

Topologically Locally Finite Groups with a CC-Subgroup

Zvi Arad and Wolfgang Herfort*

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Abstract. A proper subgroup M of a finite group G is called a CC-subgroup of G if the centralizer $C_G(m)$ of every $m \in M^\# = M \setminus \{1\}$ is contained in M . Such finite groups had been partially classified by S. WILLIAMS, A. S. KONDRAT'IEV, N. IYORI and H. YAMAKI, M. SUZUKI, W. FEIT and J. - G. THOMPSON, M. HERZOG, Z. ARAD, D. CHILLAG and others. In [6] the present authors, having taken all this work into account, classified all finite groups containing a CC-subgroup.

As an application, in the present paper, we classify totally disconnected topologically locally finite groups, containing a topological analogue of a CC-subgroup.

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1. Statement of the results

A proper subgroup M is a *CC-subgroup* of a finite group G , provided $C_G(m) \leq M$ for all $m \in M^\#$. Examples of such groups G are Frobenius groups, which are semidirect products $F \rtimes H$, where each nontrivial $h \in H$ acts by conjugation without nontrivial fixed points upon F . Then M can be taken either F or H , since both are CC-subgroups of G . Groups with a CC-subgroup became an important ingredient in classifying finite simple groups, like Suzuki groups and certain projective groups. Having taken preceding work in [23], [8], [14], [1], [2], [3], [24], [4], [18], [15] and [16] into account, the authors gave a complete classification of all finite groups containing a CC-subgroup in Theorem A of [6]. Theorem 3.1 below recalls this result in a form to be used in the present paper. A theory of Frobenius groups has been developed for the classes of profinite groups (see Chapter 4 in [22]), for locally finite groups in [17], and for certain infinite groups in [10]. In [13] the class $[\text{LF}]^-$ of all *topologically locally finite* groups has been defined to contain all locally compact groups in which every compact subset is contained in a compact subgroup - a compactness condition in the the spirit of [11]. Let $[\text{TD}]$ denote the class of all locally compact totally disconnected groups. Then the class $[\text{LF}]^- \cap [\text{TD}]$ of *totally disconnected topologically locally finite groups* contains both

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classes, the one of profinite groups and the one of locally finite groups.

It appears desirable to us to classify all groups $G \in [\text{LF}]^- \cap [\text{TD}]$ possessing a subgroup M , which contains the centralizers of all its nontrivial elements. But, without further restrictions on such M , for profinite groups G the situation can be as complicated as for arbitrary infinite groups (without topology). For instance, in the free pro- p group $G = \langle x, y \rangle$ (the pro- p completion of a free group on the set $\{x, y\}$) the closed subgroup A generated by the commutator $[x, y]$ contains the centralizers of its nontrivial elements. In fact, A is *isolated* in G (like Frobenius-complements in the situation of a Frobenius group), i.e., $A \cap A^g \neq \{1\}$ always implies $g \in A$, but contrasting the finite situation, neither does A possess a normal complement in G , nor it is a Hall-subgroup of G . Therefore, in accordance with [9, 13], for a profinite group G we define a CC-subgroup M to be a proper subgroup of G which contains all centralizers of its nontrivial elements and is a Hall-subgroup, i.e., for every open normal subgroup N of G the quotient MN/N is a Hall-subgroup of G/N (see [22] or [25]). Then the following description of profinite Frobenius groups similar to the finite ones can be given, and we quote the result for later use as well. For finite Frobenius groups see D.Passman's book [21].

Proposition 1.1. *The following statements on a profinite group G and Hall-subgroups H and K are equivalent:*

- (i) K is a CC-subgroup of G ; (then there exists a complement H of K);
- (ii) H is an isolated Hall-subgroup of G ; (then $K := \left(G \setminus \bigcup_{g \in G} H^g\right) \cup \{1\}$ is a normal CC-subgroup of G);
- (iii) $G = K \rtimes H$, a semidirect product with H, K Hall-subgroups of G , $\pi(H) \cap \pi(K) = \emptyset$ and each nontrivial element of H acts by conjugation without fixed points upon $K \setminus \{1\}$; (then H, K are CC-subgroups of G);
- (iv) G is the projective limit of an inverse system of finite Frobenius groups, where the canonical maps are all epimorphisms and K, H are projective limits of the respective Frobenius kernels and complements.

When either of these conditions holds, then G is a profinite Frobenius group, K is the Frobenius kernel, and H is a Frobenius complement. Moreover, all of the following properties hold:

- (a) K is nilpotent; there is a bound on the nilpotency class depending on $\pi(H)$ only;
- (b) H is finite. Its p -Sylow subgroups are cyclic or generalized quaternion (the latter only can happen, when $p = 2$). Moreover, $Z(H) \neq \{1\}$;
- (c) any normal subgroup $L \triangleleft G$ satisfies $L \leq K$ or $K \leq L$.

Proof. All this is taken from Section 4.6. in [22]. ■

What can be said about locally finite groups? First, when G is locally finite, it possesses a local system of finite groups [17], i.e., a set Λ of finite subgroups of G with the property that any finite subset is contained in a subgroup occurring in Λ . It is plain that for a proper subgroup M of G , which contains the centralizers of its nontrivial elements, the intersection $H \cap M$ for any $H \in \Lambda$ either is not a proper subgroup or it is a CC-subgroup of H . Finally suppose $G \in [\text{LF}]^- \cap [\text{TD}]$. Then it has been observed in [13] that such G possesses a *local system* Λ of profinite subgroups, i.e., a system of compact subgroups such that every compact subset is contained in a group of the system. The following formal definition of a CC-subgroup is appropriate.

Definition 1.2. A proper subgroup M of a group $G \in [\text{LF}]^- \cap [\text{TD}]$ is a CC-subgroup if there is a local system Λ such that $M \cap H$ is a CC-subgroup of H for every $H \in \Lambda$.

Our definition very well depends upon the local system Λ .

An announcement of the results and some of the historical background can be found in [5], however in this paper corrections to some of the statements of the results are made. For $G \in [\text{LF}]^- \cap [\text{TD}]$ any set of normal closed pronilpotent subgroups generates a normal subgroup whose compact subgroups are all pronilpotent. Thus the notion of the *Fitting subgroup*, denoted by $F(G)$, for the smallest closed normal subgroup containing all pronilpotent normal subgroups of G generalizes a well-known concept from finite group theory.

The authors would like to thank the referee for several helpful remarks. We come to stating the main result of the paper.

Theorem 1.3. *Let G be an infinite group in $[\text{LF}]^- \cap [\text{TD}]$ possessing a CC-subgroup M . Then precisely one of the following holds:*

- (i) M is locally finite; $M \cap M^g \neq \{1\}$ implies $g \in M$ (we say M is isolated), G is the semidirect product $G = F \rtimes M$ for a suitable normal subgroup F and $F := \left(G \setminus \bigcup_{g \in G} M^g \right) \cup \{1\}$ is a CC-normal subgroup of G ; (in accordance with [10, 13] call G a Frobenius group with kernel F and complement H).
- (ii) M is finite and cyclic. G possesses a normal subgroup F , and a finite cyclic subgroup R such that $FM \triangleleft G$ is a Frobenius group (F is the kernel and M a finite complement) and MR is a finite Frobenius group with kernel M and complement R (let us say that $G = FMR$ is a 2-Frobenius group).
- (iii) G is a Frobenius group with $M \triangleleft G$ an open CC-normal subgroup (the kernel) and it possesses a locally finite Frobenius-complement H , which is an isolated subgroup of G ; every complement to M in G is a conjugate of H ;
- (iv) M is locally finite; for $H := (M)_G$ the quotient $S := H/F(H)$ is a topologically locally finite simple group, $MF(H)/F(H) \cong M$ is a CC-subgroup of $G/F(G)$, and G/H is a locally finite group such that every element commutes with an involution; if $F(G)$ is open in H then $S := H/F(G)$ is either finite and as in Theorem A(iv)(b) of [6] or S is an infinite locally finite simple group.

Closer inspection of Theorem 3.1 yields the following consequence of Theorem 1.3.

Corollary 1.4. *Let $G \in [\text{LF}]^- \cap [\text{TD}]$ contain a CC-subgroup M either containing an involution or not being locally nilpotent then G is locally finite and one of the following holds:*

- (i) G is a Frobenius group with M either kernel or complement;
- (ii) $G \cong \text{PSL}(2, F)$, with F a locally finite field;
- (iii) $G = \text{Sz}(F)$, a locally finite Suzuki group as described in 4.18 Theorem of [17] and M is locally solvable;

We would like to fix notation on topological groups, in particular on profinite groups. For any topological group G subgroups are closed unless stated differently. For $X \subseteq G$ denote by $\langle X \rangle$ the smallest closed subgroup containing X (the subgroup *topologically generated by X*) and by $(X)_G$ the smallest closed normal subgroup of G containing X (the *normal closure of X in G*).

Let $G \in [\text{LF}]^- \cap [\text{TD}]$. Then for $x \in G$, recalling $\langle x \rangle$ to be a profinite group, let $\pi(x)$ denote the set of primes dividing $\langle x \rangle/N$ for some open subgroup N of $\langle x \rangle$. For $H \leq G$ let $\pi(H) := \bigcup_{x \in H} \pi(x)$. If $\pi(G) \subseteq \pi$ for a set π of primes, then G is a π -group and let π' denote the set theoretic complement, i.e., $\pi' := \pi(G) \setminus \pi$. For $A, B \leq G$ we shall find it convenient to write $(|A|, |B|) = 1$, if $\pi(A) \cap \pi(B) = \emptyset$.

The *prime graph* of G (Gruenberg-Kegel graph in [19]) is defined as follows (see [12]): the vertices are the primes in $\pi(G)$, two vertices p, q are joined by an edge if and only if G contains an element x with $\{p, q\} \subseteq \pi(x)$. Denote the connected components of the graph by $\{\pi_i \mid i := 1, \dots, t\}$ (for finite groups work cited above implies $t \leq 6$ and as a by-result of Theorem 1.3 the same bound on t applies to groups G in $[\text{LF}]^- \cap [\text{TD}]$) and if $2 \in \pi(G)$, denote the component containing 2 by π_1 .

2. Lifting CC-subgroups

During this section, unless stated differently, G belongs to $[\text{LF}]^- \cap [\text{TD}]$. Our main objective is deriving Lemma 2.7, which shows that given a CC-subgroup M of G and a normal subgroup N of G , then G/N has a CC-subgroup MN/N provided $N < MN < G$. The proof of this fact is first done for profinite groups (Lemma 2.6). The following observation is immediate:

Lemma 2.1. *Suppose a subgroup M of G and for normal subgroups $K \leq N$ of G , both, MK/K and $(MN/K)/(N/K)$, contain the centralizers of its nontrivial elements. Then so does MN/N .*

As a consequence of the Lemma, suppose G possesses a normal series

$$N = N_0 \geq \dots \geq N_k > N_{k+1} = \{1\}$$

and M as well as for $j = 1, \dots, k+1$ each $(MN_{j-1}/N_j)/(N_{j-1}/N_j)$ contains the centralizers of its nontrivial elements. Then so does MN/N .

We augment Proposition 1.1 with the following observation on profinite 2-Frobenius groups.

Lemma 2.2. (i) When $G = FM$ and $G' = F'M'$ are profinite Frobenius groups with $G \leq G'$ then $F \leq F'$ and M is contained in a G' conjugate of M' .

(ii) Let $G = FMR$ and $G' = F'M'R'$ be profinite 2-Frobenius groups (in particular, $N_G(M) = MR$, $N_{G'}(M') = M'R'$ and $R \neq \{1\}$) such that $G \leq G'$ then $F \leq F'$, a G' -conjugate of M is contained in M' , and, an M' -conjugate of R is contained in R' .

Proof. (i) The prime graphs of both, G and G' , contain exactly 2 connected components. Therefore either $\pi(F) \subseteq \pi(F')$ or $\pi(M) \subseteq \pi(F')$. The latter case cannot happen: else one can arrange $F \leq M'$ as well, so that $F \leq G = (M)_G \leq (M)_{G'} \leq F'$, a contradiction. Hence $F \leq F'$ must hold and therefore a conjugate of M is contained in M' .

(ii) Now $R \neq \{1\}$. When $R' = \{1\}$, an application of (i) first to $FM \leq F'M'$ and then to $MR \leq F'M'$ yield $\pi(M) \subseteq \pi(M')$ and $M \leq F'$, i.e., $\pi(M) \subseteq \pi(F')$, a contradiction. Hence $R' \neq \{1\}$, and, since R and R' are cyclic complements of FM and $F'M'$ respectively, find $FM = [G, G] \leq [G', G'] = F'M'$. Then, using (i), we can arrange $F \leq F'$ and $M \leq M'$. Let r be a generator of the cyclic group R , then $M^r = M$, so that $M^{r'} \cap M' \neq \{1\}$. Since M' is the Frobenius complement of F' in the normal Frobenius subgroup $F'M'$ of G' , there exists $f' \in F'$ with $M^{r'} = M'^{f'}$. Then $f' = 1$ must hold, since M' is isolated in $F'M'$. Hence $r \in N_{G'}(M')$, i.e., $R \leq M'R'$. Now application of (i) to $MR \leq M'R'$ yields the desired result. ■

Our next result in part corrects Theorem 4 in [4]. The rest of it is corrected in Theorem 1.3 and after Lemma 3.2 find an example of a profinite 2-Frobenius group which is not a profinite Frobenius group.

Lemma 2.3. Let M be a CC-subgroup of a profinite group G then either M is finite or M is an open subgroup of G .

Proof. We give a more direct proof than the one of the corresponding statement of Theorem 4 in [4]. Suppose the Lemma to be false. Then there exists a counterexample (G, M) with M an infinite non-open CC-subgroup of G . Note for later that for every open normal subgroup $N \triangleleft G$, the pair $(N, N \cap M)$ is a counterexample as well, since for every Hall subgroup L of G the intersection $L \cap N$ is a Hall subgroup of N .

Claim : For all $p \in \pi(G)$ the p -Sylow subgroup is finite. It is either cyclic or generalized quaternion (the latter only can happen when $p = 2$).

Suppose the Claim to be false. Then there exists a prime p and an infinite p -Sylow subgroup, say P of G . Observe $p \in \pi(N)$ for every open normal subgroup N of G . One must have $\pi(N) \cap \pi(M) \neq \emptyset$ else M would be finite.

We claim that $\pi(N) \setminus \pi(M) \neq \emptyset$ must hold. Suppose not, then $\pi(N) \subseteq \pi(M)$. According to the definitions, for any open normal subgroup K of G contained in N the factor group MK/K is a $\pi(M)$ -Hall subgroup of G/K . Thus the order of G/KM is not divisible by any prime in $\pi(M)$. In particular this holds for $|NM/KM|$ so that every $\pi(M)$ '-element of NM/K belongs to KM .

Thus $N \leq KM$, and since $K \leq N$ was arbitrary, conclude $N \leq M$, so that M is open, a contradiction.

Therefore, for any open normal subgroup N of G , when $p \notin \pi(M)$ we can fix a prime $q_N \in \pi(M) \cap \pi(N)$ and when $p \in \pi(M)$ we select $q_N \in \pi(N) \setminus \pi(M)$. Next fix a q_N -Sylow subgroup Q_N of N and observe $Q_N \neq \{1\}$. Taking into account that $Q_N \rtimes P_N$ with P_N a p -Sylow subgroup of $N_G(Q_N)$, is a profinite Frobenius group, infer from Proposition 1.1 that its complement P_N is either cyclic or generalized quaternion. Using the Frattini argument, namely that $G = N_G(Q_N)N$, and, that $P_N^{w_N} \leq P$ for suitable $w_N \in G$, one deduces $P_N/N \cong P_N N/N$, so that P , being the projective limit of either cyclic groups or of generalized quaternionic groups, and, containing up to conjugation the finite subgroup P_N , must itself be finite. Moreover, when $N \cap P = \{1\}$, then from $P \cong P_N$ the second statement of the Claim can be seen to hold true.

We continue proving the Lemma. By the Claim the 2-Sylow subgroup of G is finite, so one can pass to an open normal subgroup N of G containing no 2-elements. Then, as said earlier, $(N, M \cap N)$ would still be a counter-example. Call it again (G, M) , abusing language. Then, as a consequence of the Claim, G has cyclic Sylow subgroups, and therefore it is a profinite analogue of a Zassenhaus group, i.e., G possesses a normal procyclic subgroup L and a procyclic complement H , so that $G = L \rtimes H$ and $(|L|, |H|) = 1$.

We show that $L \leq M$ must hold. If not, then $L \cap M = \{1\}$, since M is a CC-subgroup of M . Then $L \rtimes M$ turns out to be a profinite Frobenius group with complement M . An application of Proposition 1.1 shows M must be finite, a contradiction. So $L \leq M$. Since G/L is procyclic, M is a normal subgroup of G , hence M is a CC-normal subgroup, and by Proposition 1.1, it possesses a finite Frobenius complement, so that M is open in G , contradicting (G, M) to be a counter-example. ■

The next Lemma, although it looks as if it were a mere consequence of the preceding proposition, is a form of stating a profinite version of the Schur-Zassenhaus Theorem. The latter was a key in proving the results on profinite Frobenius groups in [9].

Lemma 2.4. *Let $G \rtimes A$ be a semidirect product of profinite groups and $H \leq G$ be an A -invariant subgroup with $(|H|, |A|) = 1$. Suppose, for some $g \in G$, the coset Hg is A -invariant (as a set), then Hg contains an element x with $x^\alpha = x$ for all $\alpha \in A$.*

Proof. Restatement of Lemma 1.3 in [9], correcting a misprint ibidem. ■

Let \mathbb{Z}_p denote the pro- p group of p -adic integers. The following elementary result will be needed:

Lemma 2.5. *Let A be a finitely generated free \mathbb{Z}_p -module and let $H \cong C_p \times C_p$ act on it. Then there exists $h \in H \setminus \{1\}$ having a fixed point in $A \setminus \{1\}$.*

Proof. Let \mathbb{Q}_p denote the field of p -adic numbers, and consider the induced action of H on $V := A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Every fixed point $v \in V$ of an element $h \in H$

has the form $v = ap^{-l}$ for some $l \in \mathbb{N}$ and thus gives rise to the fixed point $a \in A$ for h . Thus it suffices to prove the Lemma for an arbitrary $\mathbb{Q}_p[H]$ -module V . If the Lemma were false it must be false for the induced action of H on $\hat{V} := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[\zeta_p]$, where $\zeta_p := \sqrt[p]{1}$. Then, since H is abelian, $\mathbb{Q}_p[\zeta_p]$ is a splitting field for H and every irreducible H -submodule $U \leq \hat{V}$ is 1-dimensional. Therefore, by Schur's Lemma, the kernel of the action of H on U is nontrivial. ■

Lemma 2.6. *Let M be a CC-subgroup of G . The following statements hold:*

- (i) *For any normal subgroup N of G with $N < MN < G$ the quotient MN/N is a CC-subgroup of G/N ;*
- (ii) *G is the projective limit of finite groups containing a CC-subgroup.*

Proof. (ii) is an immediate consequence of (i). Indeed, the set Σ of all open normal subgroups with $N < MN < G$ forms a fundamental system of open neighbourhoods of the identity of G , which we turn into a directed set by using the inclusion relation. Then $G = \varprojlim_{N \in \Sigma} G/N$.

Thus we need to prove (i). Suppose it to be false and let (G, M, N, g, m) be data of a counter-example, i.e., M is a CC-subgroup of G , N a normal subgroup of G , $g \in G \setminus MN$ and $m \in M \setminus N$ are elements with $[g, m] \in N$.

Claim 1: $\pi(M) \cap \pi(N) \neq \emptyset$. In particular $M \cap N \neq \{1\}$. Moreover G is infinite.

The second statement is a consequence of the first one, since M contains an r -Sylow subgroup for any $r \in \pi(M) \cap \pi(N)$. Suppose $(|N|, |M|) = 1$. Then setting in Lemma 2.4 $(H, A, g) := (N, \langle m \rangle, g)$ ensures the existence of $g_0 \in C_G(m) \cap gN \subseteq G \setminus M$, a contradiction to M being a CC-subgroup of G .

Since $M \cap N \neq \{1\}$, Theorem 1 in [7] shows that G is a Frobenius group with kernel M . Then $N \leq M$ holds and G/N is a Frobenius group with kernel M/N , i.e., M/N is a CC-subgroup of G/N , a contradiction. Hence G is infinite.

Claim 2: M is open. When H is any open subgroup of G , one can find an open normal subgroup K of G with $K \leq H$, and, (G, M, K, g, m) is a counter-example.

Given H open, select any K open normal in G with $K \leq H$ and $g \notin MK$, $m \notin KN$, $MK < G$ and $N \setminus K \neq \emptyset$. Then, if $[g, m] \notin K$, the equality $[gK/K, mK/K] \in NK/K$ shows $(G/K, MK/K, NK/K)$ to be a finite counter-example, a contradiction. Hence the second statement of the Claim holds.

When M is not open, then by Lemma 2.3 it is finite. Then, by what just has been proved, one can arrange $K \cap M = \{1\}$ contradicting Claim 1.

Claim 3: One can assume $N \leq M$. There is a prime p with $g^p = m^p = 1$. The core $C := \bigcap_{z \in G} M^z$ is nilpotent and can be assumed to be a torsion free pro- p subgroup.

By Claim 2 one can assume $N \leq M$. Since $\langle g \rangle \cap M = \{1\}$ the group $\langle g \rangle$ is finite and we may replace g by a suitable power in order to find a prime p with

$g^p = 1$. Since $N \leq M$ is open, replacing m by a suitable power, yields a prime q with $m^q \in N$.

Since M by assumption is an open Hall subgroup, there exists $h \in G$ with $\pi\langle h \rangle \cap \pi(M) = \emptyset$. Then $C \rtimes \langle h \rangle$ is a profinite Frobenius group, so that by Proposition 1.1, C is nilpotent.

Suppose C contains a nontrivial Hall- p' subgroup K . When $[g, m] \in K$ then setting $(A, H, g) := (\langle g \rangle, K, m)$ yields an element $m_0 \in mK \cap C_G(g)$, contradicting M being a CC-subgroup. Hence, using Lemma 2.1, we may factor K and assume that C is a pro- p group.

Since M is a Hall subgroup, there exists $x \in G$ with $g \in M^x$. Then our example (G, M, N, g, m) gives rise to the counter-example (G, M^x, N, m, g) , so that $m^q = 1$. Therefore one can assume C to be a pro- q subgroup, so that $p \neq q$ would imply $C = \{1\}$, and hence G to be finite, contradicting Claim 1. Hence $p = q$.

If C contains torsion, being nilpotent, there must be torsion in the center $Z(C)$ and hence the subgroup T of elements of order p in $Z(C)$ is not trivial. It is not hard to see that $L := \langle g, m, Z(C) \rangle$ is nilpotent and that $M \geq Z(C) \cap Z(L) \neq \{1\}$. Then the contradiction $g \in M$ follows. Hence C is torsion free.

Claim 4: The element m must belong to C .

Since, by Claim 3, $g^p = 1 = m^p$, and since $[g, m] \in N$, conjugation induces a $C_p \times C_p$ -action on the torsion free abelian group $Z(N)$. We claim the existence of a nontrivial $n \in Z(N)$ with $[m, n] = 1$. In order to see this, pick any $n_0 \in Z(N) \setminus \{1\}$ and consider the finitely generated $C_p \times C_p$ -module $L := \langle n_0^{g^i m^j} \mid i, j \in \{0, \dots, p-1\} \rangle$ then an application of Lemma 2.5 yields $h = g^i m^j$ with $(i, j) \neq (0, 0)$ and element $n \in L \setminus \{1\}$ with $[h, n] = 1$. Since $n \in M$, one has $h \in M$ and hence, as $g \notin M$, conclude $i = 0$. Therefore $[m, n] = 1$ holds.

Since M is a CC-subgroup of G , for any $x \in G$, $[m^x, n^x] = 1$, and $n^x \in M$, conclude $m \in C$.

For deriving a final contradiction, observe the existence of $x \in G$ with $g \in M^x$, consider (G, M, N, g, m) replaced by (G, M^x, N, m^x, g) , then Claim 4 shows $g \in C^x$ and hence $g \in C$. This contradicts $g \notin M$. ■

Lemma 2.7. Let $M \leq G$ be a CC-subgroup of $G \in [\text{LF}]^- \cap [\text{TD}]$ and $N \triangleleft G$ with $N < MN < G$. Then MN/N is a CC-subgroup of G/N .

Proof. Observe first that $\{LN/N \mid L \in \Lambda\}$ is a local system of G/N and that each $(M \cap L)N/N$ is a Hall π subgroup of LN/N . Now suppose the Lemma is false. Then there exists $x \in G \setminus MN$ and $m \in M$ with $[x, m] \in N$. Find a compact subgroup $L \in \Lambda$ (which then is profinite) with $N \cap L < (M \cap L)(N \cap L) < L$ and $\langle x, m \rangle \leq L$. Then $M \cap L$ is a CC-subgroup of L so that $[x, m] \in N \cap L$. Therefore Lemma 2.6 implies $x \in (\Gamma \cap M)(\Gamma \cap N) \leq NM$, a contradiction. ■

3. Proof of Theorem 1.3

We shall prove the Theorem in Lemma 3.2 in the profinite situation and we include Theorem 3.1, a simplified version of Theorem A in [6], sufficient for our purpose. Recall that for a finite group G , by $O_\pi(G)$ one denotes the maximal normal subgroup of G containing only π -elements.

Theorem 3.1. *Let G be a finite group containing a CC-subgroup M . Let $\pi := \pi(M)$. Then all π subgroups are conjugates of a subgroup of M in G . Furthermore we have one of the following four cases:*

(i) *M is non-nilpotent and of even order and one of the following holds:*

- (a) *G is a Frobenius group with complement M ;*
- (b) *$G \cong PSL(2, 2^n)$, $n \geq 2$ and M is solvable;*
- (c) *$G \cong Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$ and M is solvable.*

(ii) *M is nilpotent of even order and one of the following holds:*

- (a) *G is a solvable Frobenius group with complement M ;*
- (b) *G is a solvable Frobenius group with kernel M ;*
- (c) *$G \cong PSL(2, 2^n)$, $n \geq 2$ and M is a 2-Sylow subgroup;*
- (d) *$G \cong Sz(q)$, $q = 2^{2n+1}$, $n \geq 1$ and M is a 2-Sylow subgroup.*

(iii) *M is non-nilpotent of odd order and one of the following holds*

- (a) *G is a solvable Frobenius group with complement M ;*
- (b) *$G \cong PSL(2, q)$, $q \equiv 3 \pmod{4}$ and M is solvable of odd order $|M| = q^{\frac{q-1}{2}}$;*

(iv) *M is nilpotent of odd order and one of the following holds:*

- (a) *G is a Frobenius group with M either kernel or complement;*
- (b) *G is simple non-abelian and G and M are classified in Theorem II.11 of [6];*
- (c) *G is not simple; putting $H := (M)_G$, G/H and $F(G) = F(H) = O_{\pi_1}(H)$ are π_1 groups, $S := H/F(H)$ is a simple group having the CC-subgroup $MF(H)/F(H) \cong M$ and the pair (S, M) is of type (iv)(b);*
- (d) *G is a 2-Frobenius group;*

Lemma 3.2. *When G is profinite and M a CC-subgroup, then Theorem 1.3 holds.*

Proof. Suppose G is infinite. We shall find it convenient, during this proof, to say a property P holds *eventually*, if to every open subgroup U_0 of G there exists an open normal subgroup U of G inside U_0 such that G/U satisfies P . Then G is the projective limit of finite groups for which P holds. This notion at hand, Lemma 2.6 shows that G cannot *eventually belong* to the lists (i)(b) and (c), (ii)(c) and (d), (iii)(b), (iv)(b) of Theorem 3.1. If G is eventually a Frobenius group, then Proposition 1.1 implies G to be a profinite Frobenius group, whence it is described in Theorem 1.3 (i) or (iii). If M is finite, it is the complement, else it is the kernel.

From now on assume that each MU/U is odd order nilpotent then from the discussion up to now it follows that we may assume G to eventually belong to either (iv)(c) or (iv)(d) of Theorem 3.1.

In the second case for 'small' U the group G/U is as in Theorem 3.1(iv)(d). Then there exists $F \triangleleft G$ with FU/U Frobenius kernel in FMU/U . Therefore FM is a profinite Frobenius group with kernel F and finite complement M . Then $FM \triangleleft G$ must be open, as by Lemma 2.6, FM/F is a finite normal CC-subgroup of G/F possessing a cyclic complement L . Lift a generator of L to some element $c \in G$. Then, M being a Hall subgroup, one finds $M^c = M^f$ for a suitable element $f \in F$, so that $c_0 := cf^{-1}$ belongs to $N_G(M)$, so that with $R := \langle c_0 \rangle$, one has a finite Frobenius group $M \rtimes R$. Thus G satisfies Theorem 1.3 (ii).

Finally suppose that G is eventually as in Theorem 3.1(iv)(c). Setting $H := (M)_G$, a standard projective limit argument shows that G/H must be a profinite π_1 -group. Similarly it turns out that $S := H/O_{\pi_1}(H)$ is a finite simple group as described in Theorem 3.1(iv)(b). As S contains a CC-subgroup isomorphic to M , one finds that G/H must be finite. Such G is described in Theorem 1.3 (iv). ■

Theorem 1.3 (iii) corrects Theorem 4 in [4], where profinite 2-Frobenius groups are missing. Consider $F := \mathbb{Z}_2 \times \mathbb{Z}_2$ and define 2×2 -matrices

$$x := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $y^x = y^{-1}$, $1 + x + x^2 = 0$. One easily verifies that $M := \langle x \rangle \cong C_3$ is a CC-subgroup of the semidirect product $G := F \rtimes \langle x, y \rangle$. With $R := \langle y \rangle$ find $G = FMR$ to be profinite 2-Frobenius group, which is not Frobenius.

Proof of Theorem 1.3 in the general situation:

Proof. Fix $G \in [\text{LF}]^- \cap [\text{TD}]$ and a local system Λ of G with each $M \cap L$ being a CC-subgroup of the profinite group $L \in \Lambda$. Moreover we claim that all L can be assumed to be open subgroups. Indeed, when U is any open neighbourhood of the identity with compact closure, then $\langle U \rangle$ is compact. Whence there exists $L_0 \in \Lambda$ containing $\langle U \rangle$. Passing from Λ to a system with each L containing $\langle U \rangle$ serves the purpose.

In light of Theorem 3.1 and Lemma 3.2 we may assume G to be infinite and not compact. Then Theorem 3.1 and Lemma 3.2 together yield that all $L \in \Lambda$ can be assumed to satisfy exactly one of the following:

- (i) with $M_L := M \cap L$, $L = F_L M_L R_L$ is a (pro)-finite 2-Frobenius group, $F_L M_L$ and $M_L R_L$ are profinite Frobenius groups with kernels F_L, M_L respectively;

(we do not exclude that all R_L are trivial, i.e., all L are profinite Frobenius groups).

- (ii) L is a (pro)-finite Frobenius group with $M \cap L$ kernel;
- (iii) $L/(M \cap L)_L$ is a π_1 group, $O_{\pi_1}(L) = F(L)$ and $S_L := (M \cap L)_L/O_{\pi_1}(L)$ is a finite simple group as described in Theorem 3.1(iv)(b);

We discuss the three cases.

Assume that every $L \in \Lambda$ satisfies (i). Then $M \cap L$ is finite, so that M is a discrete subgroup of G . Note that M has locally cyclic Sylows so that M is at most countable. Observe that every F_L is open in G . Therefore, taking Lemma 2.2 into account, i.e., that $F_L \leq F_{L'}$, whenever $L \leq L'$, the subgroup

$$F := \langle F_L \mid L \in \Lambda \rangle$$

coincides with the group generated by all the F_L as an abstract group. Hence

$$F = \bigcup_{\Lambda_0 \subseteq \Lambda, |\Lambda_0| < \infty} \langle F_L \mid L \in \Lambda_0 \rangle,$$

showing that $\pi(F) = \bigcup_{L \in \Lambda} \pi(F_L)$. Therefore $\pi(F) \cap \pi(M) = \emptyset$ and hence $M \cap F = \{1\}$. When all R_L are trivial then G is listed in Theorem 1.3 (i). Moreover, evidently M is isolated. Since all elements in F are p -divisible for all primes in $\pi(M)$, by using Theorem 1 in [10] one finds (i) to be fully established.

When some $R_L \neq \{1\}$, we may assume $R_L \neq \{1\}$ for all $L \in \Lambda$. Since M is locally cyclic, it has a finite automorphism group and so there must be R finite cyclic with R_L isomorphic to a subgroup of R . It is not hard to see that $G = FMR$ is listed in (ii) of Theorem 1.3.

Next suppose that all $L \in \Lambda$ satisfy (ii). It is plain that M is an open normal subgroup of G . Therefore G/M is a locally finite group and $\bar{\Lambda} := \{H_L M/M \mid L \in \Lambda\}$ is a local system for G/M . Then for all $p \in \pi(G/M)$ the p -subgroups of G/M are locally cyclic, in particular they are countable. Therefore G/M is countable and so one can find a countable subset $\{L_i \mid i \in \mathbb{N}\}$ of Λ with $L_i M \leq L_{i+1} M$ for all $i \in \mathbb{N}$. Inductively we shall construct a countable subset $\{L'_i \mid i \in \mathbb{N}\}$ satisfying for all $i \in \mathbb{N}$ the conditions $L'_i \leq L'_{i+1}$ and $L_i \leq L'_i$. Set $L'_1 := L_1$. When $\{L'_j \mid j = 1, \dots, n\}$ have been constructed, let L'_{n+1} be any member of Λ containing both, L'_n and L_{n+1} . Abusing language, we would like to denote all L'_i by L_i again. Then, noting that each $L_i = M_i \rtimes H_i$ is a profinite Frobenius group and taking Lemma 2.2 into account, one finds $M_i \leq M_{i+1}$, and inductively replacing H_{i+1} by a suitable M_{i+1} -conjugate, one can arrange $H_i \leq H_{i+1}$. Then $H := \bigcup_{i \in \mathbb{N}} H_i$ turns out to be a complement to M in G . Apply Theorem 1 of [10], in order to see that G is described in Theorem 1.3 (iii).

We are left with the case when all $L \in \Lambda$ satisfy (iii) in the list. When $x \notin (M)_G$, then for all $L \in \Lambda$ one must have $x \notin (M \cap L)_L$ so that x is a π_1 -element. Therefore $\pi(G/(M)_G) \subseteq \pi_1$. Next let $N \triangleleft (M)_G$ be arbitrary. Then for all $L \in \Lambda$ either $N \cap L = L$ or $N \cap L \triangleleft L$ is nilpotent of bounded class. If N is a proper normal subgroup of $(M)_G$, it is contained in a characteristic nilpotent subgroup $F(G) \leq (M)_G$ (in the finite situation corresponding to the Fitting subgroup).

Then $S := (M)_G/F(G)$ must be a simple group in $[\text{LF}]^- \cap [\text{TD}]$, and therefore Theorem 1.3 (iv) holds. It is not hard to construct an example for each of the possibilities listed in the Theorem. ■

Proof of Corollary 1.4:

Proof. When either (i) or (iii) of Theorem 1.3 hold, G is listed in (i) of the Corollary. When (ii) holds, then M cannot contain an involution and it is nilpotent. Hence (iv) of Theorem 1.3 holds. When Λ is a local system of profinite groups for G with each $H \cap M$ a CC-subgroup of H , one can assume that all $H \cap M$ contain an involution, or all $H \cap M$ are not nilpotent. When N is an open normal subgroup of H with $1 < (M \cap H)N < H$ then by Lemma 2.6, conclude $(M \cap H)N/N$ to be a CC-subgroup of H/N . When N is small enough, either $(M \cap H)N/N$ contains an involution or it is not nilpotent. Therefore Theorem 3.1 shows H/N to be simple. Hence H is finite and hence G must be locally finite. Therefore, when M contains an involution by [4] (ii) of the Corollary must hold. Likewise, when M is not locally nilpotent, (iii) holds. ■

The class $[\text{LF}]^- \cap [\text{TD}]$ does not contain the most simple example of an algebraic Frobenius group, the affine group of the real line, Also to mention that its p -adic analogues are missing in our classifications, as their respective Frobenius-complements are not Hall subgroups. Thus indicating possible extensions of our results, we include a rough description of certain locally compact groups G with $G/G_0 \in [\text{LF}]^-$ (G_0 denoting the connected component of G) having a subgroup M containing the centralizers of its nontrivial elements. Recall from [11] that $G \in [\text{SIN}]$, if in every open neighbourhood of $1 \in G$ there exists a compact normal subgroup K of G with G/K a Lie-group.

Proposition 3.3. *Let $G \in [\text{SIN}]$, $G/G_0 \in [\text{LF}]^-$ and $M \leq G$ contain the centralizers of its nontrivial elements. Suppose $G \notin [\text{LF}]^-$. Then there is a short-exact sequence $1 \rightarrow V \times K \rightarrow G \rightarrow D \rightarrow \{1\}$ with V a vector-group (topologically isomorphic to some \mathbb{R}^n), K compact, D discrete such that the decomposition $G_1 := V \times K$ is G -invariant, and there exists an abelian subgroup of finite index in D . Moreover, one of the following happens:*

- (i) $M \cap G_1 = \{1\}$ and M is finite;
- (ii) $G_1 \leq M$. The set $G \setminus M$ consists of torsion elements only. If $K_0 \neq \{1\}$, then $M \triangleleft G$ and G/M is finite.

Proof. The first statement (about the exact sequence) is precisely Theorem (2.13) in [11]. The second one follows from the fact that the canonical epimorphic image of D in $\text{Aut}(V)$ is a torsion subgroup and hence is finite.

If (i) holds, for every $g_1 \in G_1^\#$ one finds $C_{G_1 M}(g_1) \leq G_1$ so that by Theorem 3 (ii) in [13] M must be finite.

Suppose (i) does not hold. If $g_1 \in M \cap G_1 \neq \{1\}$, then $V \leq C_G(g_1)$ and therefore $V \leq M$. Since $G \notin [\text{LF}]^-$, $V \neq \{1\}$. Then for some $v \in V \setminus \{1\}$, $K \leq C_G(v)$, whence $K \leq M$ and thus $G_1 \leq M$ as stated. Pick any $x \in G \setminus M$, then $\langle x \rangle \cap M = \{1\}$ implies $\langle x \rangle M/M \cong \langle x \rangle$ finite, i.e., x is torsion. Suppose next, $K_0 \neq \{1\}$. Fix a maximal torus $T \leq K_0$ and $x \in G \setminus M$. Then there exists

$k_x \in K_0$ with $T^x = T^{k_x} \leq M^x \cap M^{k_x} = (M \cap M^{xk_x^{-1}})^{k_x}$. As M contains the centralizers of its nontrivial elements, one has $x \in N_G(M)$. The other assertion follows from Theorem 3 (ii) in [13]. ■

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Zvi Arad
Department of Mathematics,
Bar-Ilan University, Ramat-Gan
Department of Computer Science
and Mathematics
and
Netanya Academic College
Netanya, Israel
aradtzvi@macs.biu.ac.il

Wolfgang Herfort
Institute of Analysis
and Scientific Computation
University of Technology
Vienna, Austria
wolfgang.herfort@tuwien.ac.at

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