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Constructing Homomorphisms between Verma Modules

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Abstract. We describe a practical method for constructing a nontrivial homomorphism between two Verma modules of an arbitrary semisimple Lie algebra. With some additions the method generalises to the affine case.
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A theorem of Verma, Bernstein-Gel'fand-Gel'fand gives a straightforward criterion for the existence of a nontrivial homomorphism between Verma modules. Moreover, the theorem states that such homomorphisms are always injective. In this paper we consider the problem of explicitly constructing such a homomorphism if it exists. This boils down to constructing a certain element in the universal enveloping algebra of the negative part of the semismiple Lie algebra.

There are several methods known to solve this problem. Firstly, one can try and find explicit formulas. In this approach one fixes the type (but not the rank). This has been carried out for type A_n in [18], Section 5, and for the similar problem in the quantum group case in [4], [5], [6]. In [4] root systems of all types are considered, and the solution is given relative to so-called straight roots, using a special basis of the universal enveloping algebra (not of Poincaré-Birkhoff-Witt type). In [5], [6] the solution is given for types A_n and D_n for all roots, in a Poincaré-Birkhoff-Witt basis. Our approach compares to this in the sense that we have an algorithm that, given any root of a fixed root system, computes a general formula relative to any given Poincaré-Birkhoff-Witt basis (see Section 3.).

A second approach is described in [18], which gives a general construction of homomorphisms between Verma modules. However, it is not easy to see how to carry out this construction in practice. The method described here is a variant of the construction in [18], the difference being that we are able to obtain the homomorphism explicitly.

In Section 1. of this paper we review the theoretical concepts and notation that we use, and describe the problem we deal with. In Section 2. we derive a few commutation formulas in the field of fractions of $U(\mathfrak{n}^-)$. Then in Section 3. the construction of a homomorphism between Verma modules is described. In Section 4. we briefly comment on the problem of finding compositions of such homomorphisms. In Section 5. we comment on the analogous problem for affine

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algebras, and we show how our algorithm generalises to that case. Finally in Section 6. we give an application of the algorithm to the problem of constructing irreducible modules. This is based on a result by P. Littelmann.

I have implemented the algorithms described in this paper in the computer algebra system GAP4 ([7]). Sections 3. and 6. contain tables of running times. All computations for these have been done on a PII 600 Mhz processor, with 100M of memory of GAP.

1. Preliminaries

Let $\mathfrak{g}$ be a semisimple Lie algebra, with root system Φ, relative to a Cartan subalgebra $\mathfrak{h}$. We let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a fixed set of simple roots. Let $\Phi^+ = \{\alpha_1, \dots, \alpha_s\}$ be the set of positive roots (note that here the simple roots are listed first). Then there are root vectors $y_i = x_{-\alpha_i}$, $x_i = x_{\alpha_i}$ (for $1 \le i \le s$), and basis vectors $h_i \in \mathfrak{h}$ (for $1 \le i \le l$), such that the set $\{x_1, \dots, x_s, y_1, \dots, y_s, h_1, \dots, h_l\}$ forms a Chevalley basis of $\mathfrak{g}$ (cf. [10]). We have that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$, $\mathfrak{n}^+$ are the subalgebras spanned by the y_i, x_i respectively.

In the sequel, if $\beta = \alpha_i \in \Phi^+$, then we also write y_β in place of y_i.

We let P denote the integral weight lattice spanned by the fundamental weights $\lambda_1, \dots, \lambda_l$. Also $\mathbb{Q}P = \mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_l$. For $\lambda, \mu \in \mathbb{Q}P$ we write $\mu \le \lambda$ if $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\ge 0}$. Then $\le$ is a partial order on $\mathbb{Q}P$.

For $\alpha \in \Phi$ we have the reflection $s_\alpha : \mathbb{Q}P \to \mathbb{Q}P$, given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We consider $U(\mathfrak{g})$ as a $\mathfrak{g}$-module by left multiplication. Let $\lambda = \sum a_i \lambda_i \in \mathbb{Q}P$, and let $J(\lambda)$ be the $\mathfrak{g}$-submodule of $U(\mathfrak{g})$ generated by $h_i - a_i + 1$ for $1 \le i \le l$ and x_i for $1 \le i \le s$. Then $M(\lambda) = U(\mathfrak{g})/J(\lambda)$ is a $\mathfrak{g}$-module. It is called a Verma module. As $U(\mathfrak{g}) = U(\mathfrak{n}^-) \oplus J(\lambda)$ we see that $U(\mathfrak{n}^-) \cong M(\lambda)$ (as $U(\mathfrak{n}^-)$-modules). Let v_λ denote the image of 1 under this isomorphism. Then $h_i \cdot v_\lambda = (a_i - 1)v_\lambda$, and $x_i \cdot v_\lambda = 0$. Furthermore, all other elements of $M(\lambda)$ can be written as $Y \cdot v_\lambda$, where $Y \in U(\mathfrak{n}^-)$.

Let $\nu = \sum_{i=1}^{l} k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\ge 0}$. Then we let $U(\mathfrak{n}^-)_\nu$ be the span of all $y_{i_1} \cdots y_{i_r}$ such that $\alpha_{i_1} + \cdots + \alpha_{i_r} = \nu$.

For a proof of the following theorem we refer to [1], [3].

Theorem 1.1. (Verma, Bernstein-Gel'fand-Gel'fand) *Let $\lambda, \mu \in \mathbb{Q}P$, and set*

$$R_{\mu,\lambda} = \mathrm{Hom}_{U(\mathfrak{g})}(M(\mu), M(\lambda)).$$

Then

1. $\dim R_{\mu,\lambda} \le 1$,
2. *non-trivial elements of $R_{\mu,\lambda}$ are injective,*
3. $\dim R_{\mu,\lambda} = 1$ *if and only if there are positive roots $\alpha_{i_1}, \dots, \alpha_{i_k}$ such that*

$$\mu \le s_{\alpha_{i_1}}(\mu) \le s_{\alpha_{i_2}} s_{\alpha_{i_1}}(\mu) \le \cdots \le s_{\alpha_{i_k}} \cdots s_{\alpha_{i_1}}(\mu) = \lambda.$$

The problem we consider is to construct a non-trivial element in $R_{\mu,\lambda}$ if $\dim R_{\mu,\lambda} = 1$. By Theorem 1.1, this boils down to finding an element in $R_{\mu,\lambda}$ if $\mu = s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ and $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. Suppose that we are in this situation, and set $h = \langle \lambda, \alpha^\vee \rangle$. An element $Y \cdot v_\lambda \in M(\lambda)$, where $Y \in U(\mathfrak{n}^-)$ is said to be *singular* if $x_\alpha \cdot (Y \cdot v_\lambda) = 0$ for $\alpha \in \Phi^+$. Let $\psi \in R_{\mu,\lambda}$ be a non-trivial $U(\mathfrak{g})$-homomorphism. Then $\psi(v_\mu) = Y \cdot v_\lambda$ for some $Y \in U(\mathfrak{n}^-)$ with $Y \cdot v_\lambda$ singular. We have $h_i y = y h_i - \langle \nu, \alpha_i^\vee \rangle y$ for all $y \in U(\mathfrak{n}^-)_\nu$. Hence $h_i \cdot (y \cdot v_\lambda) = (\langle \lambda - \nu, \alpha_i^\vee \rangle - 1) y v_\lambda$. So, as $h_i \cdot (Y v_\lambda) = (\langle \mu, \alpha_i^\vee \rangle - 1) Y v_\lambda$ we see that $Y \in U(\mathfrak{n}^-)_{h\alpha}$. Conversely, if we have a $Y \in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y \cdot v_\lambda$ is singular, then $\psi : M(\mu) \to M(\lambda)$ defined by $\psi(Y' \cdot v_\mu) = Y'Y \cdot v_\lambda$ will be a non-trivial element of $R_{\mu,\lambda}$. So the problem reduces to finding a $Y \in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y \cdot v_\lambda$ is singular. Note that this can be done by writing down a basis for $U(\mathfrak{n}^-)_{h\alpha}$ and computing a set of linear equations for Y. However, this algorithm becomes rather cumbersome if $\dim U(\mathfrak{n}^-)_{h\alpha}$ gets large. We will describe a more direct method.

2. The field of fractions

From [3], §3.6 we recall that $U(\mathfrak{n}^-)$ has a (non-commutative) field of fractions, denoted by $K(\mathfrak{n}^-)$. It consists of all elements ab^{-1} for $a \in U(\mathfrak{n}^-)$, $b \in U(\mathfrak{n}^-) \setminus \{0\}$. For the definitions of addition and multiplication in $K(\mathfrak{n}^-)$ we refer to [3], §3.6. They imply $aa^{-1} = a^{-1}a = 1$.

Let $\alpha, \beta \in \Phi^+$. If $\alpha + \beta \in \Phi^+$ then we let $N_{\alpha,\beta}$ be the scalar such that $[y_\alpha, y_\beta] = -N_{\alpha,\beta} y_{\alpha+\beta}$. Also set $P_{\alpha,\beta} = \{ i\alpha + j\beta \mid i, j \geq 0 \} \cap \Phi^+$. Then there are seven possibilities for $P_{\alpha,\beta}$:

(I) $P_{\alpha,\beta} = \{\alpha, \beta\}$,

(II) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta\}$

(III) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta\}$,

(IV) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$,

(V) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$

(VI) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$,

(VII) $P_{\alpha,\beta} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, \alpha+2\beta\}$.

Lemma 2.1. *In case (I) we have $y_\beta^m y_\alpha^n = y_\alpha^n y_\beta^m$ for all $m, n \in \mathbb{Z}$.*

Proof. If $n > 0$ then $y_\beta y_\alpha^n = y_\alpha^n y_\beta$. Multiplying this relation on the left and on the right by y_α^{-n} we get $y_\beta y_\alpha^{-n} = y_\alpha^{-n} y_\beta$. So we have $y_\beta y_\alpha^n = y_\alpha^n y_\beta$ for all $n \in \mathbb{Z}$. From this it follows that $y_\beta^m y_\alpha^n = y_\alpha^n y_\beta^m$ for $m > 0$, $n \in \mathbb{Z}$. If we now multiply this from the left and the right by y_β^{-m} we get the result for $m < 0$ as well. ■

Since

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},$$

these binomial coefficients are defined for arbitrary $n \in \mathbb{Q}$, and $k \in \mathbb{Z}_{\geq 0}$. In fact, we see that $\binom{n}{k}$ is a polynomial of degree k in n. Note also that if $n \in \mathbb{Z}$ and $0 \leq n < k$ then the coefficient is 0.

Lemma 2.2. *In case (II) we have for $m \geq 0$, $n \in \mathbb{Z}$,*

$$y_\beta^m y_\alpha^n = \sum_{k=0}^{m} N_{\alpha,\beta}^k \binom{m}{k}\binom{n}{k} k! y_\alpha^{n-k} y_\beta^{m-k} y_{\alpha+\beta}^k.$$

Proof. First of all, this formula is known for $m,n \geq 0$ (see, e.g., [9]). In particular, for $n > 0$ we have $y_\beta y_\alpha^n = y_\alpha^n y_\beta + N_{\alpha,\beta} n y_\alpha^{n-1} y_{\alpha+\beta}$. If we multiply this relation on the left and the right with y_α^{-n}, and use Lemma 2.1, then we get it for all $n \in \mathbb{Z}$. Now the formula for $m > 1$ is proved by induction. ∎

Lemma 2.3. *In case (III) we have for $m \geq 0$, $n \in \mathbb{Z}$,*

$$y_\beta^m y_\alpha^n = \sum_{\substack{k,l \geq 0 \\ k+2l \leq m}} c_{k,l}^{m,n} y_\alpha^{n-k-l} y_\beta^{m-k-2l} y_{\alpha+\beta}^k y_{\alpha+2\beta}^l,$$

where

$$c_{k,l}^{m,n} = N_{\alpha,\beta}^{k+l} \Big(\frac{1}{2} N_{\beta,\alpha+\beta}\Big)^l \binom{n}{k+l}\binom{m}{k+2l}\binom{k+l}{l}(k+2l)!.$$

Proof. This goes in exactly the same way as the proof of Lemma 2.2. ∎

Lemma 2.4. *In case (IV) we have for $m \geq 0$, $n \in \mathbb{Z}$,*

$$y_\beta^m y_\alpha^n = \sum_{\substack{k,l \geq 0 \\ k+l \leq m}} c_{k,l}^{m,n} y_\alpha^{n-k-2l} y_\beta^{m-k-l} y_{\alpha+\beta}^k y_{2\alpha+\beta}^l,$$

where

$$c_{k,l}^{m,n} = N_{\alpha,\beta}^{k+l} \Big(\frac{1}{2} N_{\alpha,\alpha+\beta}\Big)^l \binom{n}{k+2l}\binom{m}{k+l}\binom{k+l}{l}(k+2l)!.$$

Proof. Again we get the formula for $m,n \geq 0$ from [9]. In this case the formula for $m = 1$, $n \geq 0$ reads

$$y_\beta y_\alpha^n = y_\alpha^n y_\beta + N_{\alpha,\beta} n y_\alpha^{n-1} y_{\alpha+\beta} + N_{\alpha,\beta} N_{\alpha,\alpha+\beta} \binom{n}{2} y_\alpha^{n-2} y_{2\alpha+\beta}.$$

If we multiply this on the left and the right by y_α^{-n}, and use Lemmas 2.1, 2.2 we get the same relation with n replaced by $-n$. So the case $m = 1$, $n \in \mathbb{Z}$ follows. The formula for $m > 1$ now follows by induction. ∎

The cases (V), (VI), (VII) can only occur when the root system has a component of type G_2. We omit the formulas for these cases; they can easily be derived from those given in [9].

Now let $a = y_1^{n_1} \cdots y_s^{n_s}$ be a monomial in $U(\mathfrak{n}^-)$. For $\beta \in \Phi^+$ and $m,n \in \mathbb{Z}$ consider the element $y_\beta^m a y_\beta^{-n}$. By repeatedly applying Lemmas 2.1, 2.2, 2.3, 2.4 we see that

$$y_\beta^m a y_\beta^{-n} = \sum_{(k_1,\dots,k_t) \in I} c(k_1,\dots,k_t) y_\beta^{m-n-p_1k_1-\cdots-p_tk_t} a(k_1,\dots,k_t). \tag{1}$$

Here the $a(k_1,\dots,k_t)\in U(\mathfrak{n}^-)$, the (finite) index set I, the $p_i\in\mathbb{Z}_{>0}$ are all independent of n, they only depend on a. Only the exponents of y_β and the coefficients $c(k_1,\dots,k_t)$ (which are polynomials in n) depend on n.

Now we take $m,n\in\mathbb{Q}$ such that $m-n\in\mathbb{Z}$. Then we define $y_\beta^m a y_\beta^{-n}$ to be the right-hand side of (1), and we say that $y_\beta^m a y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$. More generally, if Y is a linear combination of monomials, and $m,n\in\mathbb{Q}$ such that $m-n\in\mathbb{Z}$ then $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$.

3. Constructing singular vectors

Here we suppose that we are given a $\lambda\in\mathbb{Q}P$ and $\alpha\in\Phi^+$ with $\langle\lambda,\alpha^\vee\rangle=h\in\mathbb{Z}_{>0}$. The problem is to find a $Y\in U(\mathfrak{n}^-)_{h\alpha}$ such that $Y\cdot v_\lambda$ is a singular vector.

We recall that $l=|\Delta|$ is the rank of the root system. Let $1\le i\le l$, then

$$x_i y_i^r\cdot v_\lambda = r(\langle\lambda,\alpha_i^\vee\rangle - r)y_i^{r-1}\cdot v_\lambda. \tag{2}$$

Lemma 3.1. *Suppose that $\alpha\in\Delta$, i.e., $\alpha=\alpha_i$, $1\le i\le l$. Then $y_i^h\cdot v_\lambda$ is a singular vector.*

Proof. This follows from (2), cf. the proof of [1], Lemma 2. ∎

Note that this solves the problem when $\mathfrak{g}=\mathfrak{sl}_2$. So in the remainder we will assume that the rank of the root system is at least 2. By an embedding $\phi: M(\mu)\hookrightarrow M(\lambda)$ we will always mean an injective $U(\mathfrak{g})$-homomorphism.

Lemma 3.2. *Suppose that $\nu,\eta\in P$, and $\beta\in\Delta$ is such that $m=\langle\nu,\beta^\vee\rangle$ is a non-negative integer. Suppose further that we have an embedding $\psi: M(\nu)\hookrightarrow M(\eta)$ given by $\psi(v_\nu)=Yv_\eta$. Set $n=\langle\eta,\beta^\vee\rangle$. Then $y_\beta^m Y y_\beta^{-n}$ is an element of $U(\mathfrak{n}^-)$ and we have an embedding $\phi: M(s_\beta\nu)\hookrightarrow M(s_\beta\eta)$ given by $\phi(v_{s_\beta\nu}) = y_\beta^m Y y_\beta^{-n}\cdot v_{s_\beta\eta}$.*

Proof. If $n\le 0$ then the first statement is clear. The embedding ϕ is the composition $M(s_\beta\nu)\hookrightarrow M(\nu)\hookrightarrow M(\eta)\hookrightarrow M(s_\beta\eta)$, where the first and the third maps follow from Lemma 3.1.

If $n>0$, then we view $M(s_\beta\eta)$ as a submodule of $M(\eta)$. We have $v_{s_\beta\eta}=y_\beta^n v_\eta$ (Lemma 3.1). Set $v=y_\beta^m Y v_\eta$; then v is a singular vector (being the image of $v_{s_\beta\nu}$ under $M(s_\beta\nu)\hookrightarrow M(\nu)\hookrightarrow M(\eta)$). We claim that $v\in M(s_\beta\eta)$. Suppose that this claim is proved. Then there is a $Y'\in U(\mathfrak{n}^-)$ such that $v=Y'v_{s_\beta\eta}$. But that means that $y_\beta^m Y = Y' y_\beta^n$, and the lemma follows.

The claim above is proved in [1]. For the sake of completeness we transcribe the argument. Set $V=M(\eta)/M(s_\beta\eta)$, and let $\bar v_\nu$ denote the image of $\psi(v_\nu)$ in V; then $\bar v_\nu = X\cdot\bar v_\eta$, for some $X\in U(\mathfrak{n}^-)$. For $k\ge 0$ write $y_\beta^k X = X_1 y_\beta^{k_1}$. By increasing k we can get k_1 arbitrarily large (cf. [3], Lemma 7.6.9; it also follows by straightforward weight considerations). By Lemma 3.1 we know that $y_\beta^n v_\eta\in M(s_\beta\eta)$. Therefore there is a $k>0$ such that $y_\beta^k\bar v_\nu=0$. Then by using (2) we see that the smallest such k must be equal to m. ∎

Proposition 3.3. *Let $\nu, \eta \in \mathbb{Q}P$ be such that $\nu = s_\gamma(\eta) = \eta - k\gamma$, where $\gamma \in \Phi^+$ and $k \in \mathbb{Z}_{>0}$. Let $Y \in U(\mathfrak{n}^-)_{k\gamma}$ be such that $Y \cdot v_\eta$ is singular. Let $\beta \in \Delta$, $\beta \neq \gamma$. Set $m = \langle \nu, \beta^\vee \rangle$, $n = \langle \eta, \beta^\vee \rangle$. Then $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$; it is even an element of $U(\mathfrak{n}^-)$. Secondly, we have an embedding $\phi : M(s_\beta \nu) \hookrightarrow M(s_\beta \eta)$ given by $\phi(v_{s_\beta \nu}) = y_\beta^m Y y_\beta^{-n} \cdot v_{s_\beta \eta}$.*

Proof. We have that $m - n = -k\langle \gamma, \beta^\vee \rangle \in \mathbb{Z}$, so $y_\beta^m Y y_\beta^{-n}$ is an element of $K(\mathfrak{n}^-)$.

Set $V = \{\mu \in \mathbb{Q}P \mid \langle \mu, \gamma^\vee \rangle = k\}$, which is a hyperplane in $\mathbb{Q}P$, containing η. Let $\{a_1, \ldots, a_t\}$ be a basis of $U(\mathfrak{n}^-)_{k\gamma}$. Take $\mu = \sum_{i=1}^l r_i \lambda_i \in V$ and set $\tilde{\mu} = s_\gamma(\mu) = \mu - k\gamma$. Then by Theorem 1.1 there is a $Y_\mu = \sum_{i=1}^t \zeta_i a_i$ such that $Y_\mu \cdot v_\mu$ is singular. Here the ζ_i are polynomial functions of the r_i. (Indeed, the ζ_i form a solution to a set of linear homogeneous equations. The coefficients of these equations depend linearly on the r_i. Hence the coefficients of a solution are polynomial functions of the r_i.)

Set $p = \langle \tilde{\mu}, \beta^\vee \rangle$, $q = \langle \mu, \beta^\vee \rangle$. Then $Y' = y_\beta^p Y_\mu y_\beta^{-q} = \sum_j c_j b_j$, where the b_j are linearly independent elements of $K(\mathfrak{n}^-)$, and the c_j are coefficients that depend polynomially on the r_i. Now Lemma 3.2 implies that if the $r_i \in \mathbb{Z}$ and $p \geq 0$, then $Y' \in U(\mathfrak{n}^-)$. Let now j be such that $b_j \notin U(\mathfrak{n}^-)$. If the $r_i \in \mathbb{Z}$ and $p \geq 0$, then $c_j = 0$. Suppose that $\beta = \alpha_{i_0}$, the i_0-th simple root. Then the condition $p \geq 0$ amounts to $r_{i_0} \geq k\langle \gamma, \beta^\vee \rangle$. We have that $\mu \in V$ if and only if $\sum_{i=1}^l u_i r_i = k$, where the u_i are certain elements of $\mathbb{Z}$. Also, since $\beta \neq \gamma$ at least one $u_i \neq 0$ with $i \neq i_0$. We see that the requirement $r_{i_0} \geq k\langle \gamma, \beta^\vee \rangle$ cuts a half space W off V. Furthermore $V \cap P$ is an $(l-1)$-dimensional lattice in V (cf. [1]). The conclusion is that $c_j = 0$ if $\mu \in W \cap P$. Since the c_j are polynomials in the r_i, it follows that $c_j = 0$ if $\mu \in V$. In particular, $y_\beta^m Y y_\beta^{-n}$ lies in $U(\mathfrak{n}^-)$.

Finally we note that $Y' \cdot v_{s_\beta \mu}$ is singular, by the same arguments. (Indeed, $x_i \cdot (Y' \cdot v_{s_\beta \mu}) = \sum_j f_j z_j \cdot v_{s_\beta \mu}$ where the f_j are polynomials in the r_i, and the z_j are elements of $U(\mathfrak{n}^-)$. Since the f_j are zero when $\mu \in W \cap P$ we have that $f_j = 0$ when $\mu \in V$.) In particular, $y_\beta^m Y y_\beta^{-n} \cdot v_{s_\beta \eta}$ is singular. ■

Example 3.4. To illustrate the argument in the preceding proof, consider the Lie algebra of type A_3, with simple roots α, β, γ (with β corresponding to the middle node of the Dynkin diagram). Then it is possible to choose a Chevalley basis such that $[y_\alpha, y_\beta] = y_{\alpha+\beta}$, $[y_\alpha, y_{\beta+\gamma}] = y_{\alpha+\beta+\gamma}$, $[y_\beta, y_\gamma] = y_{\beta+\gamma}$, $[y_\gamma, y_{\alpha+\beta}] = -y_{\alpha+\beta+\gamma}$. Set $a_1 = y_\alpha y_\beta y_\gamma$, $a_2 = y_\gamma y_{\alpha+\beta}$, $a_3 = y_\alpha y_{\beta+\gamma}$, $a_4 = y_{\alpha+\beta+\gamma}$. Then $\{a_1, a_2, a_3, a_4\}$ is a basis of $U(\mathfrak{n}^-)_{\alpha+\beta+\gamma}$.

We abbreviate a weight $r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3$ by (r_1, r_2, r_3). Let V be the hyperplane in $\mathbb{Q}P$ consisting of all weights μ such that $\langle \mu, (\alpha+\beta+\gamma)^\vee \rangle = 1$, i.e., $V = \{(r_1, r_2, r_3) \mid r_1 + r_2 + r_3 = 1\}$. Let $\mu = (r_1, r_2, r_3) \in V$ and set $\tilde{\mu} = s_{\alpha+\beta+\gamma}(\mu) = (r_1 - 1, r_2, r_3 - 1)$. Set $Y_\mu = a_1 - r_1 a_2 - (r_1 + r_2)a_3 - r_1 r_3 a_4$; then $Y_\mu \cdot v_\mu$ is singular. Set $p = \langle \tilde{\mu}, \alpha^\vee \rangle = r_1 - 1$ and $q = \langle \mu, \alpha^\vee \rangle = r_1$. Now $Y' = y_\alpha^p Y_\mu y_\alpha^{-q} = y_\beta y_\gamma - (r_1 + r_2) y_{\beta+\gamma} + r_1(1 - r_1 - r_2 - r_3) y_\alpha^{-1} y_{\alpha+\beta+\gamma}$. According to Lemma 3.2 this is an element of $U(\mathfrak{n}^-)$ whenever $(r_1, r_2, r_3) \in V$ with the r_i integral and $p \geq 0$. Therefore the coefficient of $y_\alpha^{-1} y_{\alpha+\beta+\gamma}$ has to vanish, which is indeed the case. We see that Y' lies in $U(\mathfrak{n}^-)$ for all $(r_1, r_2, r_3) \in V$.

Now we return to the situation of the beginning of the section. We have

$\lambda \in \mathbb{Q}P$, $\alpha \in \Phi^+$ with $\langle \lambda, \alpha^\vee \rangle = h \in \mathbb{Z}_{>0}$. Set $\mu = s_\alpha(\lambda) = \lambda - h\alpha$. To obtain an embedding $M(\mu) \hookrightarrow M(\lambda)$, we perform the following steps:

1. Select $\beta_1, \ldots, \beta_r \in \Delta$ and positive roots $\alpha_0, \ldots, \alpha_r$ in the following way. Set $\alpha_0 = \alpha$, and $k = 0$. Then:
   (a) If $\alpha_k \in \Delta$, then set $r = k$ and go to step 2.
   (b) Otherwise, let $\beta_{k+1} \in \Delta$ be such that $\langle \alpha_k, \beta_{k+1}^\vee \rangle > 0$, and set $\alpha_{k+1} = s_{\beta_{k+1}}(\alpha_k)$, and $k := k+1$. Return to (a).
2. Set $\beta = \alpha_r \in \Delta$. For $1 \leq k \leq r$ set $a_k = -\langle \mu, s_{\beta_1} \cdots s_{\beta_{k-1}}(\beta_k)^\vee \rangle$, and $b_k = \langle \lambda, s_{\beta_1} \cdots s_{\beta_{k-1}}(\beta_k)^\vee \rangle$.
3. Set $Y_0 = y_\beta^h$, and for $0 \leq k \leq r-1$:

$$Y_{k+1} = y_{\beta_{r-k}}^{a_{r-k}} Y_k y_{\beta_{r-k}}^{b_{r-k}}.$$

Remark 3.5. Note that the β_{k+1} in step 1 (b) exists as otherwise $\langle \alpha_k, \gamma^\vee \rangle \leq 0$ for all $\gamma \in \Delta$, and this implies that the set $\Delta \cup \{\alpha_k\}$ is linearly independent (cf. [11], Chapter IV, Lemma 1), which is not possible since $\alpha_k \notin \Delta$. Also, all α_k must be positive roots because s_γ permutes the positive roots other than γ, for $\gamma \in \Delta$. Then the loop in 1. must terminate because the height of α_k decreases every step.

Proposition 3.6. *All Y_k are elements of $U(\mathfrak{n}^-)$ and we have an embedding $M(\mu) \hookrightarrow M(\lambda)$ given by $v_\mu \mapsto Y_r \cdot v_\lambda$.*

Proof. We write $s_i = s_{\beta_i}$. For $0 \leq k \leq r$ we set $w_k = s_{r-k} \cdots s_1$ (so $w_r = 1$), and $\mu_k = w_k\mu$, $\lambda_k = w_k\lambda$. We claim that there is an embedding $M(\mu_k) \hookrightarrow M(\lambda_k)$ given by $v_{\mu_k} \mapsto Y_k \cdot v_{\lambda_k}$. First we look at the case $k = 0$. Note that $s_r \cdots s_1(\alpha) = \beta \in \Delta$. Since for w in the Weyl group we have $ws_\beta w^{-1} = s_{w\beta}$ we get $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1 = w_0^{-1} s_\beta w_0$. Therefore $\mu_0 = w_0 s_\alpha(\lambda) = s_\beta(\lambda_0)$, and $\langle \lambda_0, \beta^\vee \rangle = \langle \lambda, s_1 \cdots s_r(\beta)^\vee \rangle = \langle \lambda, \alpha^\vee \rangle = h$. The case $k = 0$ now follows by Lemma 3.1.

Now suppose we have an embedding $M(\mu_k) \hookrightarrow M(\lambda_k)$ as above. Note that $w_{k+1} = s_{r-k} w_k$ and $\alpha_k = w_k\alpha$. Also $\mu_k = w_k\mu = \lambda_k - h\alpha_k$, and $\langle \lambda_k, \alpha_k^\vee \rangle = h$, so that $\mu_k = s_{\alpha_k}(\lambda_k)$. We now apply Proposition 3.3 (with $\nu := \mu_k$, $\eta := \lambda_k$, $\beta := \beta_{r-k}$). We have $\beta_{r-k} \in \Delta$ and $\beta_{r-k} \neq \alpha_k$ as $\alpha_k \notin \Delta$. Furthermore, $m = \langle s_{r-k} \cdots s_1 \mu, \beta_{r-k}^\vee \rangle = -\langle \mu, s_1 \cdots s_{r-k-1}(\beta_{r-k})^\vee \rangle = a_{r-k}$. In the same way $n = \langle \lambda_k, \beta_{r-k}^\vee \rangle = -b_{r-k}$. So by Proposition 3.3, if we set

$$Y_{k+1} = y_{\beta_{r-k}}^{a_{r-k}} Y_k y_{\beta_{r-k}}^{b_{r-k}},$$

then we have an embedding $M(\mu_{k+1}) = M(s_{\beta_{r-k}}\mu_k) \hookrightarrow M(s_{\beta_{r-k}}\lambda_k) = M(\lambda_{k+1})$ by $v_{\mu_{k+1}} \mapsto Y_{k+1} \cdot v_{\lambda_{k+1}}$.

Finally we note that $\lambda_r = \lambda$, $\mu_r = \mu$. ■

It is possible to reformulate the algorithm in such a way that it looks more like the method from [18]. The construction described in [18] works as follows. Write $s_\alpha = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_t}}$, as a product of simple reflections. For $1 \leq k \leq t$ set $m_k = \langle s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_t}} \lambda, \alpha_{i_k}^\vee \rangle$. Then $Y = y_{i_1}^{m_1} \cdots y_{i_t}^{m_t}$ is an element of $U(\mathfrak{n}^-)$ and we have an embedding $M(\mu) \hookrightarrow M(\lambda)$ by $v_\mu \mapsto Y \cdot v_\lambda$. Now, using the same notation as in the description of the algorithm, the expression we get is

$$y_{\beta_1}^{a_1} \cdots y_{\beta_r}^{a_r} y_\beta^h y_{\beta_r}^{b_r} \cdots y_{\beta_1}^{b_1}.$$

As remarked in the proof of Proposition 3.6, $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1$ (where again we write $s_i = s_{\beta_i}$). Furthermore, $b_k = \langle s_{k-1} \cdots s_1 \lambda, \beta_k^\vee \rangle$, $h = \langle \lambda, \alpha^\vee \rangle = \langle s_r \cdots s_1 \lambda, \beta^\vee \rangle$, $a_k = \langle s_{k+1} \cdots s_r s_\beta s_r \cdots s_1 \lambda, \beta_k^\vee \rangle$. So we see that our method is a special case of the construction in [18]. However, the difference is that we have an explicit method to rewrite the element above to an element of $U(\mathfrak{n}^-)$. By the next lemma the expression we use for s_α is the shortest possible (so we cannot do essentially better by taking a different reduced expression).

Lemma 3.7. *The expression $s_\alpha = s_1 \cdots s_r s_\beta s_r \cdots s_1$ obtained by the first step of the algorithm, is reduced.*

Proof. Set $\gamma = s_1(\alpha) = \alpha - m\beta_1$, where $m > 0$. Then $s_\alpha = s_{s_1(\gamma)} = s_1 s_\gamma s_1$. By induction, the expression $s_\gamma = s_2 \cdots s_r s_\beta s_r \cdots s_2$ is reduced. We show that $\ell(s_\alpha) = \ell(s_\gamma) + 2$. For this we use the fact that the length of an element w of the Weyl group is equal to the number of positive roots that are mapped to negative roots by w. Write $\Phi^+ = A \cup \{\beta_1\}$, where $A = \Phi^+ \setminus \{\beta_1\}$. There is a positive root $\delta_0 \in \Phi$ with $s_\gamma s_1 \delta_0 = \beta_1$. Set $S = \{\delta \in A \mid s_\gamma s_1 \delta < 0\} \cup \{\delta_0, \beta_1\}$. Then s_α maps all elements of S to negative roots. Since $\langle \gamma, \beta_1^\vee \rangle = -m < 0$, also $\langle \beta_1, \gamma^\vee \rangle < 0$, and hence $s_\gamma(\beta_1) > 0$. So all roots that are mapped to negative roots by s_γ are in A. Therefore, since s_1 permutes A, there are $\ell(\gamma)$ roots $\delta \in A$ with $s_\gamma s_1(\delta) < 0$. We conclude that the cardinality of S is $\ell(\gamma) + 2$. So $\ell(\alpha) \geq \ell(\gamma) + 2$, but that means that $\ell(\alpha) = \ell(\gamma) + 2$. ∎

We can use the algorithm described in this section to construct general formulas for singular elements. More precisely, let γ be a fixed root in the root system of $\mathfrak{g}$. Then by applying the formulas of Section 2. symbolically we can derive a formula that given arbitrary weights λ, μ such that $\langle \lambda, \gamma^\vee \rangle \in \mathbb{Z}_{>0}$ and $\mu = s_\gamma(\lambda)$ produces an element $Y \in U(\mathfrak{n}^-)_{\lambda-\mu}$ such that $Y \cdot v_\lambda$ is singular. We illustrate this with an example.

Example 3.8. Suppose that $\mathfrak{g}$ is of type A_3. We use the same basis of $\mathfrak{n}^-$ as in Example 3.4. We consider the root $\alpha + \beta + \gamma$. Let $\lambda = (r_1, r_2, r_3)$ be such that $h = r_1 + r_2 + r_3$ is a positive integer. A reduced expression of $s_{\alpha+\beta+\gamma}$ is $s_\alpha s_\beta s_\gamma s_\beta s_\alpha$. The corresponding element of $U(\mathfrak{n}^-)$ is

$$Y = y_\alpha^{r_2+r_3} y_\beta^{r_3} y_\gamma^h y_\beta^{r_1+r_2} y_\alpha^{r_1}.$$

First we have

$$y_\beta^{r_3} y_\gamma^h y_\beta^{r_1+r_2} = \sum_{k=0}^{h} (-1)^k \binom{h}{k} \binom{r_1+r_2}{k} k! y_\beta^{h-k} y_\gamma^{h-k} y_{\beta+\gamma}^k.$$

Now to obtain the formula for Y we have to apply Lemma 2.2 three times (and Lemma 2.1 a few times). We get

$$Y = \sum_{k=0}^{h}\sum_{l=0}^{k}\sum_{s=0}^{h-k}\sum_{t=0}^{s}(-1)^{k+l+s}\binom{h}{k}\binom{r_1+r_2}{k}\binom{k}{l}\binom{r_1}{l}\binom{h-k}{s}\binom{r_1-l}{s}$$
$$\binom{s}{t}\binom{h-k}{t}k!l!s!t!y_\alpha^{h-l-s}y_\beta^{h-k-s}y_\gamma^{h-k-t}y_{\alpha+\beta}^{s-t}y_{\beta+\gamma}^{k-l}y_{\alpha+\beta+\gamma}^{l+t}.$$

Table 1 contains a few running times of the implementation of this algorithm in GAP4. The root γ is in each case the highest root of the root system. The

type	length	time (s)
A_6	29	0.2
D_6	109	1.6
E_6	316	2.9
E_7	2866	26.3
E_8	> 10556	∞

Table 1: Running times for the computation of a formula for a singular vector.

length of a formula is the number of summations it contains (so the length of the above formula for A_3 is 4). The computation for E_8 did not terminate in the available amount of memory (100M). When the program exceeded the memory, the expression contained 10556 summations.

Remark 3.9. It is also possible to use this method to obtain formulas for a fixed type, but variable rank. However, for that a convenient Chevalley basis needs to be chosen. We refer to [18], Section 5, for the formula for A_n.

Remark 3.10. We have chosen $\mathbb{Q}$ as the ground field, because it is easy to work with. However, from the algorithm it is clear that instead we can choose any field F of characteristic zero and construct embeddings of Verma modules with highest weights from FP.

4. Composition of embeddings

In this section we consider the problem of obtaining an embedding $M(\nu) \hookrightarrow M(\lambda)$, where $\nu = s_\alpha s_\beta(\lambda) < s_\beta(\lambda) < \lambda$. The obvious way of doing this is to set $\mu = s_\beta(\lambda)$ and obtain the embeddings $M(\nu) \hookrightarrow M(\mu)$, $M(\mu) \hookrightarrow M(\lambda)$ and composing them. This amounts to multiplying two elements of $U(\mathfrak{n}^-)$. This then corresponds to an expression for $s_\alpha s_\beta$, which is not necessarily reduced. The question arises whether in this case it is possible to do better, i.e., to start with a reduced expression for $s_\alpha s_\beta = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$, set $m_k = \langle s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_t}}\lambda, \alpha_{i_k}^\vee\rangle$, and rewrite $Y = y_{i_1}^{m_1} \cdots y_{i_t}^{m_t}$ to an element of $U(\mathfrak{n}^-)$. The next example shows that this does not always work.

Example 4.1. Let Φ be of type F_4, with simple roots $\alpha_1, \ldots, \alpha_4$ and Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3$ and $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$. We abbreviate a weight $a_1\lambda_1 + \cdots + a_4\lambda_4$ by (a_1, a_2, a_3, a_4). Set $\lambda = (\frac{5}{6}, -\frac{1}{2}, \frac{2}{3}, 0)$ and $\nu = (-\frac{1}{6}, -\frac{1}{2}, -\frac{1}{3}, 2)$. Then $\nu = s_\alpha s_\beta(\lambda)$. Write $s_i = s_{\alpha_i}$. Then a reduced expression of $s_\alpha s_\beta$ is

$$s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 s_3 s_2.$$

We get

$$Y = y_1^{\frac{1}{6}} y_2^{\frac{2}{3}} y_1^{\frac{1}{2}} y_3^{\frac{5}{3}} y_2^{\frac{3}{2}} y_1 y_3^{\frac{4}{3}} y_2^{\frac{5}{6}} y_4 y_3^{\frac{4}{3}} y_2^{\frac{1}{2}} y_1^{\frac{1}{3}} y_3^{-\frac{1}{3}} y_2^{-\frac{1}{2}}.$$

And I do not see any direct way to rewrite this as an element of $U(\mathfrak{n}^-)$.

In general we have to obtain the embedding by composition. In this example set $\mu = s_\beta(\lambda) = \lambda - \beta = (\frac{5}{6}, -\frac{3}{2}, \frac{5}{3}, 0)$. Then for the embedding $M(\mu) \hookrightarrow M(\lambda)$ we get

$$Y_1 = y_2^{\frac{2}{3}} y_3^{\frac{4}{3}} y_1^{\frac{2}{3}} y_2^{\frac{1}{2}} y_3^{-\frac{1}{3}} y_4 y_3^{\frac{4}{3}} y_2^{\frac{1}{2}} y_1^{\frac{1}{3}} y_3^{-\frac{1}{3}} y_2^{-\frac{1}{2}}.$$

For the embedding $M(\nu) \hookrightarrow M(\mu)$ we get

$$Y_2 = y_1^{\frac{1}{6}} y_3^{\frac{1}{3}} y_2 y_3^{\frac{5}{3}} y_1^{\frac{5}{6}}.$$

Then the product $Y_2 Y_1$ will provide the embedding $M(\nu) \hookrightarrow M(\lambda)$.

5. Affine algebras

In this section we comment on finding embeddings of Verma modules of affine Kac-Moody algebras. First we fix some notation and recall some facts. Our main reference for this is [12].

We let $\hat{\mathfrak{g}}$ be the (untwisted) affine Lie algebra corresponding to $\mathfrak{g}$, i.e.,

$$\hat{\mathfrak{g}} = \mathbb{Q}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{Q}K \oplus \mathbb{Q}d$$

with multiplication

$$\begin{aligned}[t^m \otimes x + a_1 K + b_1 d, t^n \otimes y + a_2 K + b_2 d] =& t^{m+n} \otimes [x, y] + b_1 n t^n \otimes y - b_2 m t^m \otimes x \\ &+ m\delta_{m,-n}\kappa(x, y)K,\end{aligned}$$

where $m, n \in \mathbb{Z}$, $x, y \in \mathfrak{g}$, $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ and $\kappa(\ ,\ )$ is the Killing form on $\mathfrak{g}$.

The Lie algebra $\hat{\mathfrak{g}}$ has a triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$. Here $\hat{\mathfrak{n}}^-$ is spanned by the $t^m \otimes y_i$ for $m \leq 0$, along with $t^n \otimes x_i$, and $t^n \otimes h_j$ for $n < 0$. The subalgebra $\hat{\mathfrak{h}}$ is spanned by the $t^0 \otimes h_i$ and K and d. Furthermore, $\hat{\mathfrak{n}}^+$ is spanned by the $t^m \otimes x_i$ for $m \geq 0$, along with $t^n \otimes y_i$ and $t^n \otimes h_j$ for $n > 0$.

The Verma module $M(\lambda)$ of highest weight λ is defined in the same way as for $\mathfrak{g}$. As vector spaces $M(\lambda) \cong U(\hat{\mathfrak{n}}^-)$. Let α be a positive root of $\hat{\mathfrak{g}}$. Then from [13] we get that $M(\lambda - n\alpha)$ embeds in $M(\lambda)$ if and only if $2(\lambda, \alpha) = n(\alpha, \alpha)$, where n is a positive integer.

Now if α is a real root with $2(\lambda, \alpha) = n(\alpha, \alpha)$, then we can construct a singular vector in $U(\hat{\mathfrak{n}}^-)_{n\alpha}$ by essentially the same method as in Section 3.. The only difference is the algorithm for rewriting $f_i^{n-r} a f_i^r$, where $r \in \mathbb{Q}$, $a \in U(\hat{\mathfrak{n}}^-)$, and f_i a basis element of $\mathfrak{n}^-$. We need commutation relations $y^m f_i^r = f_i^r y^m + \cdots$, where y runs through the basis elements of $\hat{\mathfrak{n}}^-$.

First of all, if $f_i = t^j \otimes x_\alpha$ for some $\alpha \in \Phi$, and $y = t^k \otimes x_\beta$ for some $\beta \in \Phi$ such that $\alpha+\beta \in \Phi$, then set $y_{m\alpha+n\beta} = t^{mj+nk} \otimes x_{m\alpha+n\beta}$. Set $B = \{y_{m\alpha+n\beta} \mid m\alpha+n\beta \in \Phi\}$. Then B spans a subalgebra of $\hat{\mathfrak{g}}$ isomorphic to the subalgebra of $\mathfrak{g}$ spanned by the corresponding $x_{m\alpha+n\beta}$. The isomorphism is given by $y_{m\alpha+n\beta} \mapsto x_{m\alpha+n\beta}$. So we get the same formula as in the finite-dimensional case.

Now suppose that $\alpha+\beta = 0$. Then $j+k \le 0$; so $[f_i, y] = t^{j+k} \otimes h_\alpha$, where $h_\alpha = [x_\alpha, x_{-\alpha}]$. In this case we use the following relation:

$$\begin{aligned}(t^k \otimes x_{-\alpha})(t^j \otimes x_\alpha)^r =&(t^j \otimes x_\alpha)^r(t^k \otimes x_{-\alpha}) - \\ & r(t^j \otimes x_\alpha)^{r-1}(t^{k+j} \otimes h_\alpha) - r(r-1)(t^j \otimes x_\alpha)^{r-2}(t^{k+2j} \otimes x_\alpha),\end{aligned}$$

which is easily proved by induction. If $t^k \otimes x_{-\alpha}$ occurs with an exponent > 1 then we use this formula repeatedly.

The last possibility is

$$(t^k \otimes h_q)(t^j \otimes x_\alpha)^r = (t^j \otimes x_\alpha)^r(t^k \otimes h_q) + r\langle \alpha, \alpha_q^\vee \rangle (t^j \otimes x_\alpha)^{r-1}(t^{k+j} \otimes x_\alpha).$$

Again, we use this formula repeatedly if $t^k \otimes h_q$ occurs with exponent > 1.

Now we suppose that $\alpha = m\delta$ is an imaginary root with $(\lambda, \alpha) = 0$ (here δ is the fundamental imaginary root). Then $M(\lambda - n\alpha) \hookrightarrow M(\lambda)$ for all positive integers n. In this case there are a lot of singular elements. One class of them is easily constructed. Let $u_1, \dots, u_q$, $u^1, \dots, u^q$ be two basis of $\mathfrak{g}$, dual to each other with respect to the Killing form. For $n > 0$ set

$$S_n = \sum_{i=1}^{q} \sum_{j=0}^{n} (t^{-j} \otimes u_i)(t^{j-n} \otimes u^i).$$

Lemma 5.1. *Suppose that $(\lambda, \delta) = 0$, then $S_n \cdot v_\lambda$ is a singular vector of weight $n\delta$ in $M(\lambda)$.*

Proof. From [12], 12.8 we have the Sugawara operators

$$T_s = \sum_{m \in \mathbb{Z}} \sum_{i=1}^{q} (t^{-m} \otimes u_i)(t^{m+s} \otimes u^i).$$

It is straightforward to see that $S_n \cdot v_\lambda = T_{-n} \cdot v_\lambda$. Now K acts on $M(\lambda)$ as scalar multiplication by $-h^\vee$, where $h^\vee$ is the dual Coxeter number. But also by [12], Lemma 12.8 we have for $x \in \mathfrak{g}$:

$$[t^m \otimes x, T_{-n}] = 2m(K + h^\vee)(t^{m-n} \otimes x).$$

From this it follows that $x \cdot S_n v_\lambda = 0$ for $0 \le i \le l$, $x \in \mathfrak{n}^+$. Therefore $S_n \cdot v_\lambda$ is a singular vector. ∎

Lemma 5.1 provides an infinite number of singular vectors. However, these are not the only ones. In [17] it is shown that for $n > 0$ and $1 \le i \le l$ there are independent elements $S_n^i \in U(\hat{\mathfrak{n}}^-)$ of weight $n\delta$, such that $S_n^i \cdot v_\lambda$ is a singular vector. These S_n^i are constructed from the generators of the centre of $U(\mathfrak{g})$. In this construction the S_n correspond to the Casimir operator. However, with the exception of the Casimir operator, I do not know of efficient algorithms to construct the generators of the centre of $U(\mathfrak{g})$. For example, the explicit expressions given in [8] for a generator of the centre of degree s involve sums of $(\dim \mathfrak{g})^s$ terms. So constructing generators of the centre of $U(\mathfrak{g})$ appears to be a very hard algorithmic problem in its own right.

The conclusion is that we have efficient algorithms to construct an inclusion $M(\lambda - n\alpha) \hookrightarrow M(\lambda)$ if α is a real root, or when α is imaginary. However, in the last case there are many singular vectors which at present appear to be rather difficult to construct.

6. Constructing irreducible representations

In [16], P. Littelmann proves a theorem describing a particular basis of the irreducible representations of $\mathfrak{g}$, using inclusions of Verma modules. Apart from giving a basis this result also allows one to construct the irreducible representations of $\mathfrak{g}$. In this section we first briefly indicate how this works, and then give some experimental data concerning this algorithm.

The first ingredient of the construction is Littelmann's path method. Here we only give a very rough description of that method; for the details we refer to [14], [15]. A path is a piecewise linear function $\pi : [0,1] \to \mathbb{R}P$, such that $\pi(0) = 0$. Such a path is given by two sequences $\bar{\mu} = (\mu_1, \ldots, \mu_r)$ and $\bar{a} = (a_0 = 0, a_1, \ldots, a_r = 1)$, where the $\mu_i \in \mathbb{R}P$ and the a_i are real numbers with $0 = a_0 < a_1 < \ldots < a_r = 1$. The path π corresponding to this data is given by

$$\pi(t) = (t - a_{s-1})\mu_s + \sum_{i=1}^{s-1}(a_i - a_{i-1})\mu_i \quad \text{for } a_{s-1} \le t \le a_s.$$

Let λ be a dominant weight. Then the path π_λ is given by the sequences (λ) and $(0,1)$, i.e., it is the straight line from the origin to λ. For $\alpha \in \Delta$ there is a path-operator f_α. Given a path π, $f_\alpha(\pi)$ is a new path, or 0. Set $B(\lambda) = \{f_{\alpha_{i_1}} \cdots f_{\alpha_{i_k}}(\pi_\lambda) \mid k \ge 0,\ \alpha_{i_j} \in \Delta\}$, and let $V(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight λ. Then from [14], [15] we have that the endpoints of the paths in $B(\lambda)$ are weights of $V(\lambda)$ and the number of paths with endpoint μ is equal to the dimension of the weight space in $V(\lambda)$ with weight μ.

Let $\pi \in B(\lambda)$ be given by $(\mu_1, \ldots, \mu_r)$ and $(a_0 = 0, a_1, \ldots, a_r = 1)$. Set $\mu_{r+1} = \lambda$ and $\nu_i = a_i\mu_i$ and $\eta_i = a_i\mu_{i+1}$ for $1 \le i \le r$. Then it can be shown that $M(\nu_i) \hookrightarrow M(\eta_i)$. Let $\Theta_i \in U(\mathfrak{n}^-)_{\eta_i - \nu_i}$ be such that $\Theta_i \cdot v_{\eta_i}$ is a singular vector. Then set $\Theta_\pi = \Theta_1 \cdots \Theta_r$. The element $\Theta_\pi \in U(\mathfrak{n}^-)_{\lambda - \pi(1)}$ is determined upto a multiplicative constant.

Now in [16] an inclusion $B(m\lambda) \hookrightarrow B(n\lambda)$ is described for $m < n$. With this inclusion we can view $B(m\lambda)$ as as a subset of $B(n\lambda)$. Furthermore, $B(\lambda, \infty)$ denotes the union of all $B(m\lambda)$ for $m \ge 1$. Write $\lambda = n_1\lambda_1 + \cdots + n_l\lambda_l$, and let $I(\lambda)$ be the left ideal of $U(\mathfrak{n}^-)$ generated by the elements $y_i^{n_i+1}$ for $1 \le i \le l$. Then $V(\lambda) = M(\lambda + \rho)/I(\lambda) \cdot v_\lambda$, where $\rho = \lambda_1 + \cdots + \lambda_l$. Now from [16] we have the following result.

Proposition 6.1. *Suppose that all* $n_i > 0$. *Then* $\{\Theta_\pi \mid \pi \in B(\lambda, \infty), \pi \notin B(\lambda)\}$ *is a basis of* $I(\lambda)$.

(If some $n_i = 0$ then there is a similar result, which we will omit here, cf. [16].)

In order to construct and work with the quotient $M(\lambda+\rho)/I(\lambda)$, we need a basis of $I(\lambda)$. If $\lambda - \mu$ is not a weight of $V(\lambda)$, then $I(\lambda) \cap U(\mathfrak{n}^-)_\mu = U(\mathfrak{n}^-)_\mu$. So we only need bases of the spaces $I(\lambda) \cap U(\mathfrak{n}^-)_\mu$ where $\lambda - \mu$ is a weight of $V(\lambda)$. By the above theorem we can compute those bases by first computing paths $\pi \in B(\lambda, \infty)$ with $\pi(1) = \lambda - \mu$, and then constructing the corresponding Θ_π. We call this algorithm A.

In Table 2, the running times are given of algorithm A on some sample inputs. Also listed are the running times of the algorithm described in [9], which uses a Gröbner basis method to compute bases of the spaces $I(\lambda) \cap U(\mathfrak{n}^-)_\mu$. We call it algorithm B. In order to fairly compare both algorithms, the output in both cases consisted of the representing matrices of a Chevalley basis of $\mathfrak{g}$.

type	λ	$\dim V(\lambda)$	$\sharp$ inclusions	time A (s)	time B (s)
A_2	$(2,2)$	27	64	1	1
A_2	$(3,4)$	90	296	2	5
A_2	$(5,5)$	216	788	7	14
A_3	$(1,1,1)$	64	897	17	6
A_3	$(2,1,1)$	140	2834	56	15
A_3	$(2,1,2)$	300	7837	178	40
B_2	$(2,2)$	81	807	10	6
B_2	$(3,3)$	256	3330	56	23
B_2	$(4,4)$	625	9502	347	79

Table 2: Running times (in seconds) of the algorithms A and B for the construction of $V(\lambda)$. The fourth column displays the number of inclusions of Verma modules computed by algorithm A. The ordering of the fundamental weights is as in [2].

We see that for type A_2, algorithm A competes well with algorithm B. However, for the other types considered this is not the case. In these cases huge numbers of inclusions of Verma modules have to be constructed, which slows the algorithm down considerably. I have also tried to construct $V(\lambda)$ for $\lambda = (1,1,1)$, and $\mathfrak{g}$ of type B_3. But algorithm A did not complete this calculation within the available amount of memory (100M).

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