=t_r=0} \\
&= L_{E_{\iota_1}\dots E_{\iota_q}} g^\#(\gamma_z k_x e^{\sum \tau_j E_j}),
\end{aligned}$$

so that

$$\Big|\frac{\partial^q u(\tau_1,\dots,\tau_r)}{\partial\tau_{\iota_1}\dots\partial\tau_{\iota_q}}\Big| \le \|\!|g|\!\|_{E_{\iota_1}\dots E_{\iota_q}},$$

we thus obtain

$$\left|\frac{d^{M+1}}{ds^{M+1}}(g\circ\gamma_z)(sx)\right| \le C'_M|x|^{M+1}h(x,x)^{-M-1}\sum_{q=0}^{M+1}\sum_{\iota_1,\dots,\iota_q}\|\!|g|\!\|_{E_{\iota_1}\dots E_{\iota_q}}$$
$$=: C'_M|x|^{M+1}h(x,x)^{-M-1}\,\|\!|g|\!\|_M,$$

whence

$$|\operatorname{Rem}_M(g\circ\gamma_z)(x)| \le C''_M|x|^{M+1}h(x,x)^{-M-1}\|\!|g|\!\|_M.$$

Substituting this into (5.1), we thus get the estimate

$$|G_{III}| \le C''_M\|\!|g|\!\|_M \iint_{\Omega\times\Omega}\frac{|f(z)\overline{\psi(z)}|}{h(z,z)^{\nu/2}}\,\frac{|x|^{M+1}}{h(x,x)^{M+1}}\,|U^{(\nu)}_{\gamma_z}\phi(x)|\,d\mu_\nu(x)\,d\mu_\nu(z).$$

Applying the Schwarz inequality to the x integration and using (3.42) in [3] gives

$$\begin{aligned}
|G_{III}| &\le C''_M\|\!|g|\!\|_M\int_\Omega\frac{|f(z)\overline{\psi(z)}|}{h(z,z)^{\nu/2}}\left[\int_\Omega\frac{|x|^{2M+2}}{h(x,x)^{2M+2}}\,d\mu_\nu(x)\right]^{1/2}\|U^{(\nu)}_{\gamma_z}\phi\|_\nu\,d\mu_\nu(z)\\
&\le C'''_M\|\!|g|\!\|_M\nu^{-(M+1)/2}\|\phi\|_\nu\int_\Omega|f(z)\overline{\psi(z)}|\,h(z,z)^{-\nu/2}\,d\mu_\nu(z)\\
&\le C'''_M\|\!|g|\!\|_M\nu^{-(M+1)/2}\|\phi\|_\nu\|\psi\|_\nu\left(\int_\Omega|f(z)|^2h(z,z)^{-\nu}\,d\mu_\nu(z)\right)^{1/2}\\
&= C'''_M\|\!|g|\!\|_M\nu^{-(M+1)/2}\|\phi\|_\nu\|\psi\|_\nu\,\Lambda_\nu^{1/2}\|f\|_{L^2(\Omega,d\mu)}\\
&\le C''''_M\,\nu^{(d-M-1)/2}\|\!|g|\!\|_M\|\phi\|_\nu\|\psi\|_\nu\|f\|_{L^2(\Omega,d\mu)}.
\end{aligned}$$

Putting together (5.2), (5.6) and the last inequality, we thus obtain

$$\begin{aligned}
\Big\|\,T_\nu[f]T_\nu[g]\;-\sum_{|\mathbf{m}|\le M}&\frac{1}{(\nu)_\mathbf{m}}T_\nu[A_\mathbf{m}(f,g)]\Big\|\\
&\le c'''_M[\![g]\!]_M\|f\|_{L^2(\Omega,d\mu)}\,\nu^{\frac d2-\frac M4}+C''''_M\|\!|g|\!\|_M\|f\|_{L^2(\Omega,d\mu)}\,\nu^{\frac d2-\frac{M+1}{2}}\\
&\le C_M([\![g]\!]_M+\|\!|g|\!\|_M)\|f\|_{L^2(\Omega,d\mu)}\cdot\nu^{\frac d2-\frac M4}.
\end{aligned}$$

Choose now $M=2d+4(N+1)$ and note that, by Proposition 4, $A_\mathbf{m}(f,g)\in IBC^\infty\subset L^\infty$, so $\|\frac{1}{(\nu)_\mathbf{m}}T_\nu[A_\mathbf{m}(f,g)]\|=O(\nu^{-|\mathbf{m}|})$. Thus we may throw away the terms with $|\mathbf{m}|>N$ from the sum on the left-hand side. This gives

$$\Big\|\,T_\nu[f]T_\nu[g]\;-\sum_{|\mathbf{m}|\le N}\frac{1}{(\nu)_\mathbf{m}}T_\nu[A_\mathbf{m}(f,g)]\Big\|\le C_{f,g,N}\,\nu^{-N-1},$$

which is the desired assertion. ■

Remark. The hypothesis that $f\in IBC^\infty\cap L^2(\Omega,d\mu)$ and $g\in IBC^\infty$ in Theorem 8 is rather asymmetric, but since $T_\nu[f]^*=T_\nu[\overline{f}]$, taking adjoints shows that the theorem remains in force also for $f\in IBC^\infty$ and $g\in IBC^\infty\cap L^2(\Omega,d\mu)$. It is unclear whether the theorem can be extended beyond this, e.g. to any $f,g\in IBC^\infty$.

6. Applications to quantization

Let $C^\infty(\Omega)[[h]]$ be the ring of all power series with $C^\infty(\Omega)$ coefficients in a formal parameter h. Recall that a (differential) *star-product* on Ω is a $\mathbf{C}[[h]]$-bilinear mapping $*: C^\infty(\Omega)[[h]] \times C^\infty(\Omega)[[h]] \to C^\infty(\Omega)[[h]]$ such that

(i) $*$ is associative;

(ii) there exist bidifferential operators $C_j : C^\infty(\Omega) \times C^\infty(\Omega) \to C^\infty(\Omega)$ $(j = 0, 1, \dots)$ such that $\forall f, g \in C^\infty(\Omega)$,

$$f * g = \sum_{j=0}^\infty C_j(f,g) h^j; \tag{6.1}$$

(iii) the operators C_j satisfy

$$C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}, \text{ and} \tag{6.2}$$

$$C_j(\mathbf{1}, \cdot) = C_j(\cdot, \mathbf{1}) = 0 \ \ \forall j \geq 1. \tag{6.3}$$

(Note that the last requirement means that $\mathbf{1}$ is the identity element for $*$.) The star-product is called *(G-)invariant* if

$$(f \circ \phi) * (g \circ \phi) = (f * g) \circ \phi, \qquad \forall \phi \in G, \ \forall f, g. \tag{6.4}$$

Two invariant star-products are called *G-equivalent* if there exist invariant differential operators M_j, $j = 0, 1, 2, \dots$, with $M_0 = I$ (the identity operator), such that the operator $M := \sum_{j=0}^\infty M_j h^j$ on $C^\infty(\Omega)[[h]]$ satisfies

$$M(u *' v) = (Mu) * (Mv), \qquad \forall u, v \in C^\infty(\Omega)[[h]]. \tag{6.5}$$

It is known that on any Cartan domain Ω, the bidifferential operators C_j from (1.7) determine an invariant star-product, called the *Berezin-Toeplitz* star-product. It was shown in [4] that any invariant star-product $*'$ which is G-equivalent to the Berezin-Toeplitz star-product can also be obtained by a formula akin to (1.7) but with the Toeplitz operators $T_f^{(\nu)}$ replaced by another *invariant operator calculus* $Q_f^{(\nu)}$, $f \in \mathcal{D}(\Omega)$. That is, there exists (for each $\nu > p - 1$) a linear assignment $f \mapsto Q_f^{(\nu)}$ from the space $\mathcal{D}(\Omega)$ of all compactly supported smooth functions on Ω into the bounded linear operators on A^2_ν, which is invariant in the sense that

$$U_\phi^{(\nu)} Q_f^{(\nu)} U_\phi^{(\nu)*} = Q_{f\circ\phi}^{(\nu)} \qquad \forall \phi \in G,$$

and satisfies, for any integer $N \geq 0$,

$$\left\| Q_f^{(\nu)} Q_g^{(\nu)} - \sum_{j=0}^N Q^{(\nu)}_{C'_j(f,g)} \right\| = O(\nu^{-N-1}) \qquad \text{as } \nu \to +\infty, \tag{6.6}$$

for any $f, g \in \mathcal{D}(\Omega)$. (Here C'_j are, of course, the bidifferential operators corresponding to $*'$.)

It was also noted in [4] (cf. the end of Section 3 there) that the validity of (6.6) can even be extended to the whole Schwartz space $\mathcal{S} \supset \mathcal{D}(\Omega)$, granted one can show that (1.7) is valid for $f, g \in \mathcal{S}$. Since $\mathcal{S} \subset IBC^\infty \cap L^2(d\mu)$, it follows from our Main Theorem that this is indeed the case. Thus the main result of the paper [4] (i.e. the formula (6.6)) holds not only for $f, g \in \mathcal{D}(\Omega)$, but even for all $f, g \in \mathcal{S}$. Again, it seems to be an open problem whether $\mathcal{S}$ can be replaced by some even larger function space.

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